Exercise 1 *The following stochastic volatility model [FPSS00] generalizes the well known Black-Scholes geometric Brownian motion model [BS73], improving some aspects of option pricing.*
A simplified version of the model reads

$$
\begin{aligned}
dS(t) &= rS(t)dt + e^{Y(t)}S(t)dW(t) && (1)\\
dY(t) &= \left(-\alpha(1+Y(t)) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\right)dt + 0.4\sqrt{\alpha}d\widehat{Z}(t), && (2)
\end{aligned}
$$

where W and Z are independent Wiener process and

$$\widehat{Z}(t) \equiv \rho W(t) + \sqrt{1-\rho^2}Z(t) \quad (3)$$

Note: The correlation coefficient is $\rho = -0.3$.

To solve the SDE (1), define the following:

$$f(x) = \log(x) \quad (4)$$

Solving the SDE (1) by Itô formula:

$$
\begin{aligned}
d\log S_t &= \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}(dS_t)^2\\
&= r\,dt + e^{Y_t}dW_t - \frac{1}{2}e^{2Y_t}d_t\\
&= \left(r - \frac{1}{2}e^{2Y_t}\right)dt + e^{Y_t}dW_t && (5)
\end{aligned}
$$

Integrating (5), obtain the following:

$$
\begin{aligned}
\log S_t &= \log S_0 + \int_0^t (r - \frac{1}{2}e^{2Y_s})\,ds + \int_0^t e^{Y_s}dW_s\\
S_t &= S_0 e^{\int_0^t (r-\frac{1}{2}e^{2Y_s})ds + \int_o^t e^{Y_s}dW_s} && (6)
\end{aligned}
$$

Solving the SDE (2):

$$\begin{aligned}
dY_t &= \left(-\alpha(1+Y_t) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\right) dt + 0.4\sqrt{\alpha} d\widehat{Z}_t \\
&= \left(-\alpha\, Y_t + \underbrace{0.4\sqrt{\alpha}\sqrt{1-\rho^2} - \alpha}_{=M}\right) dt + \underbrace{0.4\sqrt{\alpha}}_{\sigma} d\widehat{Z}_t \\
&= (-\alpha Y_t + M) dt + \sigma\, d\widehat{Z}_t \\
&= -\alpha \left(Y_t - \frac{M}{\alpha}\right) dt + \sigma\, d\widehat{Z}_t \qquad \alpha \neq 0 \qquad (7)
\end{aligned}$$

Define the following:

$$X_t = Y_t - \frac{M}{\alpha} \qquad (8)$$

Rewriting (7) by substituting (8), obtain the following:

$$dX_t = -\alpha\, X_t\, dt + \sigma\, d\widehat{Z}_t \qquad (9)$$

To solve (9) use the following integrating factor:

$$V_t = e^{\alpha\, t} X_t \qquad (10)$$

Differentiating (10) and subsituting (9) obtain:

$$\begin{aligned}
dV_t &= d(e^{\alpha\, t} X_t) \\
&= \alpha e^{\alpha\, t} X_t dt + e^{\alpha\, t} \underbrace{dX_t}_{=(9)} \\
&= \alpha e^{\alpha\, t} X_t dt - e^{\alpha\, t} \alpha\, X_t dt + e^{\alpha\, t} \sigma\, d\widehat{Z}_t \\
dV_t &= e^{\alpha\, t} \sigma\, d\widehat{Z}_t \qquad (11)
\end{aligned}$$

From (11), as well as using (10), obtain an equation for X_t.

$$\begin{aligned}
\underbrace{V_t}_{(10)} &= V_0 + \int_0^s e^{\alpha\, s} \sigma\, d\widehat{Z}_s \\
e^{\alpha\, t} X_t &= X_0 + \int_0^t e^{\alpha\, s} \sigma\, d\widehat{Z}_s \\
X_t &= X_0 e^{-\alpha\, t} + \int_0^t e^{-\alpha\,(t-s)} \sigma\, d\widehat{Z}_s \qquad (12)
\end{aligned}$$

From (8), and substituting into above, we obtain an equation for Y_t.

