1

Thank you.

SAFE TRAVELS

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③ Wiener process

Properties

① w/ probability 1, the continuous mapping $t \to W(t)$

w/ $W(0) = W_0$

② if $0 = t_0 < t_1 < t_2 \dots < t_N = T$, then the increments

$\Delta W_t = W(t_n) - W(t_{n-1})$ are independent

w/ quadratic variation

$\{W(t), W(t)\} = t$

③ $\forall\ t > s$, then $W(t) - W(s) \sim N(0, t-s)$

$E(W(t) - W(s)) = 0$

$E((W(t) - W(s))^2) = t - s$

that is

$$P(W_t - W_s \in \Gamma) = \int_\Gamma \frac{e^{\frac{-x^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\,dx \quad (\Gamma \in \mathbb{R})$$

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(9) State and derive Feyman Kac formula

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for smooth a, b, g, h and V, bounded functions.

$$u(t,x) = E\left[g(x(T))\, e^{\int_t^T V(s,x(s))\,ds} \,\middle|\, x_t = x\right] =$$

$$+ E\left[-\int_t^T h(s,x(s))\, e^{\int_t^s V(\tau, x(\tau))\,d\tau}\, ds\right]$$

then u solve the PDE

$$L_V^* u \equiv u_t + a u_x + \frac{b^2}{2} u_{xx} + Vu = h \qquad t < T \qquad (1)$$

$$u(\cdot, T) = g(\cdot)$$

Correct enunciate

Proof

① define $\hat{u}$ to solve (1) then

$$L_V^* \hat{u} = h$$

and

let $G(s) \equiv e^{\int_t^s V(s,x(s))\,ds}$

better to use another variable to avoid confusion with the upper limit s!

$\hat{U}$ cannot be solution to both these different PDE's (unless V and h are zero)

② $\hat{u}$ solves the Kolmogorov's BWD PDE

$$\begin{cases} L\hat{u} = u_t + a u_x + \frac{1}{2} b^2 u_{xx} = 0 & t < T \\ \hat{u}(\cdot, T) = g(\cdot) \end{cases}$$

NO

In step 1 you defined $\hat{U}$ to solve another PDE ??

Apply Ito to $\hat{u}$ and use ?

③ integrate from t to T and take expected value

Note $L^*\hat{u} + V\hat{u} = L_V^* \hat{u} \overset{=h}{}$

$$L^* \hat{u} = h - V\hat{u}$$

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$$d\left(\hat{u}(t,x)\, e^{\int_t^s V(s,x(s))\,ds}\right) = d(\hat{u}\,G)$$

these should have been $(s, X(s))$

Otherwise you have constant term!

$$= G\,d\hat{u} + \hat{u}\,dG$$

$$= G\left[L\hat{u}\,dt + b u_x\,dW\right] + \hat{u} V G\,dt$$

then from t to T

determinstic

$$E\left[g(x(T)) \mid x_t = x\right] - u(t,x) =$$

$$= \int_t^T E\left[G L\hat{u}\,ds\right] + \cancel{\int_t^T E\left[b G u_x\,dW_s\right]} + \int_t^T E\left(\hat{u} V G\right] ds$$

$$= \int_t^T E\left[G h\right] ds - \cancel{\int_t^T E(\hat{u} V G)\,ds} + \cancel{\int E(\hat{u} V G)\,ds}$$

$$u(t,x) = E\left(g(x(T)) \mid x_t = x\right] - E\left[\int_t^T G h\,ds\right]$$

Missing $G(X_T)$

Not correct

Does not correspond to what you enunciated before on the previous page.

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STATE how to apply MC to compute an integral

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Define $I = \int_a^b g(x)\,dx$ $\qquad g(x) : [0,1]^d \to \mathbb{R}$

this is different from your domain of integration !!

NOT CONSISTENT

represent the integral as a expected value.

$$I = \int_a^b \frac{g(x)}{f_x(x)} f_x(x)\,dx \qquad f_x > 0 \qquad f_x \text{ is density function}$$

$$g(x) \neq 0$$

$$I = E\left[\frac{g(x)}{f_x(x)}\right]$$

assum $[a,b]^d \to \mathbb{R}$

are you working with d=1 or with a general dimension?

$$f_x = \begin{cases} \frac{1}{(b-a)^d} & a < x < b \\ 0 & \text{else} \end{cases}$$

NOT TRUE

$$E(g(x)) = \frac{1}{b-a}\int_a^b g(x)dx \quad I = (b-a)\frac{1}{N}\sum_{n=1}^{N} g(x_n)$$

unbiased estimator of μ is sample mean

there is no time dependence in the original problem, why should there be any sum here?

$$E(g(x(T))) = \sum_{j=1}^{N} \frac{g(\bar{x}(T; w_j))}{N}$$

when $\bar{x}$ is approx of x

ie. Euler Method ???

Ex let $I = \int_{[0,1]^d} f(x)\,dx$

$$I = \int_{[0,1]^d} f(x)\,p(x)\,dx \qquad p(x) \text{ is uniform density function}$$

$$I = E(f(x))$$

$$I \simeq \sum_{i=0}^{N} \frac{f(x_i)}{N}$$

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Error in M.C. method

2 parts to error

$\boxed{\text{① Time discretization error (TD error)}}$ ← X

② Statistical error

★ come back to #13.

- There will be NO TD error when ? (converges weakly)
  ↑ in the case of this problem!!
- STATISTICAL error will go to zero b/c of CLT.

$$E\left(g(x(T))\right) - \frac{\sum^{N} g(\bar{x}(T;w_i))}{N} = E\left(g(x(T)) - g(\bar{x}(T))\right]$$

X

NO time dependence in the problem I asked!

$$+ \frac{\sum E(g(\bar{x}(T)) - g(\bar{x}(T;w_i))}{N}$$

Ex. computation of error from previous example NO TD ERROR

$$\epsilon_N = \int_{[0,1]} f^2(x)\,dx - \left(\int_{[0,1]} f(x)\,dx\right)^2$$

this does not depend on N at all!

this is not the M.C. error!

$$= \int_{[0,1]} \left(f(x) - \int f(x)\,dx\right)^2 dx$$

as $\sqrt{N}\,\epsilon_N \to 0$ by CLT $\sim N(0,1)$

X Not with your def. of ϵ_N

$\sigma^2 =$

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