Universität Bern
Doing Economics with the Computer

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Session 5: The Cake-Eating Problem

1 The Topic

Once upon a time there was a little girl who got a cake. The girl decided to eat the cake all alone. But she was not sure when she wanted to eat the cake. First, she thought of eating the whole cake right away. But then, nothing would be left for tomorrow and the day after tomorrow. Well, on the one hand, eating cake today is better than eating eat it tomorrow. On the other hand, eating too much at the same time might not be the best. She imagined that the first mouthful of cake is a real treat, the second is great, the third is also nice. But the more you eat, the less you enjoy it. In the end you're almost indifferent, she thought. So, she decided to eat only a bit of the cake everyday. Then, she could eat everyday another first mouthful of cake. The girl knew that the cake would be spoiled if she kept it more than nine days. Therefore, she would eat the cake in the first ten days. Yet, how much should she eat everyday? She thought of eating everyday a piece of the same size. But if eating cake today is better than waiting for tomorrow, how can it possibly be the best to do the same today as tomorrow? If I ate just a little bit less tomorrow and a little bit more today I would be better off, she concluded. — And she would eat everyday a bit less than the previous day and the cake would last ten days long and nothing would be left in the end.

1.1 The Formal Structure of the Story

Let's solve the girl's problem formally. Assume that the preferences for cake are in any period described by the following utility function:

$$U(c_t) = ln\, c_t \tag{1}$$

where, c_t is the amount of cake consumed in period t. This utility function yields decreasing marginal utility. Future consumption is discounted with the time-preference factor $\beta < 1$. The present value in period 0 of the whole consumption path $(c_0, c_1, ..., c_T)$ is, therefore,

$$V(c_0, c_1, ..., c_T) = \sum_{t=0}^{T} \beta^t U(c_t) \quad , \tag{2}$$

where T is the last day with consumption. In the above story T is 9 as "today" is 0. These preferences are called intertemporally additive. The person tries to maximise (2) by choosing its consumption in period $t = 0, ..., T$. The cake size in period t is the previous size less the previous consumption:

$$k_t = k_{t-1} - c_{t-1} . \tag{3}$$

Consumption in any period must not exceed the cake size in that period. Therefore, the cake size must be nonnegative in any period:

$$k_t \geq 0 \text{ for } t = 1 \ldots T+1 . \tag{4}$$

1.2 The Relevance of the Story

The cake-eating problem yields the basic mathematical structure of the optimal growth models in modern macroeconomics. The famous Ramsey

Model (see e.g. Romer 1996, 39) can in principle be reduced to the above cake-eating problem. A society has to choose between consumption or investments. The more a society invests, the less it can consume instantaneously but the more it can produce, and hence consume, in the future. Models of optimal growth determine the optimal level of investment in any period.

1.3 Advanced: The Analytical Solution

The Cake-Eating problem can be solved analytically. The optimal consumption path satisfies the so-called *Euler equation*:

$$U'(c_t) = \beta U'(c_{t+1}) \tag{5}$$

The Euler equation has an intuitive interpretation: at a utility maximum, the consumer cannot gain from feasible shifts of consumption between periods. A one-unit reduction in period t consumption lowers U_t by U_t'. This unit saved can be shifted to period $t + 1$ where it raises utility by U'_{t+1}. Discounted to period t this is $\beta \cdot U'_{t+1}$. In the optimum, these two quantities must be equal. Using utility function (1), the Euler-equation (5) is

$$c_{t+1} = \beta c_t . \tag{6}$$

In the optimum, nothing is left in the end:

$$k_{T+1} = 0 . \tag{7}$$

Recursive insertion of (3) and (6) in (7) yields the optimal initial period consumption:

$$c_0 = \frac{1-\beta}{1-\beta^{T+1}} k_0 . \tag{8}$$

Together with (6) the optimal consumption path is exactly described.

2 The Method

The general method to solve this kind of problems is called *Dynamic Optimisation*. In dynamic optimisation problems a path of variables is choosen in order to maximise a criterion function subject to a set of intertemporal constraints. This is very advanced mathematics and its foundations are hidden from most economists (the authors self-evidently included). Several textbooks show how dynamic optimisation is applied in economics. See e.g. Kamien and Schwartz (1995) or Stokey and Lucas (1989). We will leave this to theorists.

If the problem is small and nice it may be solved numerically. This is an example of numerical constrained optimisation. Using an already programmed algorithm, this is a handy way to solve the problem. (You will learn how to program such an algorithm in session 10.) However, numerical solutions have several disadvantages: they may not be feasible, the algorithm may not find a solution, the algorithm may yield inaccurate results, the algorithm may find a local rather than a global maximum or the algorithm may be very time consuming.

3 The Software

Excel contains a powerful solution algorithms for solving both linear and nonlinear programming problems. Read the separate handout if you are not already familiar with the solver in Excel.

4 References

Kamien, Morton I. and Nancy L. Schwartz (1995), *Dynamic Optimization : The Calculus of Variations and Optimal Control in Economics and Management*, Amsterdam: North-Holland.

Romer, David (1996), *Advanced Macroeconomics*, New York: McGraw-Hill.

Stokey, Nancy L. and Robert E. Lucas (1989), *Recursive Methods in Economic Dynamics*, Cambridge: Harvard University Press.

