

Exercise 1 *The following stochastic volatility model [FPSS00] generalizes the well known Black-Scholes geometric Brownian motion model [BS73], improving some aspects of option pricing. A simplified version of the model reads*

$$dS(t) = rS(t)dt + e^{Y(t)}S(t)dW(t) \quad (1)$$

$$dY(t) = \left(-\alpha(1 + Y(t)) + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2} \right) dt + 0.4\sqrt{\alpha}d\widehat{Z}(t), \quad (2)$$

where W and Z are independent Wiener process and

$$\widehat{Z}(t) \equiv \rho W(t) + \sqrt{1 - \rho^2}Z(t) \quad (3)$$

Note: The correlation coefficient is $\rho = -0.3$.

To solve the SDE (1), define the following:

$$f(x) = \log(x) \quad (4)$$

Solving the SDE (1) by Itô formula:

$$\begin{aligned} d \log S_t &= \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} (dS_t)^2 \\ &= r dt + e^{Y_t} dW_t - \frac{1}{2} e^{2Y_t} dt \\ &= \left(r - \frac{1}{2} e^{2Y_t} \right) dt + e^{Y_t} dW_t \end{aligned} \quad (5)$$

Integrating (5), obtain the following:

$$\begin{aligned} \log S_t &= \log S_0 + \int_0^t \left(r - \frac{1}{2} e^{2Y_s} \right) ds + \int_0^t e^{Y_s} dW_s \\ S_t &= S_0 e^{\int_0^t (r - \frac{1}{2} e^{2Y_s}) ds + \int_0^t e^{Y_s} dW_s} \end{aligned} \quad (6)$$

Solving the SDE (2):

$$\begin{aligned}
dY_t &= \left(-\alpha(1 + Y_t) + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2} \right) dt + 0.4\sqrt{\alpha}d\widehat{Z}_t \\
&= \left(-\alpha Y_t + \underbrace{0.4\sqrt{\alpha}\sqrt{1 - \rho^2} - \alpha}_{=M} \right) dt + \underbrace{0.4\sqrt{\alpha}}_{\sigma} d\widehat{Z}_t \\
&= (-\alpha Y_t + M)dt + \sigma d\widehat{Z}_t \\
&= -\alpha \left(Y_t - \frac{M}{\alpha} \right) dt + \sigma d\widehat{Z}_t \quad \alpha \neq 0
\end{aligned} \tag{7}$$

Define the following:

$$X_t = Y_t - \frac{M}{\alpha} \tag{8}$$

Rewriting (7) by substituting (8), obtain the following:

$$dX_t = -\alpha X_t dt + \sigma d\widehat{Z}_t \tag{9}$$

To solve (9) use the following integrating factor:

$$V_t = e^{\alpha t} X_t \tag{10}$$

Differentiating (10) and subsituting (9) obtain:

$$\begin{aligned}
dV_t &= d(e^{\alpha t} X_t) \\
&= \alpha e^{\alpha t} X_t dt + e^{\alpha t} \underbrace{dX_t}_{=(9)} \\
&= \alpha e^{\alpha t} X_t dt - e^{\alpha t} \alpha X_t dt + e^{\alpha t} \sigma d\widehat{Z}_t \\
dV_t &= e^{\alpha t} \sigma d\widehat{Z}_t
\end{aligned} \tag{11}$$

From (11), as well as using (10), obtain an equation for X_t .

$$\begin{aligned}
\underbrace{V_t}_{(10)} &= V_0 + \int_0^t e^{\alpha s} \sigma d\widehat{Z}_s \\
e^{\alpha t} X_t &= X_0 + \int_0^t e^{\alpha s} \sigma d\widehat{Z}_s \\
X_t &= X_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} \sigma d\widehat{Z}_s
\end{aligned} \tag{12}$$

From (8), and substituting into above, we obtain an equation for Y_t .

$$\begin{aligned}
\underbrace{X_t}_{(12)} &= X_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} \sigma d\widehat{Z}_s \\
Y_t - \frac{M}{\alpha} &= \left(Y_0 - \frac{M}{\alpha} \right) e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} \sigma d\widehat{Z}_s \\
Y_t &= \left(Y_0 - \frac{M}{\alpha} \right) e^{-\alpha t} + \frac{M}{\alpha} + \int_0^t e^{-\alpha(t-s)} \sigma d\widehat{Z}_s
\end{aligned} \tag{13}$$

Compare the following explicit and implicit method for the computation of the option value below.

