Jaime Frade
ECO5282-Dr. Garriga
Homework #2

1. Consider an economy with a single consumer. There is one good in the economy, which arrives in the form of an exogenous endowement obeying, $y_{t+1} = \lambda_{t+1}\, y_t$, where $y_t$ is the endowment at time $t$ and $\{\lambda_{t+1}\}$ is governed by a two-state Markov chain with transition matrix, $P = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}$, and intial distribution $\pi_\lambda = [\pi_0, 1-\pi_0]$. The value of $\lambda_t$ is given by $\overline{\lambda_1} = 0.98$ in state 1 and $\overline{\lambda_2} = 1.03$ in state 2. Assume that the history of $y_s$ and $\lambda_s$, up to $t$ is observed at time $t$. The consumer has endowment process $\{y_t\}$ and has preferences over consumption streams that are ordered by $E_0 \sum_{t=0}^{\infty} \beta^t \dfrac{c^{1-\gamma}}{1-\gamma}$, here $\beta \in (0,1)$ and $\gamma \geq 1$.

**(a)** Define a competitive equilibrium.

**Definition** A allocation is said to be a feasible allocation if it satisfies

$$\sum_{i=1}^{2} c_t^i(s^t) = \sum_{i=1}^{2} y_t^i(s_t) \tag{1}$$

$$\text{Since one agent} \rightarrow c_t(s^t) = y_t(s_t) \tag{2}$$

**Definition** A competitive equilibrium is a feasible allocation, $\{c^i\}_{i=1}^2 = \{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i=1}^2$, and a price system, $\{p_t^0(s^t)\}_{t=0}^{\infty}$, such that the allocation solves each household problem.

For a given household $i$ solves

$$U(c^i) = \max_{\{c_t^T(s^t)\}} E_0 \sum_{t=0}^{\infty} \beta^t\, u(c_t^i)$$

$$U(c^i) = \max_{\{c_t^T(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t\, \pi(s^t|s_0)\, u(c_t^i(s^t)) \tag{3}$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} p_t^0(s^t) y_t^i(s_t), \quad \text{and} \quad c_t^1(s^t), c_t^2(s^t) \geq 0 \tag{4}$$

**(b)** From (2), **consumption** = **endowment**, since only one agent obtain the following:

$$c_t = y_t \tag{5}$$

In (5), $y$ is defined as $y_{t+1} = \lambda_{t+1}\, y_t$, where $\lambda_{t+1}$ is defined as a two state Markov Chain. Using the properites of Markov process,

$$y_{t+1} = \prod_{i=0}^{t} \lambda_i\, y_0 \tag{6}$$

Substituting (5) into the first order conditions of the household problem, into (**??**), will obtain an equation that will solve the price system for any asset that will lead to competitive equilibrium.

$$p_t^0(s^t) = \beta^t\, \pi(s^t|s_0)\, \frac{u'(c_t^i(s^t))}{u'(c_0^i(s^0))} \qquad \text{from (??)} \tag{7}$$

$$p_t^0(s^t) = \beta^t\, \pi(s^t|s_0)\, \frac{u'(y_t)}{u'(y_0)} \qquad \text{using (5)} \tag{8}$$

Suppose $p_{11} = .8, p_{22} = .85, \pi_0 = .5, \beta = .96$, and $\gamma = 2$. Suppose economy starts off with $\lambda_0 = .98$ and $y_0 = 1$.

Also, $P = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix} = \begin{bmatrix} .8 & .2 \\ .15 & .85 \end{bmatrix}$ and intial distribution $\pi_\lambda = [\pi_0, 1-\pi_0] = [0.5, .5]$. The value of $\lambda_t$ is given by $\overline{\lambda_1} = 0.98$ in state 1 and $\overline{\lambda_2} = 1.03$ in state 2.

**(c)** I will use the given conditions in the setup of the problem, stated above, and using the properties of a Markov chain, (that is, as time $\to \infty$, the probability of each state occuring will converge to a stationary distrubition) to solve for the average growth rate of consumption.

