3

Introduction to Stability Analysis

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This chapter provides an introduction to the stability analysis of discretized ODEs. It is a tutorial review of some basic definitions and techniques distributed over many books. The next Chapter discusses tools for symbolic analysis of stability. As noted in Chapter 2, using partitioned analysis gives high flexibility of implementation. The downside of this freedom is the large number of possibilities. The motivation for doing stability analysis first is to filter out unsuitable choices.

§3.1. Stability

The term "stable" informally means *resistant to change*. For technical use the term has to be defined more precisely in term of the mathematical model, but the same connotation applies.

§3.1.1. Mathematical Models

By *mathematical model* is meant the governing differential equations used for the simulation of a mechanical system. Generally we will deal with a semidiscrete model: discrete in space and continuous in time. In the time domain the model is given by ordinary differential equations (ODE) in time. For such models the following definition of stability, due to Dirichlet,[1] has survived the test of time:

> *"The equilibrium* [of a mechanical system] *is stable if, in displacing the points of the system from their equilibrium positions by an infinitesimal amount and giving each one a small initial velocity, the displacements of different points of the system remain, throughout the course of the motion, contained within small prescribed limits"*

Although the definition is dynamic in nature, it addresses *equilibrium solutions*. Definitions beyond that point become more difficult. But most look only at *steady state solutions*. An exception is chaotic systems, as in turbulence models, but we will not look at that class of problems in this course.

One extension is looking at deterministic periodic solutions: say the Earth orbiting around the Sun (or vice-versa if you believe in Aristotle). The underlying idea is the same. Suppose that a system is undergoing periodic motion: $\mathbf{u}(t+T) = \mathbf{u}(t)$. Perturb it by applying an arbitrary but tiny initial displacement or velocity. Study the subsequent motion. If the motion remains within small prescribed limits of the unperturbed motion "orbit" the steady state motion is stable, and unstable otherwise. The static steady-state case addressed by Dirichlet is of course included.

§3.1.2. Difference Equations

An integrator applied for temporal discretization produces a difference system. The foregoing definitions can be briefly recapitulated by looking at the sequence of computed solutions. Precise definitions are provided in texts cited in the **Notes and Bibliography** Section. Here we are interested in the stability of the *time integrator*. Informally, we want the time discretization

1. To preserve the stability of stable mathematical models, and
2. To manifest the instability of unstable mathematical models.

The first condition is classical. The second one is not. In the context of coupled systems, overstability can be as damaging as instability. For example, suppose one tries to determine the flutter boundary of an aircraft by doing simulations at increasing speeds. If the time integrator is strongly dissipative, the onset of flutter may be masked when running at finite step sizes, leading to erroneous conclusions.

[1] As it appears in his Appendix to the German translation of Lagrange's *Mécanique Analytique* [3.52].

§3.1.3. Stability Analysis Methods

There are two general classes of stability analysis methods:

Amplification Methods. Also called *von Neumann stability analysis*. Based on decomposition of motion into normal modes, often using Fourier analysis, and superposition. The analysis looks at the growth or decay of perturbations from one step to the next, and can be implemented using standard linear algebra procedures. It is *local* in nature, but so is the concept of stability. A more severe restriction is that it strictly applies only to *linear* systems. Despite this limitation it is frequently applied to nonlinear systems through linearization.

Energy Methods. Also known, notably in control theory, as Lyapunov methods. These look at the variation of certain function (or functional) measures of the motion amplitude. Often these are related to energy measures of various kinds, hence the name. Energy methods are not restricted to linear systems, but require the construction of suitable measures, and this has to be done case by case.

Because the systems examined in these lectures are linear, the amplification method is used.

§3.1.4. Test Equations

In practice stability analysis is not performed on the discrete systems of actual applications but on *test equations*, which are highly simplified models of the real thing. The idea is similar to the patch test in space discretizations: time integrators that do not do well on the test equations can be discarded right away. Those that survive can be subjected to further tests. Of course the ultimate test comes on the actual applications, but by then the choices are hopefully narrowed.

The selection of test equations ranges from established procedures (for standard ODE integrators) to more of a black art (in coupled systems). In this Chapter a scalar test equation is taken as given, and the stability analysis done on it. This introduces the reader to commonly used time integrators.