Explicit Method:

$$\begin{aligned} S_{n+1} - S_n &= r S_n \Delta t + e^{Y_n} S_n \Delta W_n, \\ Y_{n+1} - Y_n &= \left(-\alpha(1 + \underbrace{Y_n}) + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2} \right) \Delta t + 0.4\sqrt{\alpha} \Delta \hat{Z}_n \\ \hat{Z}_n &= \rho W_n + \sqrt{1 - \rho^2} Z_n \end{aligned}$$

Inside the code: Explicit method

```
S1(n+1) = (1+ dt*r + dW(n)*exp(V(n)))*S1(n);
V(n+1) = V(n)+(-alpha*V(n)-alpha+tmp*sqrt(1-rho^2))*dt + tmp * dhZ(n);
```

Implicit Method:

$$\begin{aligned} S_{n+1} - S_n &= r S_n \Delta t + e^{Y_n} S_n \Delta W_n, \\ Y_{n+1} - Y_n &= \left(-\alpha(1 + \underbrace{Y_{n+1}}) + 0.4\sqrt{\alpha}\sqrt{1 - \rho^2} \right) \Delta t + 0.4\sqrt{\alpha} \Delta \hat{Z}_n \\ \hat{Z}_n &= \rho W_n + \sqrt{1 - \rho^2} Z_n \end{aligned}$$

Inside the code: Implicit method

```
S(n+1) = (1+ dt*r + dW(n)*exp(Y(n)))*S(n);
Y(n+1) = (Y(n) + (-alpha+tmp*sqrt(1-rho^2))*dt + tmp * dhZ(n))/(1+alpha*dt);
```

Recall: The implicit Euler is absolutely stable; the explicit Euler method is not.

Computation of the option value. Use $\alpha = 200$, $r = 0.04$, $T = \frac{3}{4}$ and $S_0 = K = 100$.

$$V_T = e^{-rT} \mathbb{E}[\max(S_T - K, 0)] \quad (14)$$

- a. Motivate which above numerical methods is best in this case. Provide a numerical approximation, based on uniform time steps, to the option value with accuracy $TOL=10^{-2}$.

Setup for numerical approximation:

Pseudocode

Selection of Δt and M

- choose Δt , fix $M = 10^3$
 - while** $E[g(X_{\Delta t} - g(X_{\frac{\Delta t}{2}}))] + \frac{C\sigma_{\text{error}}}{\sqrt{M}} > \frac{1}{3}\text{TOL}$
 - output:** $\Delta t = \Delta t^*$

- choose M , fix Δt^*
 - while** $\hat{\sigma}^2 = E(g(X)^2) - E(g(X))^2$
 - check:** $|\text{error}_{\text{statistical}}| < \frac{C_{\alpha}\hat{\sigma}}{\sqrt{M}} \leq \frac{2}{3}\text{TOL}$

b. *In the limit $\alpha \rightarrow \infty$ one obtains the results from geometric Brownian motion. Can you verify this by numerical experiments?*

Note: In geometric Brownian motion with constant volatility, we can compute the value of the option exactly using the the Black-Scholes formula. The numerical results obtained approach this thoeretical price.

From (13), let the limit $\alpha \rightarrow \infty$,

$$\begin{aligned}
Y_t &= \left(Y_0 - \frac{M}{\alpha}\right) e^{-\alpha t} + \frac{M}{\alpha} + \int_0^s e^{-\alpha(t-s)} \sigma d\widehat{Z}_s \\
\lim_{\alpha \rightarrow \infty} \frac{Y_0}{e^{\alpha t}} &= 0 \\
\lim_{\alpha \rightarrow \infty} -\frac{0.4\sqrt{\alpha}\sqrt{1-\rho^2} + \alpha}{\alpha e^{\alpha t}} &= 0 \\
\lim_{\alpha \rightarrow \infty} \frac{0.4\sqrt{\alpha}\sqrt{1-\rho^2} - \alpha}{\alpha} &= -1 \\
\lim_{\alpha \rightarrow \infty} -\frac{0.4\sqrt{\alpha}}{e^{\alpha(t-s)}} &= 0
\end{aligned} \tag{15}$$

From above, (15), we obtain as $\alpha \rightarrow \infty$, $Y_t \rightarrow -1$.