$$\begin{aligned}
\underbrace{X_t}_{(12)} &= X_0 e^{-\alpha\, t} + \int_0^t e^{-\alpha\,(t-s)} \sigma\, d\widehat{Z}_s \\
Y_t - \frac{M}{\alpha} &= \left(Y_0 - \frac{M}{\alpha}\right) e^{-\alpha\, t} + \int_0^t e^{-\alpha\,(t-s)} \sigma\, d\widehat{Z}_s \\
Y_t &= \left(Y_0 - \frac{M}{\alpha}\right) e^{-\alpha\, t} + \frac{M}{\alpha} + \int_0^t e^{-\alpha\,(t-s)} \sigma\, d\widehat{Z}_s \qquad (13)
\end{aligned}$$

Compare the following explicit and implicit method for the computation of the option value below.

Explicit Method:

$$
\begin{aligned}
S_{n+1} - S_n &= rS_n \Delta t + e^{Y_n} S_n \Delta W_n, \\
Y_{n+1} - Y_n &= \Big(-\alpha(1 + \underbrace{Y_n}) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\Big) \Delta t + 0.4\sqrt{\alpha}\, \Delta \widehat{Z}_n \\
\widehat{Z}_n &= \rho W_n + \sqrt{1-\rho^2} Z_n
\end{aligned}
$$

Inside the code: Explicit method

```
S1(n+1) = (1+ dt*r + dW(n)*exp(V(n)))*S1(n);
V(n+1) = V(n)+(-alpha*V(n)-alpha+tmp*sqrt(1-rho^2))*dt + tmp * dhZ(n);
```

Implicit Method:

$$
\begin{aligned}
S_{n+1} - S_n &= rS_n \Delta t + e^{Y_n} S_n \Delta W_n, \\
Y_{n+1} - Y_n &= \Big(-\alpha(1 + \underbrace{Y_{n+1}}) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\Big) \Delta t + 0.4\sqrt{\alpha}\, \Delta \widehat{Z}_n \\
\widehat{Z}_n &= \rho W_n + \sqrt{1-\rho^2} Z_n
\end{aligned}
$$

Inside the code: Implicit method

```
S(n+1) = (1+ dt*r + dW(n)*exp(Y(n)))*S(n);
Y(n+1) = (Y(n) + (-alpha+tmp*sqrt(1-rho^2))*dt + tmp * dhZ(n))/(1+alpha*dt);
```

Recall: The implicit Euler is absolutely stable; the explicit Euler method is not.

Computation of the option value. Use $\alpha = 200, r = 0.04, T = \frac{3}{4}$ and $S_0 = K = 100$.

$$
V_T = e^{-rT} \mathrm{E}[\max(S_T - K, 0)] \tag{14}
$$

a. *Motivate which above numerical methods is best in this case. Provide a numerical approximation, based on uniform time steps, to the option value with accuracy TOL=10^{-2}.*

Setup for numerical approximation:

Pseudocode

Selection of Δt and M

- choose Δt, fix $M = 10^3$

  while $\mathrm{E}[g(X_{\Delta t} - g(X_{\frac{\Delta t}{2}})] + \frac{C\sigma_{\text{error}}}{\sqrt{M}} > \frac{1}{3}\text{TOL}$

  output: $\Delta t = \Delta t^*$

- choose M, fix Δt^*

  while $\widehat{\sigma}^2 = E(g(X)^2) - E(g(X))^2$

  check: $|\text{error}_{statistical}| < \frac{C_\alpha \widehat{\sigma}}{\sqrt{M}} \le \frac{2}{3}\text{TOL}$

b. *In the limit $\alpha \to \infty$ one obtains the results from geometric Brownian motion. Can you verify this by numerical experiments?*

Note: In geometric Brownian motion with constant volatity, we can compute the value of the option exactly using the the Black-Scholes formula. The numerical results obtained approach this thoerotical price.

