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Homework #4

1. **(Equilibrium with Exogenous Borrowing Constraints)** Consider an economy with a continuum [0,1] of consumers of symmetric typed who live forever. Consumers have utility

$$\sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

Consumers of type 1 have an endowment steam of the single good in each period $(w_0^1, w_1^1, w_2^1, \ldots) = (8, 1, 8, 1, \ldots)$ while consumers of type 2 have $(\omega_0^2, \omega_1^2, \omega_2^2, \ldots) = (1, 8, 1, 8, \ldots)$. In addition there is one unit of trees that produses $d = 1$ units of good at each period. Each consumer of type i owns $\bar{s}_0^i$ of such a trees in period $t = 0$, $\bar{s_0^i} > 0$, $\bar{s}_0^1 + \bar{s}_0^2 = 1$. Trees do not grow or decay over time.

(a) (i) Define an Arrow-Debreu equilibrium.

Definition An <u>Arrow-Debreu equilibrium</u> is an allocation $\{\{c^i, \bar{s}_{t+1}^i\}_{t=0}^{\infty}\}_{i=1}^{2}$ and a sequence of prices $\{q_t, r_t\}_{t=0}^{\infty}$ such that the allocation solves each household problem. For a given household, the consumer solves

$$\max \sum_{t=0}^{\infty} \beta^t \, u(c_t^i) \tag{1}$$

Subject to the following budget constraint

$$c_t^i + q_t(\bar{s}_{t+1}^i - \bar{s}_t^i) \leq \omega_t^i + d\, \bar{s}_t^i \;\; \forall t \tag{2}$$

$$\bar{s}_t^i \geq 0$$

Goods and financial markets clear

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega \;\; \forall t \tag{3}$$

$$\bar{s}_t^1 + \bar{s}_t^2 \geq 1 \;\; \forall t \tag{4}$$

From the initial problem, a symmetric allocation implies the following:

$$c^* = \widehat{c}_t^1 = \widehat{c}_t^2 = \frac{\omega^g + \omega^b + d}{2} = \frac{8+1+1}{2} = 5 \;\; \forall t \tag{5}$$

(ii) Compute the equilibrium prices.

The Euler equation for a symmetric equilibrium is also satisfied. Since $u'(c^g) = u'(c^b)$ and $u'(c^g) = u'(c^b)$ obtain the following:

$$\frac{u'(c^g)}{\beta\, u'(c^b)} = \frac{q+d}{q} = \frac{u'(c^g)}{\beta\, u'(c^b)}$$

$$\frac{1}{\beta} = \frac{q+d}{q}$$

Then, using $d = 1$, the equilibrium prices satisfies:

$$q = \frac{\beta}{1-\beta} d = \frac{\beta}{1-\beta} \tag{6}$$

(iii) Given the endowment, find the intial asset holdings $\bar{s}_0^1$ and $\bar{s}_0^2$ such that in equilibrium, $\widehat{c}_t^1 = \widehat{c}_t^2 \ \forall\ t$. [**Hint:** You have to becareful when writing down the consolidated budget constraint for the individuals.]

When the financial markets clear, (13), then, the aggregate resource constraint as well as the consumer budget constraint are satisfied. Compute the steady state trade associated to the optimal consumption level:

$$\begin{aligned} \frac{\omega}{2} - \omega^g &= (p+d)\bar{s}_0^2 - p\bar{s}_0^1 \\ \omega^b - \frac{\omega}{2} &= (p+d)\bar{s}_0^1 - p\bar{s}_0^2 \end{aligned}$$

Solve for the optimal share distribution by solving a linear system of equations:

$$\begin{bmatrix} p+d & -p \\ -p & p+d \end{bmatrix} \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} = \begin{bmatrix} \frac{\omega}{2} - \omega^g \\ \omega^b - \frac{\omega}{2} \end{bmatrix} \tag{7}$$

Consumers of type 1 will recieve the a good shock in the intial period, while consumers of type 2 recieve a bad shock.