However, when we calculate the $Var(Y_t)$, we see that similiar behavior as above (15), does not occur:

$$\begin{aligned}
E \left[\int_0^t \sqrt{\alpha} e^{-\alpha(t-s)} d\widehat{Z}_s \right]^2 &= \int_0^t \alpha e^{-2\alpha(t-s)} ds \\
&= \frac{e^{-2\alpha(t-s)}}{2\alpha} \alpha \Big|_0^t \\
&= \frac{1}{2} - \frac{e^{-2\alpha t}}{2}
\end{aligned} \tag{16}$$

Therefore, from(16), let the limit $\alpha \rightarrow \infty$, obtain:

$$\begin{aligned}
\lim_{\alpha \rightarrow \infty} E \left[\int_0^t \sqrt{\alpha} e^{-\alpha(t-s)} d\widehat{Z}_s \right]^2 &= \lim_{\alpha \rightarrow \infty} \left(\frac{1}{2} - \underbrace{\frac{e^{-2\alpha t}}{2}}_{=0} \right) \\
&= \frac{1}{2}
\end{aligned} \tag{17}$$

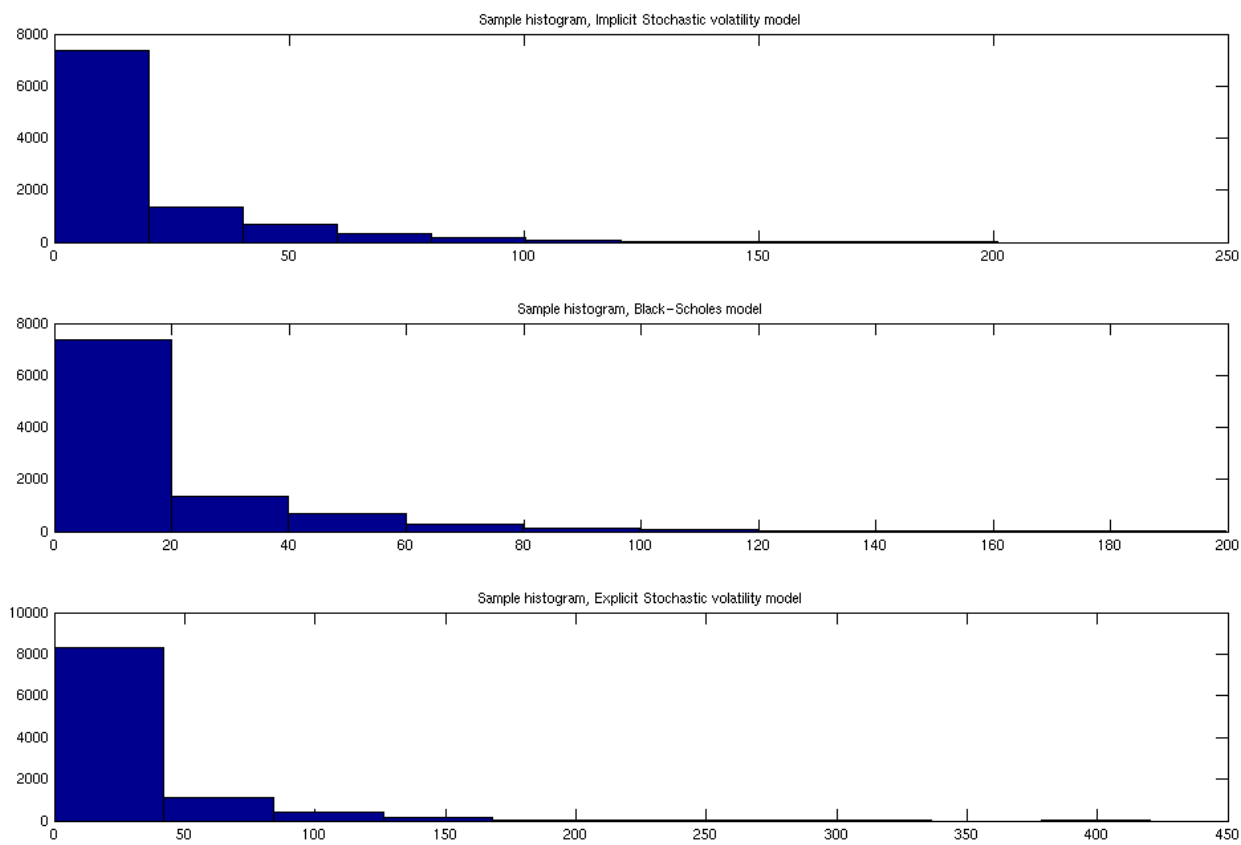
Using this results, we can see that the volatility from (1):