§3.2. A First-Order ODE Test Equation

The most commonly used scalar test equation for first-order ODEs, and also the simplest, is

$$\boxed{\dot{y} = \lambda y.} \tag{3.1}$$

Here $y = y(t)$ and λ is a coefficient independent of time, which is generally complex. To simplify the following study we will restrict its real part to be nonpositive: $\Re[\lambda] \leq 0$. If so the solutions $y(t) = Ce^{\lambda t}$ of (3.1) are stable in the sense of being bounded for $t > 0$. We therefore expect the time integrator to preserve this attribute. Gear [3.28, p. 9] defines *absolute stability* on the test equation (3.1), treated by an integrator of stepsize h, as follows:

> "*The region of absolute stability is that set of values of h (real nonnegative) and λ for which a perturbation in a single value y_n will produce a change in subsequent values which does not increase from step to step.*"

This definition is relevant for $\Re[\lambda] \leq 0$ and will be used in the next two sections. It fails for $\Re[\lambda] > 0$, as discussed in §3.6.

§3.2.1. A One-Step Integrator

The general one-step integrator in the class of Linear Multistep Methods (LMS) is

$$y_{n+1} = y_n + h\,[\beta \dot{y}_{n+1} + (1-\beta)\dot{y}_n]\,, \tag{3.2}$$

where β is a coefficient in the range $[0, 1]$. Three well known members of this family are:

Forward Euler (FE) integrator (also known as explicit Euler): $\beta = 0$.

Backward Euler (FE) integrator (also known as implicit Euler): $\beta = 1$.

Trapezoidal Rule (TR) integrator: $\beta = \frac{1}{2}$.

To get a difference equation, state that (3.1) is verify at $t = t_n$: $\dot{y}_n = \lambda y_n$, and at $t = t_{n+1}$: $\dot{y}_{n+1} = \lambda y_{n+1}$. Replacing those derivatives into (3.2) yields $y_{n+1} = y_n + h\big[\beta\lambda y_{n+1} + (1-\beta)\lambda y_n\big]$. Move all terms pertaining to the next time:t_{n+1} to the left hand side to get the difference system

$$(1-\beta\lambda h)y_{n+1} = \big[1 + (1-\beta)\lambda h\big]y_n. \tag{3.3}$$

Solve for y_{n+1} to obtain the *amplification form*:

$$y_{n+1} = \frac{1+(1-\beta)\lambda h}{1-\beta\lambda h}y_n = \frac{1+(1-\beta)\mu}{1-\beta\mu}y_n = z\,y_n, \quad \text{where} \quad \mu = \lambda h. \tag{3.4}$$

Here z denotes the *amplification factor*. This is the amount by which the solution at one step is multiplied to get to the next step, hence the name. If λ is complex, so is z. The criterion for stability is that its modulus does not exceed unity:

$$|z| = \left|\frac{1+(1-\beta)\mu}{1-\beta\mu}\right| \le 1. \tag{3.5}$$

Note that this is verified for stepsize $h = 0$ because if so $\mu = \lambda h = 0$ for any λ, and $z = 1$. So the question is: what happens for finite $h > 0$? There are three possibilities:

(C) If (3.5) is verified for $h = [0, h_{max}]$, with $h_{max} > 0$, the integrator is called *conditionally stable*. In this case h_{max} is the *stability limit* and $[0, h_{max}]$ the *stability range*.

(U) If (3.5) is violated for any stepsize $h > 0$, the integrator is called *unconditionally unstable*.

(A) If (3.5) is verified for all $h \ge 0$ (meaning that $h_{max} \to \infty$) the integrator is called *A-stable*.

Obviously (A) is the most desirable objective, followed by (C), whereas (U) is unacceptable.

```
z=(1+(1-β)*μ)/(1-β*μ);
Plot[{z/.β->0,z/.β->1/2,z/.β->1},{μ,-4,0}, Frame->True,
     PlotStyle->{{AbsoluteThickness[2],RGBColor[0,0,0]},
                 {AbsoluteThickness[2],RGBColor[1,0,0]},
                 {AbsoluteThickness[2],RGBColor[0,0,1]}}];
```

Figure 3.1. Script to generate the plot of Figure 3.2.

