1. Consider the following version of the Lucas's tree economy. There are two kinds of trees. The first kind is ugly and gives no direct utility to consumers, but yields a stream of fruit $\{d_{1t}\}$, where d_{1t} denotes a positive random process obeying a first-order Markov process. The second tree is beautiful and yields utility on itself. This tree also yields a stream of the same kind of fruit d_{2t}, where it happens that $d_{2t} = d_{1t} = \left(\frac{1}{2}\right) d_t \; \forall \, t$, so that the physical yields of the two kinds of trees are equal. There is one of each tree for each N individuals in the economy. Trees last forever, but the fruit is not storable. Trees are the only source of fruit.

   Each of the N individuals in the economy has preferences described by

$$E_0 \sum_{t=0}^{\infty} \beta^t \, u(c_t, s_{2t}) \tag{1}$$

   where $u(c_t, s_{2t}) = \ln c_t + \gamma \ln(s_{2t})$ where $\gamma \geq 0$, c_t denotes consumption of the fruit in period t and $s2t$ is the stock of beautiful trees owned at the beginning of the period t. The owner of a tree of either kind i at the start of the period receives the fruit d_{it} produced by the tree during that period.

   Let p_{it} be the price of a tree of type i (where $i = 1, 2$) during period t. Let R_{it} be the gross rate of returns of tree i during that period held from period t to $t + 1$.

   (a) Write down the consumer optimization problem in sequential and recursive form.

   Consumer optimization in a recursive form

   The Bellman's equation is given by

$$v(d_t, s_{1t}, s_{2t}) = \max_{\{c_t, s_{1t+1}, s_{2t+1}\}} (\ln(c_t) + \gamma \ln(s_{2t}) + E_t \beta v(d_t, s_{1t+1}, s_{2t+1}))$$

   where $c_t + p_{1t}s_{1t+1} + p_{2t}s_{2t+1} \leq (d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t}$.

   Consumer optimization in a sequential form

The sequential form is given by

$$\max_{\{c_t, s_{1t+1}, s_{2t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln(c_t) + \gamma \ln(s_{2t})\right)$$

where $c_t + p_{1t}s_{1t+1} + p_{2t}s_{2t+1} \leq (d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t}$.

(b) Define a rational expectations equilibrium.

Definition The following is called the *market clear condition*

$$\sum_{i=1}^{I} c_t^i = \sum_{i=1}^{I} d_t^i \tag{2}$$

$$\begin{aligned} \sum_{i=1}^{I} s_{1t}^i &= \sum_{i=1}^{I} s_{10}^i = I \\ \sum_{i=1}^{I} s_{2t}^i &= \sum_{i=1}^{I} s_{20}^i = I \end{aligned}$$

where s_{10}^i and s_{20}^i are each agent's number of trees at initial time.

Definition *A sequential household problem* is defined by each agent's utility optimization problem:

$$\max_{\{c_t, s_{1t+1}, s_{2t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln(c_t) + \gamma \ln(s_{2t})\right)$$

where $c_t + p_{1t}s_{1t+1} + p_{2t}s_{2t+1} \leq (d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t}$.

Definition A *rational competitive equilibrium* is an allocation,

(c) Find the pricing functions mapping the state of the economy at t unto p_{1t} and p_{2t} (give precise formulas). [Hint: You should be able to directly derive p_{1t} from the example seen in class, then since pricing function have to be linear you can guess a pricing function $p_{2t} = kd_t$ and solve for k parameter using Euler equation of the second stock.]

I am going to use the sequential form to find a solution. Using the Lagrange Multiplier,

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t \left((\ln(c_t) + \gamma \ln(s_{2t})\right) \\ &+ \lambda_t((d_{1t} + p_{1t})s_{1t} + (d_{2t} + p_{2t})s_{2t} - c_t - p_{1t}s_{1t+1} - p_{2t}s_{2t+1})) \end{aligned}$$

By taking differentiations with respect to $\{c_t, s_{1t+1}, s_{2t+1}\}$.

$$\begin{aligned} c_t &: E_0\beta^t(\frac{1}{c_t} - \lambda_t) = 0 \\ s_{1t+1} &: E_0\beta^t(\beta\lambda_{t+1}(d_{1t+1} + p_{1t+1}) - \lambda_t p_{1t}) = 0 \\ s_{2t+1} &: E_0\beta^t\left(\beta\lambda_{t+1}(d_{2t+1} + p_{2t+1}) - \lambda_t p_{2t} + \beta\frac{\gamma}{s_{2t+1}}\right) = 0 \end{aligned}$$

Using the Markov property $E_0 = E_t$.