Compute the intial asset holdings $\bar{s}_0^1$ and $\bar{s}_0^2$, using the following subsitutions for the equilibrium price (6), $d = 1$, $\omega^g = 8$, $\omega^b = 1$, and $\frac{\omega}{2} = 5$ into (7), to obtain:

$$\begin{aligned} \begin{bmatrix} p+d & -p \\ -p & p+d \end{bmatrix} \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} &= \begin{bmatrix} \frac{\omega}{2} - \omega^g \\ \omega^b - \frac{\omega}{2} \end{bmatrix} \\ \begin{bmatrix} \frac{1}{1-\beta} & -\frac{\beta}{1-\beta} \\ -\frac{\beta}{1-\beta} & \frac{1}{1-\beta} \end{bmatrix} \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} &= \begin{bmatrix} -3 \\ 4 \end{bmatrix} \end{aligned}$$

Solving for $\bar{s}_0^1$ and $\bar{s}_0^2$ in the previous equation will obtain the following:

$$\begin{aligned} \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{1-\beta} & -\frac{\beta}{1-\beta} \\ -\frac{\beta}{1-\beta} & \frac{1}{1-\beta} \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{1+\beta} & \frac{\beta}{1+\beta} \\ \frac{\beta}{1+\beta} & \frac{1}{1-\beta} \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ \begin{bmatrix} \bar{s}_0^2 \\ \bar{s}_0^1 \end{bmatrix} &= \begin{bmatrix} \frac{4\beta-3}{1+\beta} \\ \frac{4-3\beta}{1+\beta} \end{bmatrix} \end{aligned} \tag{8}$$

From (8), obtain the initial asset holdings for $\bar{s}_0^2 = \frac{4\beta-3}{1+\beta}$ and $\bar{s}_0^1 = \frac{4-3\beta}{1+\beta}$

(b) Suppose that consumers cannot borrow or lend. Define an equilibrium for this liquidity constraint economy.

Definition An Arrow-Debreu equilibrium is an allocation $\{\{c^i, \bar{s}^i_{t+1}\}_{t=0}^{\infty}\}_{i=1}^{2}$ and a sequence of prices $\{q_t, r_t\}_{t=0}^{\infty}$ such that the allocation solves each household problem. For a given household, the comsumer solves

$$\max \sum_{t=0}^{\infty} \beta^t\, u(c^i_t) \tag{9}$$

Subject to the following budget constraint

$$c^i_t + q_t(\bar{s}^i_{t+1} - \bar{s}^i_t) \leq \omega^i_t + d\,\bar{s}^i_t \;\forall t \tag{10}$$

$$\bar{s}^i_t \geq -A$$

Goods and financial markets clear

$$c^1_t + c^2_t = \omega^g + \omega^b + d \;=\; \omega \;\forall t \tag{11}$$

$$\tag{12}$$

$$\bar{s}^1_t + \bar{s}^2_t \;\geq\; 1 \;\forall t$$

(c) Compute the stochastic discount factor for this economy and compare it with the complete markets solution

(d) Consider the function

$$F^L(c^g) = u'(c^g)(c^g - 8) + \beta\, u'(10 - c^g)(9 - c^g) \tag{13}$$

(i) Show that when $\beta = 0.9$, $F^L(c^g) > 0$ and that $\widehat{c}_t^i = 5$ satisifies all of the equilibrium conditions for the liquidity constraint economy for the right choice of $\bar{s}_0^i$.

From the initial problem and above conditions, plug each of the following: $u(c^i) = \log(c_t^i)$, $\widehat{c}_t^i = 5 \Rightarrow c^g = 5$, and $\beta = .9$ into (13) to obtain:

$$\begin{aligned} F^L(c^g) &= \frac{c^g - 8}{c^g} + \frac{\beta\,(9 - c^g)}{10 - c^g} \\ F^L(5) &= \frac{5 - 8}{5} + \frac{(.9)(9 - 5)}{10 - 5} \\ F^L(5) &= .12 > 0 \end{aligned} \tag{14}$$

From the above calculations, (14), obtain $F^L(c^g) = .12 > 0$ when $\widehat{c}_t^i = 5$ and $\beta = .9$

(ii) Show that when $\beta = 0.2$, the solution of $F^L(c^g) = 0$ and $c^g \in [5, 8]$ is such that

$$\widehat{c}_t^i = \begin{cases} c^g & \text{if } \omega_t^i = 8 \\ 10 - c^g & \text{if } \omega_t^i = 1 \end{cases}$$

is an equilibrium allocation for the right choice of $\bar{s}_0^i$. (**Hint:** You can find the solution of $F^L(c^g) = 0$ by solving a quadratic equation)

From the initial problem and above conditions, plug each of the following: $u(c^i) = \log(c_t^i)$ and $\beta = 0.2$ into (13) to obtain:

$$F^L(c^g) = \frac{c^g - 8}{c^g} + \frac{\beta\,(9 - c^g)}{10 - c^g} \tag{15}$$

To solve (15), using the given hint can solve $F^L(c^g) = 0$ by applying quadratic formula.

