Exercise 1 *Formulate and motivate a forward Euler method for approximation of the Stratonovich SDE*

$$dX_t = a(t, X_t)dt + b(t, X_t) \circ dW_t \tag{1}$$

First approach: (Incorrect approach!) We approximated $b(t, X_t)$ by the midpoint rule:

$$X_{t_{n+1}} = X_n + a(t_n, X_{t_n})\Delta t_n + b\left(\frac{1}{2}(t_{n+1} + t_n), \frac{1}{2}(X_{t_{n+1}} + X_{t_n})\right)\Delta W_n \tag{2}$$

What is wrong with approach (2)?

Consider the following process as an example to illustrate why formulation (2) was incorrect.

$$dX(t) = bX_t dW(t) \quad \text{with a constant}\, b, \tag{3}$$

Using (2) to approximate (3) we obtain:

$$\begin{aligned} X_{t_{n+1}} &= X_{t_n} + b\left(\frac{1}{2}(X_{t_{n+1}} + X_{t_n})\right)\Delta W_n \\ X_{t_{n+1}}(1 - \frac{1}{2}b\Delta W_n) &= X_{t_n}(1 + \frac{1}{2}b\Delta W_n) \\ X_{t_{n+1}} &= X_{t_n}\left[\frac{\left(1 + \frac{1}{2}b\Delta W_n\right)}{\left(1 - \frac{1}{2}b\Delta W_n\right)}\right] \end{aligned} \tag{4}$$

Consider the computation of $E(X_1^2)$

$$E(X_1^2) = \frac{(X_0)^2}{\sqrt{2\pi\Delta t}}\int_{-\infty}^{\infty}\frac{(1 + \frac{1}{2}bz)^2}{(1 - \frac{1}{2}bz)^2}\, e^{\frac{-z^2}{2\Delta t}}\, dz \tag{5}$$

where we have used the fact that $E(f(x)) = \int(f(x)dp(x))$ where $p(x)$ is the probability density of x.

From (5), consider a similiar integral:

$$\int_{-\infty}^{\infty}\frac{(1+x)^2}{(1-x)^2}\, e^{-x^2}dx \tag{6}$$

However, the integral in (6) does not converge.

Formulation:

Rewrite (1), so that may discretize using the forward Euler method.

$$X_t = X_0 + \int_0^t a(s, X(s))ds + \frac{1}{2}\int_0^t b(s, X_s)b_x(s, X_s)ds + \int_0^t b(s, X_s)dW_s \tag{7}$$

From (7), considering one interval, $[t_n, t_{n+1}]$

$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} a(x, X(s))ds + \frac{1}{2}\int_{t_n}^{t_{n+1}} b(s, X_s)b_x(s, X_s)ds + \int_{t_n}^{t_{n+1}} b(s, X_s)dW_s \tag{8}$$

Discretization of (8) to obtain the following:

$$X_{t_{n+1}} = X_{t_n} + a(t_n, X_{t_n})\Delta t_n + \frac{1}{2}b(t_n, X_{t_n})\frac{\partial b(t_n, X_{t_n})}{\partial x}\Delta t_n + b\Delta t_n \Delta W_{t_n} \tag{9}$$

Exercise 2 *Consider the deterministic differential equation*

$$\begin{aligned} dZ_t &= a(Z_t)dt \\ Z(0) &= x_0 \qquad\qquad 0 \leq t \leq T, \end{aligned} \tag{10}$$

and consider a peturbation of (10), *the Itô stochastic differential equation*

$$\begin{aligned} dX_t &= a(X_t)dt + bdW_t \\ X(0) &= x_0 \qquad\qquad 0 \leq t \leq T, \end{aligned} \tag{11}$$

where $\boldsymbol{a}$ *is a smooth function and* $\boldsymbol{b} > 0$ *is a positive constant. The aim of this exercise is to compare the solution of both equations. Define then the difference:*

$$e_t = X_t - Z_t \tag{12}$$

a) *Consider* $\mathbf{a}(\mathbf{x}) = \mathbf{ax}$ *(linear case) and compute* $\mathbf{E}(\mathbf{e_t})$ *and* $\mathbf{Var}(\mathbf{e_t})$.
Hint: Use Itô's formula when necessary.
From (10), obtain the following solution

$$Z_t = x_0 e^{at} \tag{13}$$

Applying the intergrating factor, define the following:

$$\begin{aligned} Y_t &= e^{-at} X_t \qquad\qquad 0 \leq t \leq T \\ Y_0 &= x_0 \end{aligned} \tag{14}$$