§3.2.2. Real Negative λ

If λ is real and negative, the amplification factor behavior for different β can be simply obtained by the *Mathematica* script of Figure 3.1. The generated plot of z versus $\mu = \lambda h \in [0, -4]$ for FE, BE and TR is shown in Figure 3.2. This is called an *amplification plot*. Clearly BE ($\beta = 1$) and TR ($\beta = \frac{1}{2}$) are in the stable region $|A| \le 1$, and it is not difficult to show that this happens for all $h > 0$. On the other hand, FE clearly goes unstable for $h > -2/\lambda$. If fact, setting $\beta = 0$ in (3.5) gives $A = 1 + \mu$, which crosses the stability boundary $A = -1$ at $\mu = -2$ and goes to $-\infty$ as $\mu \to \infty$.

§3.2.3. Complex λ

The complex λ case is of particular interest when the test equation comes from the reduction of a second order equation with oscillatory solutions. An amplification plot such as that in Figure 3.2 can be done via a 3D display that shows $\Re(\mu)$ and $\Im(\mu)$ and z in the x, y and z directions, respectively. This has the advantages of showing the magnitude of z but can be difficult to visualize. The alternative is to stick with a 2D plot over the complex μ plane, with $\Re(\mu)$ and $\Im(\mu)$ along x and y, respectively, and displaying only the stable and unstable regions. This is called a *stability chart*.

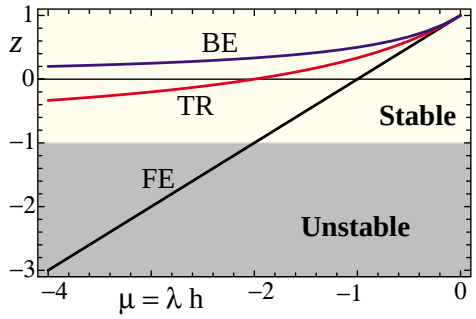

Figure 3.2. Amplification factor z for real λ, $\mu = \lambda h \in [0, -4]$ and three integrators.

```
StabilityIndicator[μ_,β_]:=If[Abs[(1+(1-β)*μ)/(1-β*μ)]<=1,1,0];
ContourPlot[StabilityIndicator[μR+I*μI,0],{μR,-4,0},{μI,-2,2},
            PlotPoints->200,ContourLines->False];
```

Figure 3.3. Script to generate the plot of Figure 3.4.

The *Mathematica* script shown in Figure 3.3 produces a stability chart for (3.5). The result is shown in Figure 3.4 for the FE ($\beta = 0$) integrator over the left part of the μ plane. The plot technique deserve some explanation. The function

$$\texttt{StabilityIndicator}[\mu,\beta] \tag{3.6}$$

evaluates $z = [1 + (1 - \beta)\mu]/[1 - \beta\mu]$, which is complex if μ is. It returns 1 if $|z| \le 1$ and 0 otherwise. So given a $\{\mu, \beta\}$ pair, (3.6) gives 1 if stable, and 0 if unstable. When this discontinuous function is displayed as a contour plot in the complex μ plane with sufficiently high resolution (requested by saying `PlotPoints->200`), the only contour lines are very close to stability boundaries.

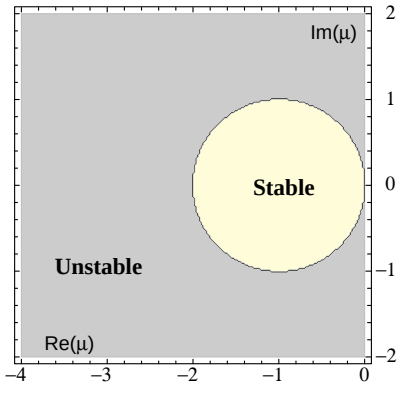

Figure 3.4. Stability chart for complex $\mu = \lambda h$, and the Forward Euler integrator.

The results can be clearly observed in Figure 3.4, which shows the stability boundary to be the circle $|1+\mu| \le 1$. The option `ContourLines->False` asks that contour lines not be shown to reduce clutter, making contrast between stable vs. unstable region colors sharper.