From (13), let the limit $\alpha \to \infty$,

$$Y_t = \left(Y_0 - \frac{M}{\alpha}\right) e^{-\alpha t} + \frac{M}{\alpha} + \int_0^s e^{-\alpha (t-s)} \sigma \, d\widehat{Z}_s$$

$$\begin{aligned} \lim_{\alpha\to\infty} \frac{Y_0}{e^{\alpha t}} &= 0 \\ \lim_{\alpha\to\infty} -\frac{0.4\sqrt{\alpha}\sqrt{1-\rho^2}+\alpha}{\alpha\, e^{\alpha t}} &= 0 \\ \lim_{\alpha\to\infty} \frac{0.4\sqrt{\alpha}\sqrt{1-\rho^2}-\alpha}{\alpha} &= -1 \\ \lim_{\alpha\to\infty} -\frac{0.4\sqrt{\alpha}}{e^{\alpha (t-s)}} &= 0 \end{aligned} \tag{15}$$

From above, (15), we obtain as $\alpha \to \infty$, $Y_t \to -1$.

However, when we calculate the $Var(Y_t)$, we see that similiar behavior as above (15), does not occur:

$$\begin{aligned} E\left[\int_0^t \sqrt{\alpha}\, e^{-\alpha(t-s)} d\widehat{Z}_s\right]^2 &= \int_0^t \alpha e^{-2\alpha(t-s)} ds \\ &= \frac{e^{-2\alpha(t-s)}}{2\alpha}\alpha\Big|_0^t \\ &= \frac{1}{2} - \frac{e^{-2\alpha t}}{2} \end{aligned} \tag{16}$$

Therefore, from(16), let the limit $\alpha \to \infty$, obtain:

$$\begin{aligned} \lim_{\alpha\to\infty} E\left[\int_0^t \sqrt{\alpha}\, e^{-\alpha(t-s)} d\widehat{Z}_s\right]^2 &= \lim_{\alpha\to\infty}\left(\frac{1}{2} - \underbrace{\frac{e^{-2\alpha t}}{2}}_{=0}\right) \\ &= \frac{1}{2} \end{aligned} \tag{17}$$

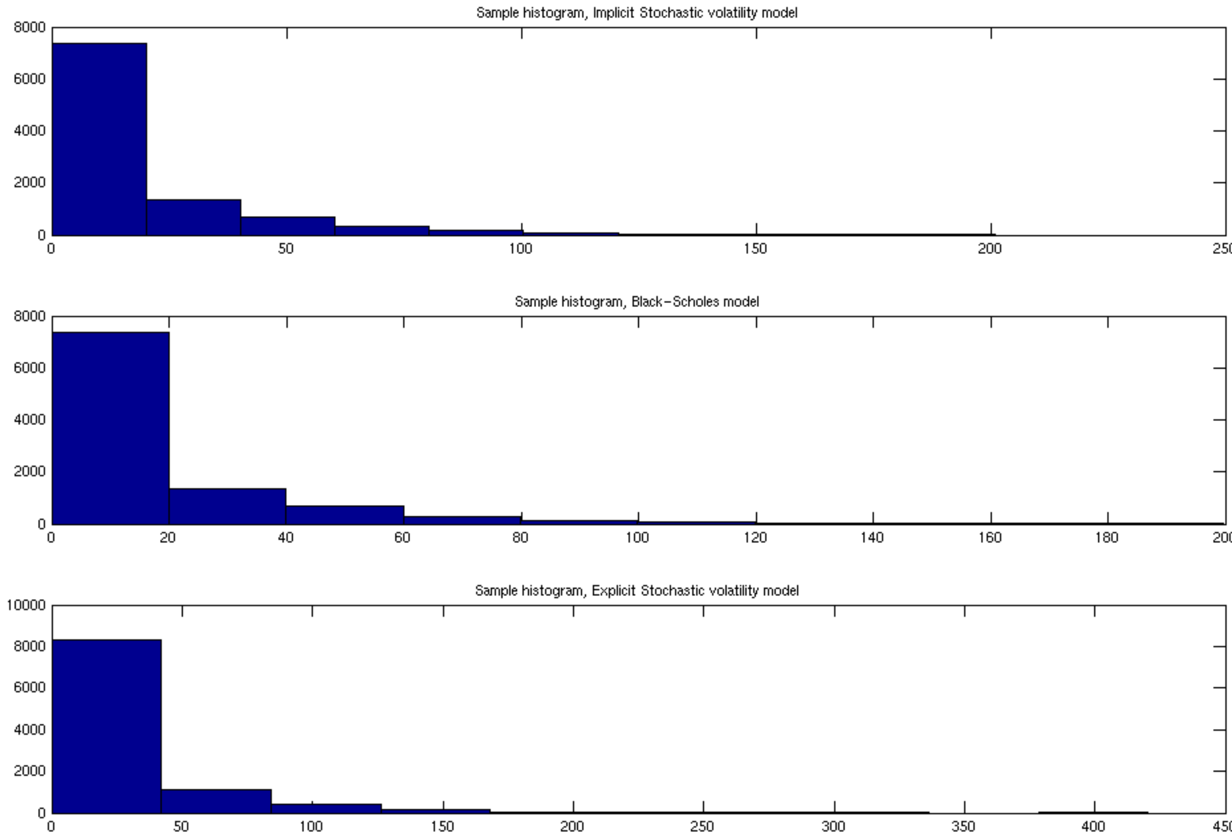
Using this results, we can see that the volatilty from (1):