Itô's formula. Let f be a smooth function, assume that X_t is the solution of an SDE, then $g_t = f(X_t, t)$ satisfies the SDE

$$df(X_t, t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2(X_t, t)dt, \tag{15}$$

Applying (15) to (14), to obtain the following:

$$\begin{aligned} dY_t &= -ae^{-at} X_t dt + e^{-at} \underbrace{dX_t}_{\text{see (11)}} + \underbrace{\text{higher order terms}}_{=0} \\ &= -ae^{-at} X_t dt + ae^{-at} X_t dt + be^{-at} dW_t \\ dY_t &= be^{-at} dW_t \end{aligned} \tag{16}$$

integrating (16), will obtain:

$$\begin{aligned} Y_t &= Y_0 + \int_0^t be^{-as} dW_s \\ e^{-at} X_t &= X_0 + \int_0^t be^{-as} dW_s \\ X_t &= e^{at} X_0 + \int_0^t be^{a(t-s)} dW_s \end{aligned} \tag{17}$$

Therefore, now that we have obtained an equation for X_t, see (17), revisit the definition for the difference, (12).

$$\begin{aligned} e_t &= \underbrace{X_t}_{\text{see (17)}} - \underbrace{Z_t}_{\text{see (13)}} \\ e_t &= \int_0^t be^{a(t-s)}dW_s \end{aligned} \tag{18}$$

Solution: Now derived an equation for e_t, (18), compute $\mathbf{E(e_t)}$ and $\mathbf{Var(e_t)}$.

- Compute $E(e_t)$. Will use, **Theorem 2.15 (ii)**, using the basic properities of Itô's integrals.

$$\textbf{Recall:} \quad E\left[\int_0^t f(s,\cdot)dW_s\right] = 0$$

$$\begin{aligned} E(e_t) &= E\left(\int_0^t be^{a(t-s)}dW_s\right) \\ &= 0 \quad \textbf{by Thm 2.15} \end{aligned} \tag{19}$$

- Compute $Var(e_t)$.

$$\begin{aligned} Var(e_t) &= E[e_t^2] - [\underbrace{E[e_t])}_{\text{see (19)}}]^2 \\ &= E[e_t^2] \\ &= E\left[\left(\int_0^t be^{a(t-s)}dW_s\right)^2\right] \end{aligned} \tag{20}$$

Will use, **Itô isometry** in order to simplify (20).

$$\textbf{Recall:} \quad E\left[\int_0^t f(W_s,s)dW_s\right]^2 = \int_0^t Ef^2(W_s,s)ds$$

From above, (20) will become,

$$\begin{aligned} Var(e_t) &= E\left[\left(\int_0^t be^{a(t-s)}dW_s\right)^2\right] \\ &= E\left(\int_0^t b^2e^{2a(t-s)}ds\right) \\ &= \frac{b^2}{2a}\left(e^{2at}-1\right) \end{aligned}$$

b. *Assume now that* $|\mathbf{a}(\mathbf{x}) - \mathbf{a}(\mathbf{y})| \leq \mathbf{C_a}|\mathbf{x} - \mathbf{y}|$ *with a positive constant* C_a. *Find bounds for the expectation,* $\mathbf{E}(|\mathbf{e_t}|^\mathbf{2})$ *and use it to bound the variance* $\mathbf{Var}(\mathbf{e_t})$.
Discuss what happens as $b \to 0$.