**Goal:** Find a stationary distribution such that,

$$\sum a_i = 1 \text{ and } a = aP, \text{ where } a = [\alpha, 1-\alpha] \tag{9}$$

Using $P$, setup a system of equations to solve for $\alpha$.

$$a = aP \longrightarrow [\alpha, 1-\alpha] = [\alpha, 1-\alpha] \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}$$

**From top equation, (Eq1)**

$$\alpha = \alpha\, p_{11} + (1-\alpha)(1-p_{22})$$
$$\alpha = \alpha\, p_{11} + 1 - \alpha - p_{22} + \alpha\, p_{22}$$
$$\alpha + \alpha - \alpha\, p_{11} - \alpha\, p_{22} = 1 - p_{22}$$
$$\alpha(2 - p_{11} - p_{22}) = 1 - p_{22}$$
$$\alpha = \frac{1-p_{22}}{(2-p_{11}-p_{22})}$$

and

**From bottom equation, (Eq2)**

$$(1-\alpha) = \alpha\,(1-p_{11}) + (1-\alpha)p_{22}$$
$$(1-\alpha) + (1-\alpha)p_{22} = \alpha\,(1-p_{11})$$
$$(1-\alpha)(1-p_{22}) = \alpha\,(1-p_{11})$$
$$\frac{(1-\alpha)}{\alpha} = \frac{(1-p_{11})}{(1-p_{22})}$$
$$\frac{1}{\alpha} = \frac{(1-p_{11})}{(1-p_{22})} + 1 = \frac{(1-p_{11}+1-p_{22})}{(1-p_{22})}$$

Solving in both equations for $\alpha$ obtain equalvalent realtions, which is suppose to happen since it is a stationary distribution.

$$\textbf{From Eq1: } \alpha = \frac{1-p_{22}}{2-p_{11}-p_{22}} \qquad \textbf{From Eq2: } \alpha = \frac{1-p_{22}}{2-p_{11}-p_{22}} \tag{10}$$

Therefore subsituting the values for $p_{11} = .8$ and $p_{22} = .85$ in (10) will obtain:

$$\alpha = \frac{1-p_{22}}{2-p_{11}-p_{22}} = \frac{1-.85}{2-.80-.85} = \frac{1-.85}{2-.80-.85} = \frac{3}{7}$$

$$a = [\alpha, 1-\alpha] = \left[\frac{3}{7}, \frac{4}{7}\right] \tag{11}$$

By properties of Markov process, as time $\to \infty$, the probability of the each state occuring will converge to the corresponding probabilities in the stationary distribution. Hence:

$$P(\lambda_t = \overline{\lambda_i}\,) = a \quad \text{where i=1,2} \tag{12}$$

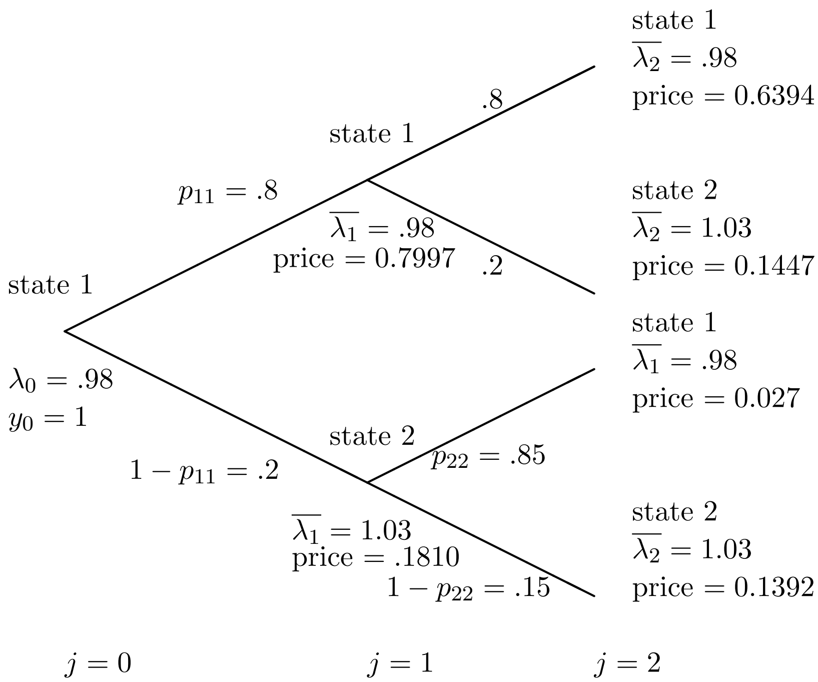
Therefore will obtain

$$P(\lambda_t = \overline{\lambda_1}\,) = \frac{3}{7} \quad \text{and} \quad P(\lambda_t = \overline{\lambda_2}\,) = \frac{4}{7} \tag{13}$$

Using (13), able to compute the average growth rate of consumption, before having observed $\lambda_0$.

$$\begin{aligned} E[\lambda] &= P(\lambda_t = \overline{\lambda_1}\,)\overline{\lambda_1} + P(\lambda_t = \overline{\lambda_2}\,)\overline{\lambda_2} \\ &= \left(\frac{3}{7}\right)(.98) + \left(\frac{4}{7}\right)(1.03) \\ &= 1.0086 \end{aligned} \tag{14}$$

In order to compute the time-0 prices for the for each of the three bonds promising to pay one unit of time-$j = 0, 1, 2$.
Assume that given intial state, $\lambda_0 = .98 = \overline{\lambda_1}$. Thus, has corresponding values, $p_{11} = .8$, and $1 - p_{11} = .2$ on first branch of tree.