$$e^{Y_t} \to e^{-1} \quad \text{as} \quad \alpha \to \infty$$

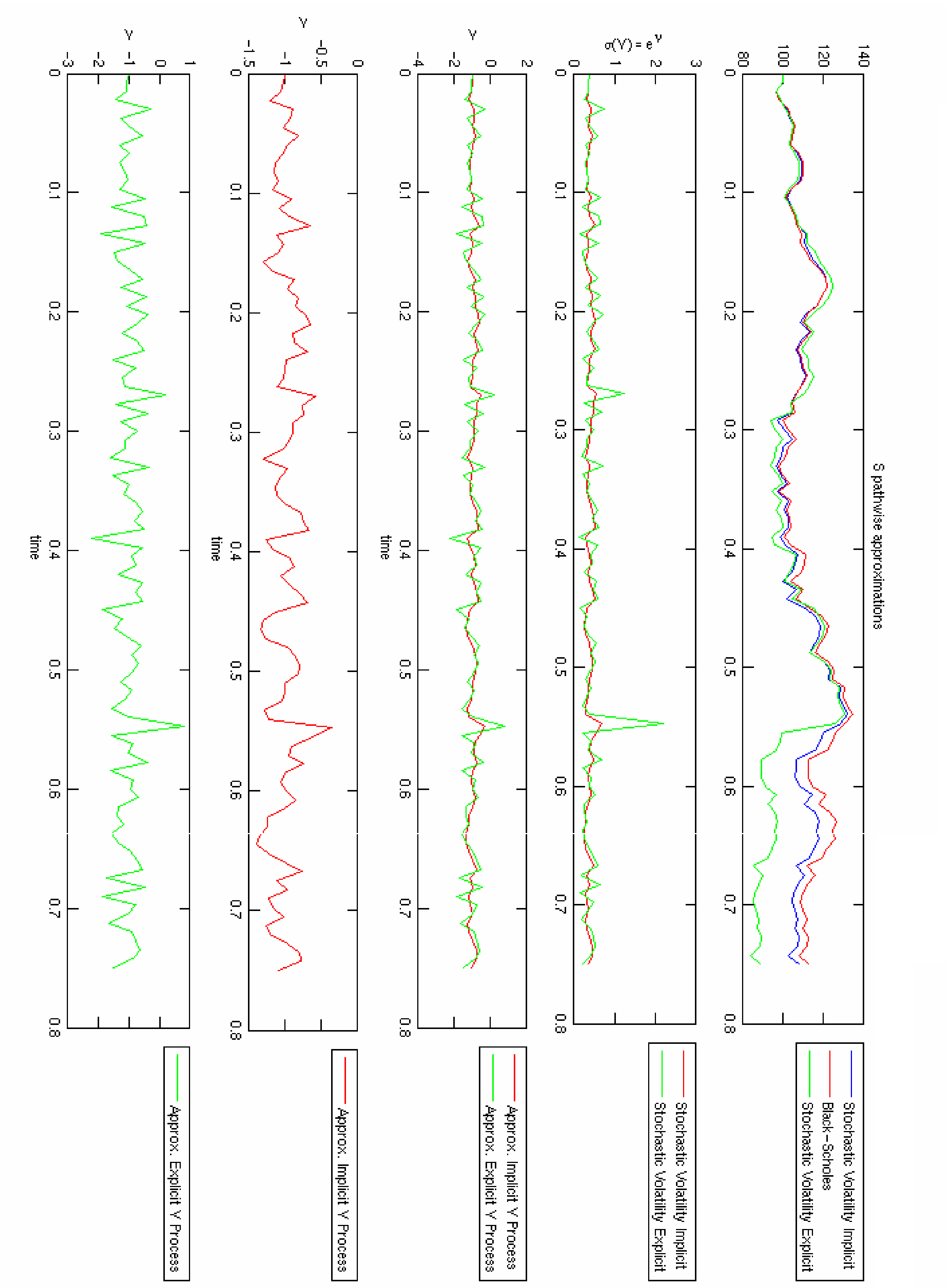
From (6), as limit $\alpha \to \infty$

$$\begin{aligned} S_t &= S_0 e^{\int_0^t (r - \frac{1}{2} e^{2Y_s}) ds + \int_o^t e^{Y_s} dW_s} \\ &= S_0 \, exp \left\{ \int_0^t \left(r - \frac{1}{2e^2} \right) ds + \int_0^t \frac{1}{e} \, dW_s \right\} \\ &= S_0 \, exp \left\{ \left(r - \frac{1}{2e^2} \right) t + \frac{1}{e} W_t \right\} \end{aligned} \tag{18}$$

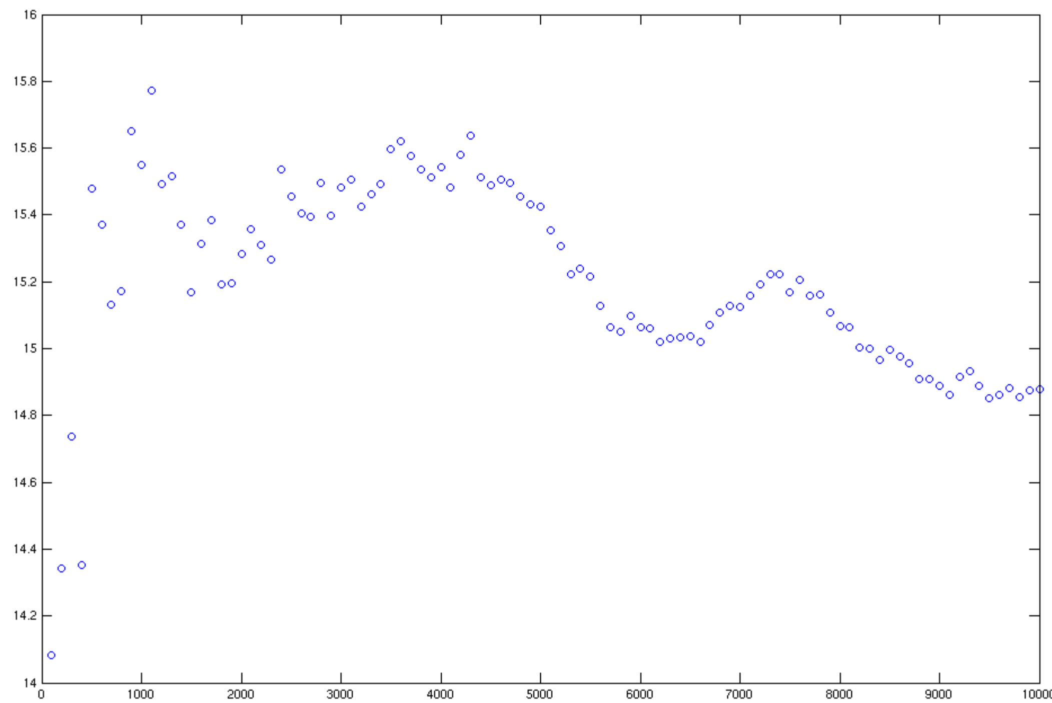
- **Figure one:** Histogram: Option Prices



- **Figure two:** Numerical Solutions



- **Figure three:** Convergence of option price



- **Figure four:** Time discretization error