$$\begin{aligned} 0 &= \frac{c^g - 8}{c^g} + \frac{\beta\,(9 - c^g)}{10 - c^g} \\ \frac{c^g - 8}{c^g} &= \frac{(0.2)(c^g - 9)}{10 - c^g} \\ (c^g - 8)(10 - c^g) &= 0.2c^g(c^g - 9) \\ -c^{2g} + 18c^g - 80 &= 0.2c^{2g} - 1.8c^g \\ 1.2(c^g)^2 - 19.8c^g + 80 &= 0 \end{aligned} \tag{16}$$

Apply the quadratic formula to (16) to find solution(s) for for c^g.

$$c_{1,2}^g = \frac{19.8 \pm \sqrt{(-19.8)^2 - 4(12)(80)}}{2(1.2)}$$

From above will obtain the following solutions:

$$c_1^g \approx 9.43 \tag{17}$$

$$c_2^g \approx 7.069 \tag{18}$$

Since must apply $c^g \in [5, 8]$ from the intial problem setup, choose $\mathbf{c_2^g \approx 7.069.}$

2. **(Equilibrium with Endogenous Borrowing Constraints)** Consider the economy from the previous question. However, we assume that consumers face borrowinf constraints of the for

$$\sum_{j=0}^{\infty} \beta^i \log c^i_{t+j} \geq \sum_{j=0}^{\infty} \beta^i \log \omega^i_{t+j} \;\; \forall t, i$$

but otherwise the definition of equilibrium is the same as in **part (a)**.

(a.) Provide a motivation for this environment and explain the economic initution of the constraint.

$$\underbrace{\sum_{j=0}^{\infty} \beta^i \log c^i_{t+j}}_{(A)} \geq \underbrace{\sum_{j=0}^{\infty} \beta^i \log \omega^i_{t+j}}_{(B)} \;\; \forall t, i \tag{19}$$

At no time should the borrower see deafualting as optimal choice. At any time each household may choose to default on credit payments and walk away from the credit market. If the consumer chooses to default, the punishment is an exclusion from all future activity in the credit market. At no time should the value of choosing to default (see **(B)**) should not exceed the value of staying in the market (see **(A)**). Lender will never loan to the borrower so that the borrower will choose to default.

(b.) Define an equilibrium for this debt constaint economy.

Definition An equilibrium is an allocation $\{c^1_t, c^2_t\}_{t=0}^{\infty}$ and a sequence of prices $\{q_t\}_{t=0}^{\infty}$ such that each consumer i solves

$$\max \sum_{t=0}^{\infty} \beta^t \, u(c^i_t) \tag{20}$$

Subject to the following budget constraint

$$\sum_{t=0}^{\infty} p_t \, c^i_t \;\; = \;\; \sum_{t=0}^{\infty} p_t(\omega^i_t + s^i_0 d) \; \forall \text{ all periods} \tag{21}$$

Financial markets clear

$$c^1_t + c^2_t = \omega^g + \omega^b + d = \omega \;\; \forall t \tag{22}$$

(c.) Consider the function

$$F^D(c^g) = u(c^g) - u(8) + \beta[u(10 - c^g) - u(1)] \tag{23}$$

(i.) Show that when $\beta = 0.9$, $F^D(c^g) > 0$ and that $\widehat{c}^i_t = 5$ satisifies all of the equilibrium conditions for the debt constraint economy for the right choice of $\bar{s}^i_0$.

From the initial problem and above conditions, plug each of the following: $u(c^i) = \log(c^i_t)$, $\widehat{c}^i_t = 5 \Rightarrow c^g = 5$, and $\beta = .9$ into (23) to obtain:

$$\begin{aligned} F^D(c^g) &= u(c^g) - u(8) + \beta\,(u(10 - c^g) - u(1)) \\ F^D(5) &= u(5) - u(8) + (.9)(u(10-5) - u(1)) \\ F^D(5) &= \log(5) - \log(8) + (.9)(\log(5) - \log(1)) \\ F^D(5) &= 0.97849 \end{aligned} \tag{24}$$

From the above calculations, (14), obtain $F^L(c^g) = 0.97849 > 0$ when $\widehat{c}^i_t = 5$ and $\beta = .9$

(ii.) Show that when $\beta = 0.2$, the solution of $F^D(c^g) = 0$ and $c^g \in [5,8]$ is such that

$$\hat{c}_t^i = \begin{cases} c^g & \text{if } \omega_t^i = 8 \\ 10 - c^g & \text{if } \omega_t^i = 1 \end{cases}$$

is an equilibrium allocation for the right choice of $\bar{s}_0^i$. (**Hint:** If $\beta = .2$, the solution to $F^D(c^g) = 0$ and $c^g \in [5,8]$ is approximately $c^g \approx 6.09076$) From the initial problem and above conditions, plug each of the following: $u(c^i) = \log(c_t^i)$ and $\beta = 0.2$ into (23) to solve $F^D(c^g) = 0$:

$$F^D(c^g) = u(c^g) - u(8) + \beta[u(10 - c^g) - u(1)] \tag{25}$$

After subsitutions, will use log properities and algebra to covert (25) into a higher order polynomial in which **Maple** will find the solution(s).