- Find bounds for the expectation, $\mathbf{E}(|\mathbf{e_t}|^\mathbf{2})$.

$$\begin{aligned} e_t &= X_t - Z_t \\ &= \int_0^t (a(X_s) - a(Z_s))ds + \int_0^t b dW_s \end{aligned} \tag{21}$$

Taking the absolute value, squaring both sides, and taking the expectation we will obtain the left hand solution. To obtain a upper bound, we will use various methods involving the triangle inequality, Cauchy-Schwarz inequality, Fubini's theorem, and Grönwall's lemma.

$$\begin{aligned} |e_t| &= \left| \int_0^t (a(X_s) - a(Z_s))ds + \int_0^t b\, dW_s \right| \\ &\leq \int_0^t C_a |X_s - Z_s| ds + b\,|W_t| \qquad (b > 0, \text{by assumption}) \\ &= \int_0^t C_a |e_s| ds + b\,|W_t| \\ |e_t| &\leq \int_0^t C_a |e_s| ds + b\,|W_t| \end{aligned} \tag{22}$$

From (22), we will square both sides, and take the expectation to obtain:

$$\begin{aligned} E|e_t|^2 &\leq E\left[\left(\int_0^t C_a |e_s| ds + b\,|W_t| \right)^2 \right] \\ &\leq 2E\left[\left(\int_0^t C_a |e_s| ds \right)^2 \right] + 2E\left[(b\,|W_t|)^2 \right] \qquad \textbf{Recall: } (a+b)^2 \leq 2(a^2 + b^2) \\ &= 2E\left[\left(\int_0^t C_a |e_s| ds \right)^2 \right] + 2E\left[b^2\, W_t^2 \right] \\ &\leq 2E\left[\left(\int_0^t C_a^2 ds \right) \left(\int_0^t |e_s|^2 ds \right) \right] + 2b^2 t \qquad \textbf{Recall: Cauchy-Schwarz inequality} \\ &\leq 2C_a^2 t E\left[\int_0^t |e_s|^2 ds \right] + 2b^2 t \\ E|e_t|^2 &\leq 2C_a^2 t \left[\int_0^t E|e_s|^2 ds \right] + 2b^2 t \qquad \textbf{Recall: Fubini's theorem} \end{aligned} \tag{23}$$

To simplify the integral in (23), we will use Grömwall's lemma.

Lemma. Assume that there exist positive constants A and K such that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies:

$$\begin{aligned} f(t) &\leq K \int_0^t f(s) ds + A \\ &\text{then} \\ f(t) &\leq A e^{Kt} \end{aligned} \tag{24}$$

Also, to use Grönwall's lemma, define the following:

$$f(t) = E|e_t|^2 \tag{25}$$

From (25), and substituting into (23), we will obtain:

$$\begin{aligned} f(t) &\leq 2C_a^2 t \int_0^t f(s)ds + 2b^2 t, \\ &\leq 2C_a^2 T \int_0^t f(s)ds + 2b^2 T \end{aligned} \tag{26}$$

Let $K = 2C_a^2 T$ and $A = 2b^2 T, \quad$ (note: $A > 0$)
Then by Grönwall's lemma, obtain a bound on $\mathbf{E}(|\mathbf{e_t}|^\mathbf{2})$.

$$f(t) \leq 2b^2 T\, e^{2C_a^2 T t} \tag{27}$$

- Using answer from part 2a), (27), obtain a bound on the variance $\mathbf{Var(e_t)}$.

$$\begin{aligned}
\mathrm{Var}(e_t) &= E(e_t^2) - \underbrace{[E(e_t)]}_{\text{see (19)}}^2 \\
&\leq E(e_t^2) \qquad (Note: [E(e_t)]^2 \geq 0) \\
\mathrm{Var}(e_t) &\leq 2b^2T\, e^{2C_a^2Tt}
\end{aligned} \tag{28}$$

- Discuss what happens as $b \to 0$.