From (8), using the following given information, $u(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$, where $\gamma = 2$, $\beta = .96$, and $y_0 = 1$.
Also known from (6), $y_t = \prod_{i=1}^{t} \lambda_i \, y_0 \quad \rightarrow \quad y_t = \prod_{i=1}^{t} \lambda_i$

From the given conditions, will obtain

$$\begin{aligned} p_t^0(s^t) &= \beta^t \, \pi(s^t|s_0) \, \frac{u'(y_t)}{u'(y_0)} \\ p_t^0(s^t) &= (.96)^t \, \pi(s^t|s_0) \left(\frac{y_0}{y_t}\right)^2 \qquad (15) \end{aligned}$$

I will use (15), to compute the price at each time $t = 0, 1, 2$

t=1

- $p^0_1(\overline{s^1}|s_0) = (.96)\,(.8)(.2)\,\left(\frac{1}{.98}\right)^2 = .7997$
- $p^0_1(\underline{s^1}|s_0) = (.96)\,(.2)\,\left(\frac{1}{1.03}\right)^2 = .1810$

t=2

- $p^0_2(\overline{s^2}|s_0) = (.96)^2\,(.8)(.8)\,\left(\frac{1}{(.98)(.98)}\right)^2 = .6394$
- $p^0_2(\underline{\overline{s^2}}|s_0) = (.96)^2\,(.2)(.15)\,\left(\frac{1}{(1.03)(.98)}\right)^2 = .027$
- $p^0_2(\underline{s^2}|s_0) = (.96)^2\,(.8)(.2)\,\left(\frac{1}{(1.03)(.98)}\right)^2 = .1447$
- $p^0_2(\underline{\underline{s^2}}|s_0) = (.96)^2\,(.85)(.15)\,\left(\frac{1}{(1.03)(1.03)}\right)^2 = .1392$

From Cochrane, (pg.52 1ed.), using to what he refers to as the **happy-meal theorem (3.1):**

$$p(x) = \sum_s pc(s)x(s) \tag{16}$$

where $x(s)$ denotes an asset's payoff in state of nature $s$, $pc$ is the price of a contingent claim and $(s)$ is used to denote which state $s$ the claim pays off. In problems **(d)-(f)**, the bond promises to pay one unit, $\mathbf{x(s) = 1}$. From (16), able to compute the price of the bond promising to pay one unit of time-$j$ comsumption for $j = 0, 1, 2$.

**(d)** Compute the time-0 prices of three risk-free discount bonds, in particular, those promising to pay one unit of time-$j$ consumption for $j = 0, 1, 2$, respectively.

t= 0

- $p(x) = 1$

t= 1

- $p(x) = .7997 + .1810 = .9807$

t= 2

- $p(x) = .6394 + .027 + .1447 + .1392 = .9508$

**(e)** Compute the time-0 prices of three bonds, in particular, those promising to pay one unit of time-$j$ contingent consumption on $\lambda_j = \overline{\lambda_1}$ for $j = 0, 1, 2$, respectively.

t= 0

- $p(x) = 1$ from $\lambda_0 = \overline{\lambda_1}$

t= 1

- $p(x) = .7997$

t= 2

- $p(x) = .6394 + .027 = .666$

**(f)** Compute the time-0 prices of three bonds, in particular, those promising to pay one unit of time-$j$ contingent consumption on $\lambda_j = \overline{\lambda_2}$ for $j = 0, 1, 2$, respectively.

t= 0

- $p(x) = 0$ from $\lambda_0 = \overline{\lambda_1}$

t= 1

- $p(x) = .1810$

t= 2

- $p(x) = .1447 + .1392 = .2839$

**(g)** Compare the prices that you computed in parts **(d)-(f)**.

From above, can see that at each time $t$, the sum of each price in answers **(e)** and **(f)**, is equal to the price of the risk free discount bond, **(d)**.

At each time, $t = 0, 1, 2$ **(e)+(f)=(d)**.