For real λ the stable interval is $\mu = h\lambda = [0, -2]$ as previously found. For purely imaginary λ there is no stable interval since the circle does not intersect the imaginary axis and FE becomes unconditionally unstable. Doing this plot for TR and BE gives no stability boundaries over the $\Re(\lambda) \in [-4, 0]$ range: the whole plotted region is stable. It is easily shown that both methods are A-stable for any complex λ with $\Re(\lambda) < 0$.

§3.3. A First-Order ODE System

Suppose that instead of one equation we have the (admittedly contrived) homogeneous system of four first-order ODEs with constant coefficients:

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{bmatrix} = - \begin{bmatrix} 85 & 51 & -75 & -45 \\ 51 & 85 & -45 & -75 \\ -75 & -45 & 85 & 51 \\ -45 & -75 & 51 & 85 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}. \quad \text{or} \quad \dot{\mathbf{u}} = \mathbf{L}\,\mathbf{u}. \tag{3.7}$$

where the u_i are functions of time.

§3.3.1. Reduction to Spectral Form

Computing the eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of $\mathbf{L}$ furnishes the spectral decomposition $\mathbf{L} = \mathbf{V}\mathbf{\Lambda}\,\mathbf{V}^T$ and $\mathbf{V}^T\mathbf{L}\mathbf{V} = \mathbf{\Lambda}$, in which

$$\mathbf{\Lambda} = \mathbf{diag}[\lambda_i] = - \begin{bmatrix} 1024 & 0 & 0 & 0 \\ 0 & 256 & 0 & 0 \\ 0 & 0 & 64 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{V} = \tfrac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}. \tag{3.8}$$

Since $\mathbf{L}$ is real symmetric, the eigenvalues are real and the matrix of normalized eigenvectors $\mathbf{V}$ is orthogonal: $\mathbf{V}^T\mathbf{V} = \mathbf{I}$. Make the change of variables:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \text{or} \quad \mathbf{u} = \mathbf{V}\,\mathbf{y}, \quad \mathbf{y} = \mathbf{V}^T\mathbf{u}, \quad \dot{\mathbf{u}} = \mathbf{V}\,\dot{\mathbf{y}}, \quad \dot{\mathbf{y}} = \mathbf{V}^T\dot{\mathbf{u}}. \tag{3.9}$$

This change converts (3.7) to $\mathbf{V}\,\dot{\mathbf{y}} = \mathbf{V}\,\mathbf{\Lambda}\mathbf{y}$. Since $\mathbf{V}$ is nonsingular, premultiplying by $\mathbf{V}^T = \mathbf{V}^{-1}$ yields

$$\dot{\mathbf{y}} = \mathbf{\Lambda}\mathbf{y}, \quad \text{or} \quad \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = - \begin{bmatrix} 1024 & 0 & 0 & 0 \\ 0 & 256 & 0 & 0 \\ 0 & 0 & 64 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}. \tag{3.10}$$

This is called the *spectral form* of (3.7). Note that the four equations in y_i uncouple. Thus they can be written as four scalar equations

$$\dot{y}_1 = -1024\,y_1, \quad \dot{y}_2 = -256\,y_2, \quad \dot{y}_3 = -64\,y_3, \quad \dot{y}_4 = -4\,y_4. \tag{3.11}$$

These have the form of the model system (3.1) with λ set to the four eigenvalues λ_i of $\mathbf{L}$ in turn.

§3.3.2. FE Stability Analysis

To fix the ideas suppose that (3.7) is treated by the Forward Euler (FE) integrator: $\mathbf{u}_{n+1} = \mathbf{u}_n + h\dot{\mathbf{u}}_n$. Making the change of variables (3.9) shows that this is equivalent to treating (3.10) by the FE integrator in $\mathbf{y}$, that is, $\mathbf{y}_{n+1} = \mathbf{y}_n + h\dot{\mathbf{y}}_n$, which holds for each uncoupled equation. From the results of the last Section, for each of the uncoupled equations (3.11) the stable stepsize is bounded by $|\lambda_i h| \le 2$. The stable stepsize for the system is the minimum of the four:

$$h_{max} = \min\left(\frac{2}{1024}, \frac{2}{256}, \frac{2}{64}, \frac{2}{4}\right) = \frac{2}{1024} = \frac{2}{\lambda_{max}}, \quad \text{where} \quad \lambda_{max} = \max_{i=1}^{4} |\lambda_i| = 1024. \tag{3.12}$$

For the TR and BE integrators it is easy to verify that A-stability is retained.