  From (28), as $b \to 0$, will lead to:

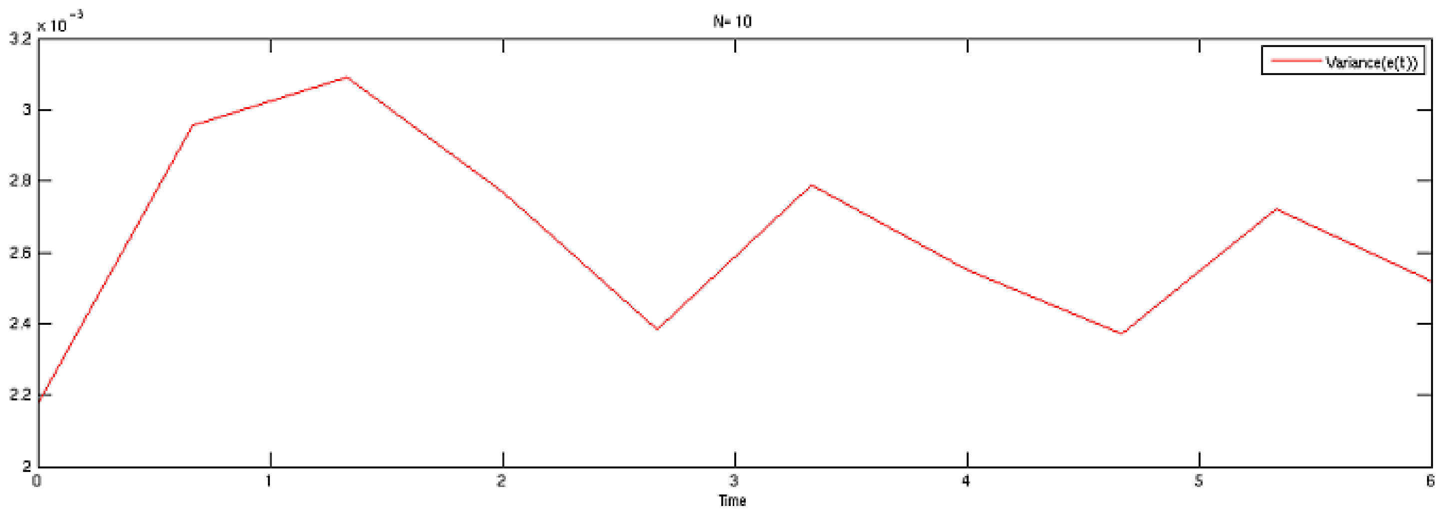
  $$0 \leq \mathrm{Var}(e_t) \leq 0 \Longrightarrow \mathrm{Var}(e_t) = 0 \tag{29}$$

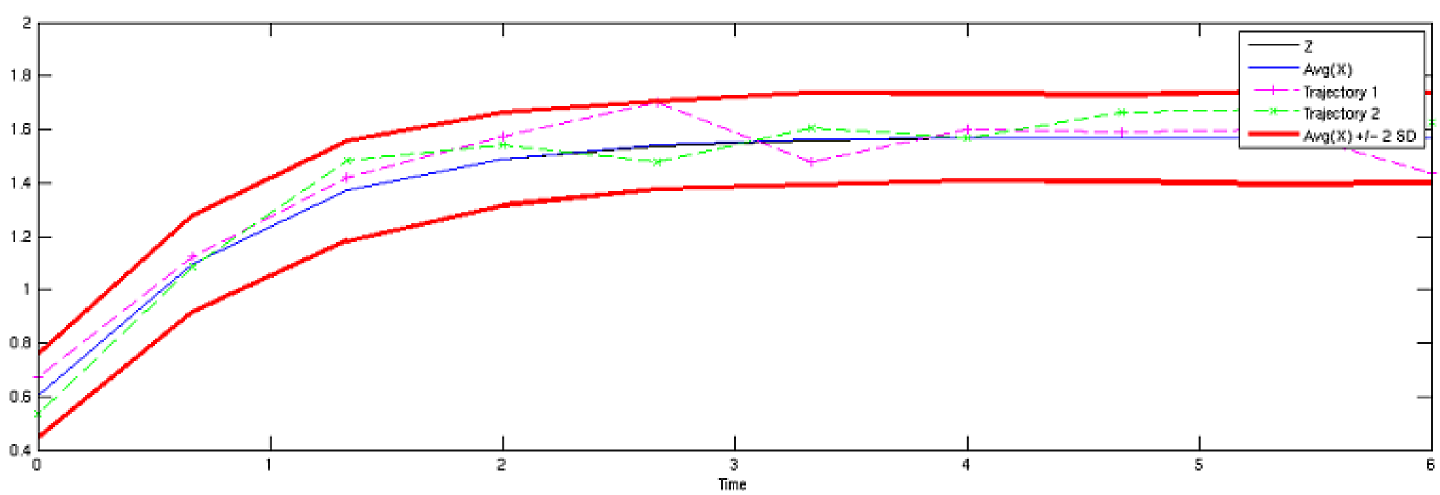
  Therefore, the stochastic term disappears as $b \to 0$ in the SDE of X_t. Thus, matching the deterministic function, Z_t