$$e^{Y_t} \rightarrow e^{-1} \quad \text{as} \quad \alpha \rightarrow \infty$$

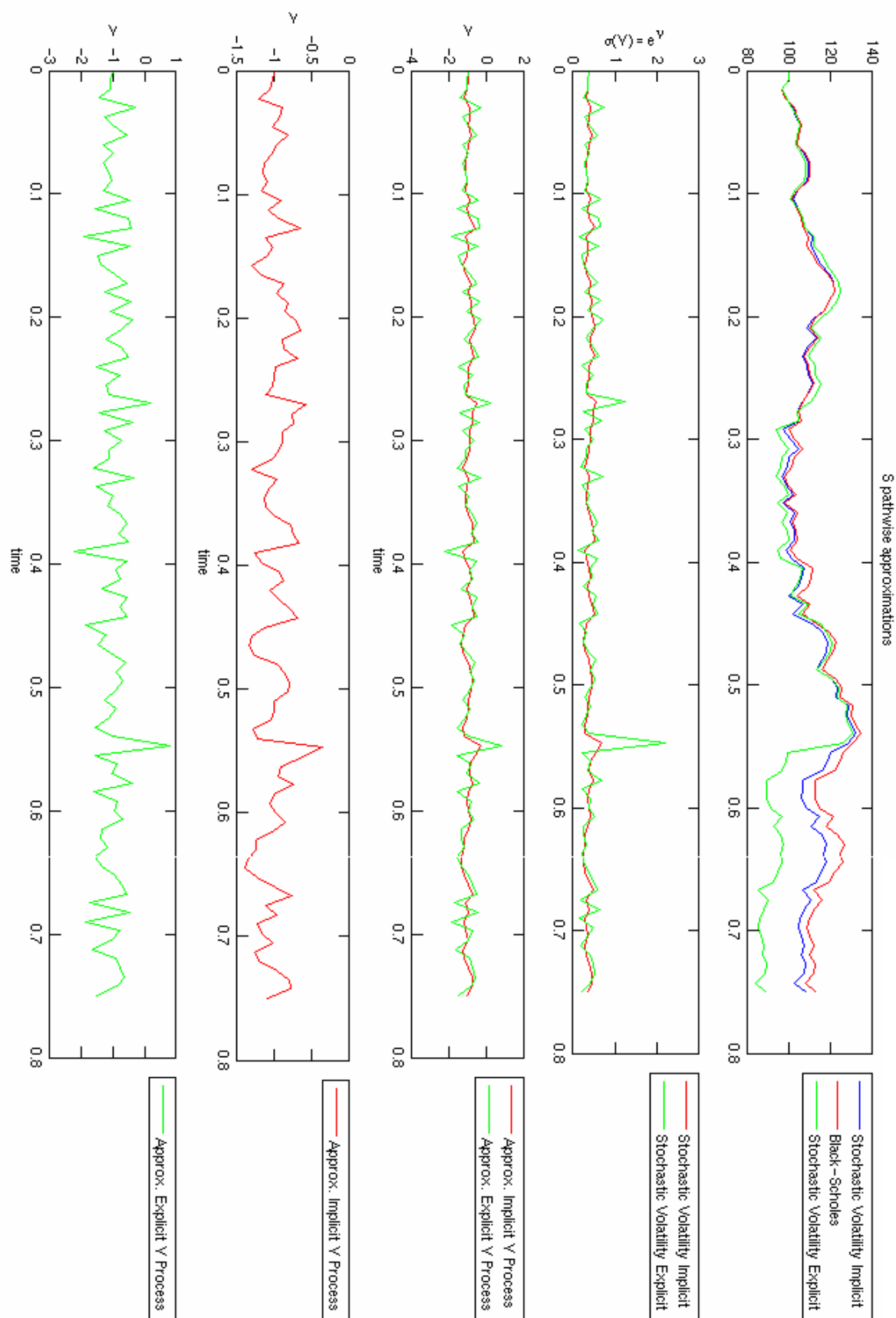
From (6), as limit $\alpha \rightarrow \infty$