5 Today's Task

Exercise 1: A Spreadsheet of the Basic Problem

We will implement the above Cake-Eating problem in an Excel spreadsheet. The appendix shows how such a spreadsheet might look like.

a) Make a sketch of what you want to put where on your Excel sheet.
b) Choose a cell in which you can put the value of the time-preference factor β and the initial size of the cake k_0. Describe the contents of the cell in another cell next to it. You may assign names to the cells with the value.
c) Find a place for a table with 11 columns and 4 rows. The first row contains the period numbers from 0 to 10. The second row contains the corresponding period consumption c_t. The third row is the size of the cake

at the beginning of the period t. The fourth and the last row contain the period utility and the discounted period utility respectively.
d) Fill the consumption path with an arbitrary series of number, for example ones. Why is there no consumption in period 10?
e) Assign the initial cake size to period 0. Use equation (3) to compute the cake size in all further periods including period 10.
f) Use the utility function (1) to calculate the period utility.
g) Get the discounted utility from each period applying the inner part of equation (2).
h) Assign the sum of the discounted period utilities to a cell.
i) Which cell should be maximal in the optimal solution of the cake-eating problem?
j) Which cells can be changed in order to maximise the discounted utility.
k) Which cells are concerned with the constraints in (4).
l) Solve the Cake-eating problem using the Solver: Extras/Sover... .
m) Try different starting values. The Solver uses the current values in the cells changed as starting values. Fill, for example, the consumption path with zeros. Explain.
n) Advanced: Add a table with the analytical solution from section 1.3 and compare the result with the numerical optimisation. Try different convergence criterion in the Solver options to obtain more precise results.

Exercise 2: Understand and use the Model

a) How should the consumption path change as the discount factor rises. Check it in your spreadsheet.
b) How changes the solution with different initial cake sizes.
c) What are two fundamental assumptions that drive the resulting

consumption path. Do you think they are reasonable. Do you agree with the outcome.

Exercise 3: The Cake Gets Worse

So far, we assumed that the quality of the cake is constant until day 9 and then suddenly the cake is rotten. What happens if the cake gets worse during the first nine days. We capture this by assuming that the cake's quality depreciates at a constant rate δ. The 'quality adjusted size' of the cake follows therefore:

$$k_t = k_{t-1} - c_{t-1} - \delta(k_{t-1} - c_{t-1}) = (1-\delta)\cdot(k_{t-1} - c_{t-1}). \qquad (9)$$

Again, the cake is rotten after ten days, whatever its size at day 9 was.

d) Change your Excel spreadsheet to include this new feature.

e) Try different depreciation rates and study the reaction of the consumption path.

Exercise 4: A More General Utility Function

Consider the following constant relative risk aversion (CRRA) or constant elasticity of intertemporal substitution utility function:[1]

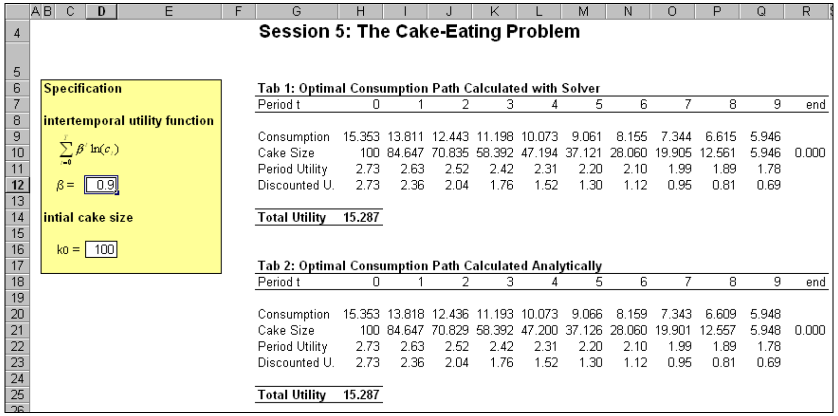
$$u(c) = \frac{c^{1-\theta}}{1-\theta}$$

f) Implement this utility function in your Spreadsheet.

g) Try different parameter values for β, δ, θ.

[1] The initial utility function ln(c) is the limit case of this new utility function as θ goes to 1, when the constant $-1/(1-\theta)$ is added. Remember from microeconomic theory that adding a constant does not change the preference structure.

6 Appendix: The Resulting Excel Sheet and the Solver Parameters



Session 5: The Cake-Eating Problem

Specification

intertemporal utility function

$\sum_{t=0}^{T} \beta^t \ln(c_t)$

β = 0.9

intial cake size

k0 = 100

Tab 1: Optimal Consumption Path Calculated with Solver

Period t	0	1	2	3	4	5	6	7	8	9	end
Consumption	15.353	13.811	12.443	11.198	10.073	9.061	8.155	7.344	6.615	5.946	
Cake Size	100	84.647	70.835	58.392	47.194	37.121	28.060	19.905	12.561	5.946	0.000
Period Utility	2.73	2.63	2.52	2.42	2.31	2.20	2.10	1.99	1.89	1.78	
Discounted U.	2.73	2.36	2.04	1.76	1.52	1.30	1.12	0.95	0.81	0.69	
Total Utility	**15.287**										

Tab 2: Optimal Consumption Path Calculated Analytically

Period t	0	1	2	3	4	5	6	7	8	9	end
Consumption	15.353	13.818	12.436	11.193	10.073	9.066	8.159	7.343	6.609	5.948	
Cake Size	100	84.647	70.829	58.392	47.200	37.126	28.060	19.901	12.557	5.948	0.000
Period Utility	2.73	2.63	2.52	2.42	2.31	2.20	2.10	1.99	1.89	1.78	
Discounted U.	2.73	2.36	2.04	1.76	1.52	1.30	1.12	0.95	0.81	0.69	
Total Utility	**15.287**