§3.4. A Second Order ODE

ODEs with second or higher time derivatives can be reduced to a system of first order equations. As an example, consider the second-order ODE that governs the behavior of an undamped, unforced linear oscillator of circular frequency ω:

$$\ddot{d} + \omega^2 d = 0. \tag{3.13}$$

where $d = d(t)$ is the displacement from equilibrium and ω is real nonnegative. This can be reduced to a first order system by introducing the velocity $v = \dot{d}$ as auxiliary variable:

$$\begin{bmatrix} \dot{d} \\ \dot{v} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ \omega^2 & 0 \end{bmatrix} \begin{bmatrix} d \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \dot{\mathbf{u}} = -\mathbf{L}\mathbf{u} = \mathbf{0}, \quad \mathbf{u} = \begin{bmatrix} d \\ v \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}. \tag{3.14}$$

Suppose that this is treated by the FE integrator: $\mathbf{u}_{n+1} = \mathbf{u}_n + h\dot{\mathbf{u}}_n$, or equivalently $d_{n+1} = d_n + h\dot{d}_n$ and $v_{n+1} = v_n + h\dot{v}_n$. the resulting difference system is

$$\mathbf{u}_{n+1} = \mathbf{P}\mathbf{u}_n, \quad \mathbf{P} = \begin{bmatrix} 1 & h \\ -h\omega^2 & 1 \end{bmatrix} \tag{3.15}$$

where $\mathbf{P}$, called the *amplification matrix*, is the generalization of the amplification factor found in (3.4). Since $\det(\mathbf{P}) = 1 + \omega^2 h^2$, the eigenvalues of $\mathbf{P}$ are $z_1 = 1 + h\omega i$ and $z_2 = 1 - h\omega i$, with $i = \sqrt{-1}$. The largest eigenvalue modulus is called the *spectral radius* and is denoted by ρ. Both eigenvalues have the same modulus; thus $\rho = |z_1| = |z_2| = \sqrt{1 + h^2\omega^2}$. A necessary condition for stability is that the spectral radius do not exceed unity:

$$\rho = |\sqrt{1 + h^2\omega^2}| \le 1, \quad \text{or} \quad 1 + h^2\omega^2 \le 1. \tag{3.16}$$

This condition (3.16) is only satisfied for $h = 0$, and otherwise violated. Consequently the FE method is *unconditionally unstable* for the oscillator problem (3.13), a conclusion that generalizes to undamped mechanical systems. The result could also be discerned graphically from Figure 3.4, by moving along the imaginary axis because the eigenvalues of $\mathbf{L}$ are purely imaginary: $\pm\omega i$.

§3.5. General ODE Systems

The rule illustrated by (3.12) can be generalized to a homogeneous first-order linear system with

$$\dot{\mathbf{u}} = \mathbf{L}\,\mathbf{u}, \quad \text{or} \quad \dot{\mathbf{u}} - \mathbf{L}\,\mathbf{u} = \mathbf{0}, \tag{3.17}$$

where $\mathbf{L}$ is a $m \times m$ square matrix. For a linear system, the entries of $\mathbf{L}$ cannot be functions of the state $\mathbf{u}$, but may depend on time.

§3.5.1. Constant Coefficients

If $\mathbf{L}$ is independent of t, the system (3.17) is said to have *constant coefficients*. To simplify matters, it is convenient to assume that $\mathbf{L}$ is diagonalizable with distinct nonzero eigenvalues λ_i through the similarity transformation

$$\mathbf{V}^{-1}\,\mathbf{L}\,\mathbf{V} = \mathbf{diag}(\lambda_i) = \mathbf{\Lambda}, \quad \text{or} \quad \mathbf{L} = \mathbf{V}\mathbf{\Lambda}\,\mathbf{V}^{-1}. \tag{3.18}$$