$$\begin{aligned} c_t &: \frac{1}{c_t} = \lambda_t \\ s_{1t+1} &: p_{1t} = E_t\beta\frac{\lambda_{t+1}}{\lambda_t}(d_{1t+1} + p_{1t+1}) \\ s_{2t+1} &: p_{2t} = E_t\beta\frac{\lambda_{t+1}}{\lambda_t}(d_{2t+1} + p_{2t+1}) + \beta\frac{\gamma}{s_{2t+1}\lambda_t} \end{aligned}$$

Combining the first-order conditions, we can obtain the pricing formula.

$$\begin{aligned} p_{1t} &= E_t\beta\frac{c_t}{c_{t+1}}(d_{1t+1} + p_{1t+1}) \\ p_{2t} &= E_t\beta\frac{c_t}{c_{t+1}}(d_{2t+1} + p_{2t+1}) + \beta\frac{\gamma c_t}{s_{2t+1}} \end{aligned}$$

Recall $c_t = d_t$ in a general equilibrium.

$$\begin{aligned} p_{1t} &= E_t \beta \frac{d_t}{d_{t+1}} (d_{1t+1} + p_{1t+1}) \\ p_{2t} &= E_t \beta \frac{d_t}{d_{t+1}} (d_{2t+1} + p_{2t+1}) + \beta \frac{\gamma d_t}{s_{2t+1}} \end{aligned}$$

Let us assume the linear form of the first tree's pricing function.

$$p_{1t} = k_{1t} d_t$$

Next apply the above linear form to the Euler equation.

$$\begin{aligned} k_{1t} d_t &= E_t \beta \frac{d_t}{d_{t+1}} (d_{1t+1} + k_{1t+1} d_{t+1}) \\ k_{1t} &= E_t \beta k_{1t+1} + E_t \beta \frac{d_{1t+1}}{d_{t+1}} \end{aligned}$$

Recursively we obtain k_{1t}.

$$k_{1t} = E_t \sum_{j=1}^{\infty} \beta^j \frac{d_{1t+j}}{d_{t+j}}$$

Since we are given $d_{1t} = \frac{1}{2} d_t$.

$$\begin{aligned} k_{1t} &= \frac{1}{2} E_t \sum_{j=1}^{\infty} \beta^j \\ &= \frac{\beta}{2(1-\beta)} \end{aligned}$$

Let us assume the linear form of the second tree's pricing function.

$$p_{2t} = k_{2t} d_t$$

Next apply the above linear form to the Euler equation.

$$\begin{aligned} k_{2t} d_t &= E_t \beta \frac{d_t}{d_{t+1}} (d_{2t+1} + k_{2t+1} d_{t+1}) + \beta \frac{\gamma d_t}{s_{2t+1}} \\ k_{2t} &= E_t \beta k_{2t+1} + E_t \beta \left(\frac{d_{2t+1}}{d_{t+1}} + \frac{\gamma}{s_{2t+1}} \right) \end{aligned}$$

Recursively we obtain k_{2t}.

$$k_{2t} = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{d_{2t+j}}{d_{t+j}} + \frac{\gamma}{s_{2t+1}} \right)$$

Since we are given $d_{2t} = \frac{1}{2} d_t$ and $s_{2t+1} = 1$ in equilibrium.

$$\begin{aligned} k_{2t} &= E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{1}{2} + \gamma \right) \\ &= \left(\frac{1}{2} + \gamma \right) \frac{\beta}{1-\beta} \end{aligned}$$

Finally, we have got the pricing equations.

$$\begin{aligned} p_{1t} &= \left(\frac{\beta}{2(1-\beta)} \right) d_t \\ p_{2t} &= \left(\left(\frac{1}{2} + \gamma \right) \frac{\beta}{1-\beta} \right) d_t \end{aligned}$$

(d) Prove that if $\gamma > 0$, then $R_{1t} > R_{2t}\ \forall\ t$

The returns R_{1t}, R_{2t} are defined as:

$$R_{1t} = \frac{p_{1t+1} + d_{1t+1}}{p_{1t}}$$

$$R_{2t} = \frac{p_{2t+1} + d_{2t+1}}{p_{2t}}$$

From the derived pricing equations,

$$\begin{aligned} R_{1t} &= \frac{\frac{1}{2}\left(\frac{\beta}{1-\beta} \right) d_{t+1} + \frac{1}{2} d_{t+1}}{\frac{1}{2}\left(\frac{\beta}{1-\beta} \right) d_t} \\ R_{2t} &= \frac{\left(\frac{1}{2} + \gamma \right)\left(\frac{\beta}{1-\beta} \right) d_{t+1} + \frac{1}{2} d_{t+1}}{\left(\frac{1}{2} + \gamma \right)\left(\frac{\beta}{1-\beta} \right) d_t} \end{aligned}$$

By rearranging the equations,

$$R_{1t} - R_{2t} = \left(1 - \frac{1}{1+2\gamma}\right) \frac{1-\beta}{\beta} \frac{d_{t+1}}{d_t}$$

If $\gamma > 0$, then

$$R_{1t} > R_{2t}$$