$$\begin{aligned} S_t &= S_0 e^{\int_0^t (r - \frac{1}{2} e^{2Y_s}) ds + \int_0^t e^{Y_s} dW_s} \\ &= S_0 \exp \left\{ \int_0^t \left(r - \frac{1}{2e^2} \right) ds + \int_0^t \frac{1}{e} dW_s \right\} \\ &= S_0 \exp \left\{ \left(r - \frac{1}{2e^2} \right) t + \frac{1}{e} W_t \right\} \end{aligned} \tag{18}$$

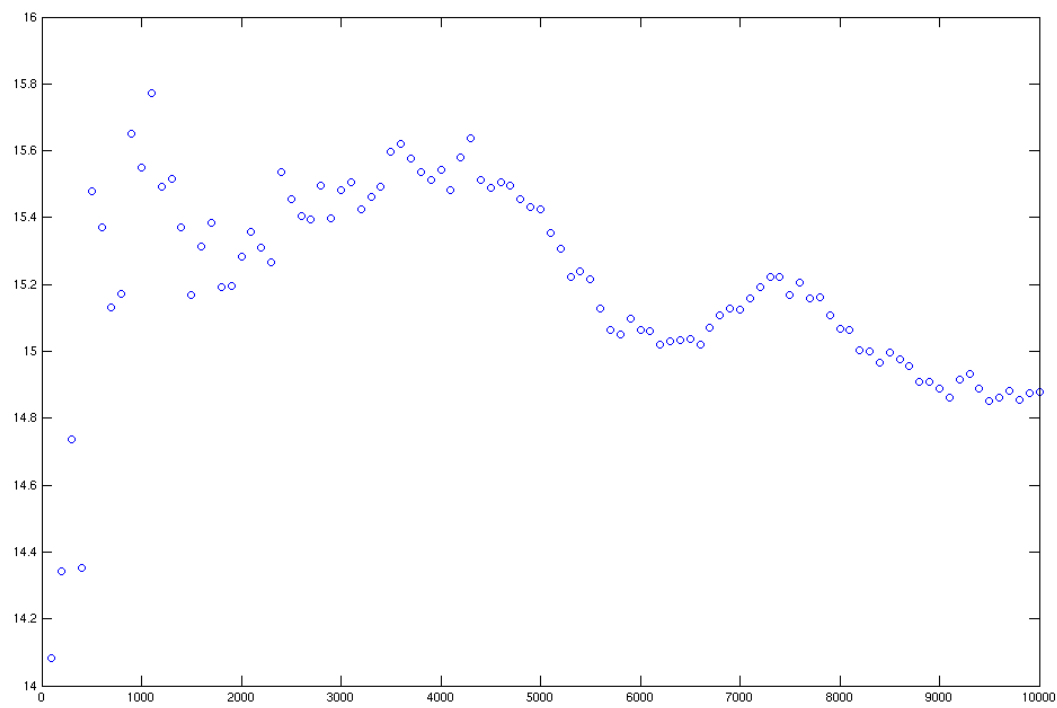
• **Figure one: Histogram: Option Prices**



• **Figure two:** Numerical Solutions



• **Figure three:** Convergence of option price



• **Figure four:** Time discretization error