Here $\mathbf{\Lambda}$ is the diagonal matrix of eigenvalues λ_i, $\mathbf{V}$ a matrix with the right eigenvectors of $\mathbf{L}$ as columns, and $\mathbf{V}^{-1}$ a matrix with the left eigenvectors as rows. Making the change of variables

$$\mathbf{u} = \mathbf{V}\,\mathbf{y}, \quad \mathbf{y} = \mathbf{V}^{-1}\mathbf{u}, \quad \dot{\mathbf{u}} = \mathbf{V}\,\dot{\mathbf{y}}, \quad \dot{\mathbf{y}} = \mathbf{V}^{-1}\dot{\mathbf{u}}, \tag{3.19}$$

reduces (3.17) to $\mathbf{V}\,\dot{\mathbf{y}} = \mathbf{V}\,\mathbf{\Lambda}\,\mathbf{y}$, which premultiplied by $\mathbf{V}^{-1}$ yields the uncoupled system $\dot{\mathbf{y}} = \mathbf{\Lambda}\,\mathbf{y}$. This system is treated by an integrator that produces k uncoupled difference equations, the stability of which can be investigated by the amplification method. Suppose that $\bar{h}_i$ is the maximum stable stepsize for the i^{th} equation. The stability limit for the system is the minimum of the stable stepsizes:

$$h_{max} = \min_{i=1}^{k} \bar{h}_i. \tag{3.20}$$

If $\mathbf{L}$ is singular or has a defective eigensystem, additional precautions are necessary. Those are discussed in several of the texts cited in the **Notes and Bibliography** Section.

§3.5.2. Variable Coefficients

If the entries of $\mathbf{L}$ depend on t, the foregoing technique can still be used by diagonalizing $\mathbf{L}$ at each time step. In this cases the stability analysis is *local*, that is, it represents the local behavior of the system to a first approximation.

§3.5.3. Forced First-Order Linear System

A generalization of (3.17) results on adding a forcing term

$$\dot{\mathbf{u}} - \mathbf{L}\,\mathbf{u} = \mathbf{f}(t) \tag{3.21}$$

But it turns out that for stability analysis the forcing term $\mathbf{f}(t)$ is irrelevant and can be dropped . Consequently it is sufficient to consider the homogeneous forms analyzed before.

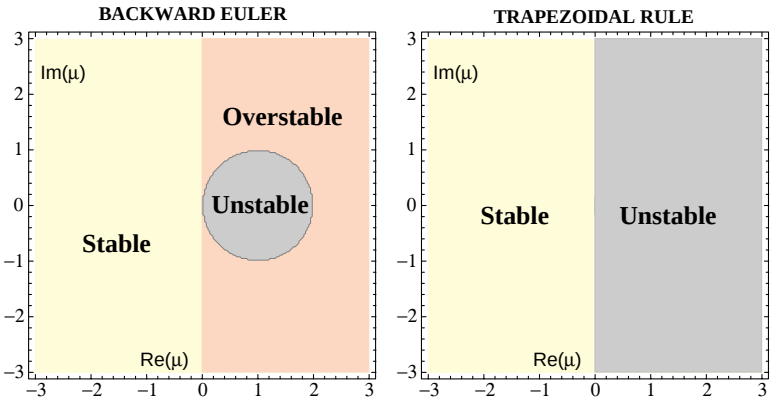

Figure 3.5. Stability charts for BE (left) and TR (right) over the complex μ plane. The overstability region for BE is shown in red.

§3.5.4. Linearization of General First-Order System

A fairly general form[2] of a system of first-order nonlinear differential equations is

$$\dot{\mathbf{u}} = \mathbf{g}(\mathbf{u}, t) \tag{3.22}$$

where $\mathbf{u}$ is a vector of k components $u_i(t)$ and $\mathbf{g}$ a vector of k functions $g_i(\mathbf{u}, t)$. To investigate the local behavior of (3.22) at $\mathbf{u} = \mathbf{u}_0$ and $t = t_0$ for deviations $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0$ and $\Delta t = t - t_0$, linearize it by Taylor series

$$\mathbf{u} \approx \frac{\partial \mathbf{g}}{\partial \mathbf{u}}\,\Delta\mathbf{u} + \frac{\partial \mathbf{g}}{\partial t}\Delta t = \mathbf{L}\,\Delta\mathbf{u} + \mathbf{q}\Delta t, \quad \mathbf{L} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_0, t=t_0}, \mathbf{q} = \left.\frac{\partial \mathbf{g}}{\partial t}\right|_{\mathbf{u}=\mathbf{u}_0, t=t_0}. \tag{3.23}$$

so matrix $\mathbf{L}$ appears as the Jacobian of $\mathbf{g}$ with respect to $\mathbf{u}$. The amplification analysis may be carried out on (3.23) as first approximation. The results, however, may be suspect for highly nonlinear systems. The energy method cited previously does not suffer from that drawback but has to reworked for different problems.