$$\begin{aligned}
0 &= \log(c^g) - \log(8) + (.2)(\log(10 - c^g) - \log(1)) \\
0 &= \log(c^g) - \log(8) + \log(10 - c^g)^{0.2} \\
0 &= \log_{10}\left(\frac{c^g(10 - c^g)^{0.2}}{8}\right) \\
1 &= \left(\frac{c^g(10 - c^g)^{0.2}}{8}\right) \\
8 &= c^g(10 - c^g)^{0.2} \\
8^5 &= (c^g)^5((10 - c^g)^{\frac{1}{5}})^5 \\
32768 &= 10c^{5g} - c^{6g} \\
(c^g)^6 - 10(c^g)^5 + 32768 &= 0 \qquad \text{Let } c^g \to x \\
x^6 - 10x^5 + 32768 &= 0 \qquad (26)
\end{aligned}$$

Using Maple, (see code below), obtain the following solution for $F^D(c^g) = 0$ in which $c^g \in [5,8]$.

```
> z:= x^6-10*x^5+32768;

                    6        5
          z := x   - 10 x  + 32768

> fsolve(z, x, 5..8);

                 6.090760678
```

Therefore, when $\beta = .2$, the solution to $F^D(c^g) = 0$ and $c^g \in [5,8]$ is $c^g \approx 6.09076$

(d.) Compute the stochastic discount factor for this economy and compit it with the liquiidty constraint model. Do you think these two economies can predict the same pattern for asset price movements?

(e.) Unfortunatly, the value of $\bar{s}_0^i$ that you caculated in the previous section when $\beta = 0.2$ is negative. Can you think of another way to make the proposed steady state an equilibrium? Explain.

(f.) Write down the endogenous borrowing constraint if the agent is allowed to save in an storage technology with a return $R > 0$, following a default period.

(g.) Bankruptcy law only preclude individuals from trading just for a finite number of periods. How would you omodify this constraint to accomodate this institutional feature.

3. **(Equilibrium with Endogenous Borrowing Constraints and Uncertainity)** We want to take the model one step furteher and include uncertiainty along two dimensions. First, we assume thar income shocks are persistent, and evolve according to a symmetric Markov process with a probability distrubution

$$\prod_{\omega'|\omega} = \begin{bmatrix} \pi_{gg} & \pi_{bg} \\ \pi_{gb} & \pi_{bb} \end{bmatrix} = \begin{bmatrix} 1-\pi & \pi \\ \pi & 1-\pi \end{bmatrix}$$

Second we assume i.i.d. dividend shock where γ is the probability of recieving a good dividend shock $d_g = 3$, and $1-\gamma$ is the probability of recieving a bad dividend or aggregrate shock, $d_b = 1$. Since we now have a aggregate uncertainity we can still solve for a symmetric equilibrium, but consumption across states of nature will differ due to aggregrate uncertainity.

(a.) Write down the market equilibrium in the endogenous borrowing constraint economy. [**Hint:** You should be very careful when writing the endogenous borrowing constraint, since it could depend on aggregate uncertainity.]

Definition An equilibrium in the stochastic economy is a contingent consumption allocation $\{\{c_t^i\}_{t=0}^{\infty}\}_{t=1}^{2}$ and a state of contingent prices $\{p_t\}_{t=0}^{\infty}$ such that each consumer i solves

$$\max E_0 \sum_{t=0}^{\infty} \beta^t\, u(c_t^i) \tag{27}$$

Subject to the following budget constraint

$$E_0 \sum_{t=0}^{\infty} p_t\, c_t^i \;\; \leq \;\; E_0 \sum_{t=0}^{\infty} p_t(\omega_t^i + s_0^i d) \;\forall\; t \tag{28}$$

Financial markets clear

$$c_t^1 + c_t^2 = \omega^g + \omega^b + d = \omega \;\; \forall t \tag{29}$$

(b.) Solve for the equilibrium allocations when the borrowing constraint binds, and calculate the stochastic discount factor in the presence of aggregate shocks. Will the borrowing constraint always bind? [**Hint:** You can use the social planner problem to solve for efficent allocations.]