§3.6. Overstability

Overstability means getting a bounded numerical solution for a differential equation or system that has unbounded solutions as t grows. As previously remarked, this can be dangerous in some situations such as flutter simulations, since physical instability is masked. The phenomenon can be easily illustrated for Backward Euler (BE) applied to the test equation $\dot{y} = \lambda y$ with $\Re(\lambda) > 0$. The analytical solutions $y(t) = Ce^{\lambda t}$ grow up exponentially. Applying BE: $y_{n+1} = y_n + h\dot{y}_{n+1}$ gives the amplification factor

$$z = \frac{1}{1-\mu}, \quad \mu = \lambda h. \tag{3.24}$$

The BE stability chart can be generated by the script of 3.3, in which the second argument of `StabilityIndicator` is 1 whereas the plot range is $\Re(\mu) \in [-3, 3]$ and $\Im(\mu) \in [-3, 3]$ so as

[2] This form assumes an explicit solution for $\dot{\mathbf{u}}$. most general form is implicit: $\mathbf{F}(\dot{\mathbf{u}}, \mathbf{u}, t) = \mathbf{0}$. But there are few results and integrators available for the most general case.

to include portion of the right hand μ plane. The chart is shown on the left of Figure 3.5. Note the large region of overstability, for example $\mu = \lambda h > 2$ for real λ. This is caused by the numerical damping injected by the method. On the other hand, the stability chart for TR, shown on the right of Figure 3.5, indicates perfect agreement between physical and numerical stability.

Overstability is not always dangerous. If the physical system is known to preclude instability and if a key objective is to converge rapidly to a steady-state solution, the large numerical damping added by an overstable integrator such as BE may help in attaining that goal. This scenario arises in pseudo dynamical simulations.

Notes and Bibliography

The notion of stability as resistance to change is universal. Specific definitions vary with the field. For dynamics and control, see [3.55,3.56]. For dynamical systems from a broader and more recent perspective, see [3.83], which has a comprehensive reference list. For chaotic systems exhibiting "orbit shadowing" see [3.51].

There are many textbooks and monographs that focus on the numerical solution of ordinary differential equations. Of the old (pre-1980) crop, by far the best organized and written is Gear's textbook [3.28]. As can be expected from its publication date it is outdated in some topics but still a gem. Lapidus and Seinfield [3.54] contains useful material for chemical engineering. Lambert's textbook [3.53] has well chosen examples. Henrici's monograph [3.39] for scalar ODEs was groundbreaking in the 1960s, and the sequel [3.40] for systems is still worth keeping as theoretical backup. One of the numerical analysis classics is Dalhquist and Björck [3.9], recently reprinted by Dover. This text contains a tutorial introduction to computational methods for ODEs, skipping proofs.

Of the more recent crop, one may cite Butcher [3.7], Hairer and colleagues [3.36,3.37], Sewell [3.80] and Shampine [3.81]. The most comprehensive and ambitious is the two-volume set [3.36,3.37]. Sample code for solving ODEs is provided in the multi-language Numerical Recipes series; for example [3.74] in Fortran (a dead language, but there is a companion book for C). The implementation quality, as typical of NR, ranges from mediocre to laughable to dismal; however for one shot student projects it might be acceptable.

There are fewer books dealing with the energy method, as it is highly problem dependent. The book by Straughan [3.82] focuses on convective fluid mechanics. Others can be found in the bibliography of [3.83]. For the algebraic eigenproblem the "bible" is Wilkinson [3.91]. Nothing else comes close in comprehensiveness and understanding, although as can be expected from a 1965 publication date several eigensolution methods are outdated or missing.

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