c) *Implement a uniform time step forward Euler discretization of the above equations taking* $\mathbf{a(x) = \cos(x), b = 0.1,}$ *and* $\mathbf{T = 6}$.
Plot the sample estimator for $\mathbf{Var(e_t)}$ *vs. time.*
Compare it with the bound obtained in part (b).
Use $\mathbf{M = 10^3}$ *sample paths and different number of time steps:* $\mathbf{N = 10, 20, 40}$.

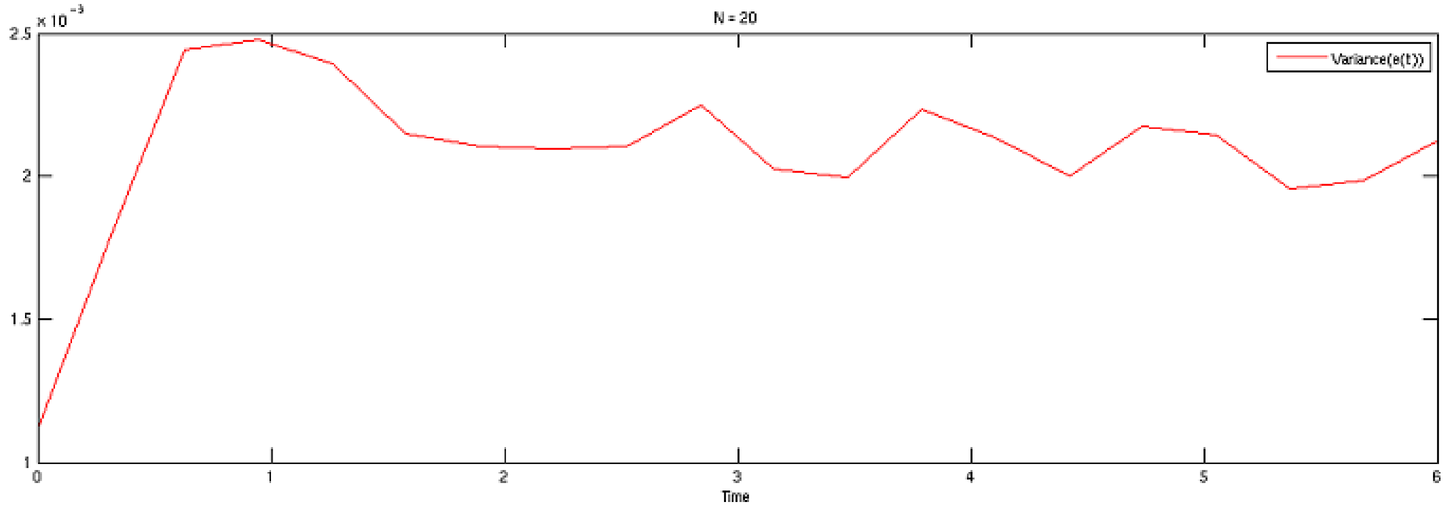
- Each graph below was graphed using Matlab, please see appendix for code.
- For each figure, the deterministic function, trajectories, and the bounds were plotted.

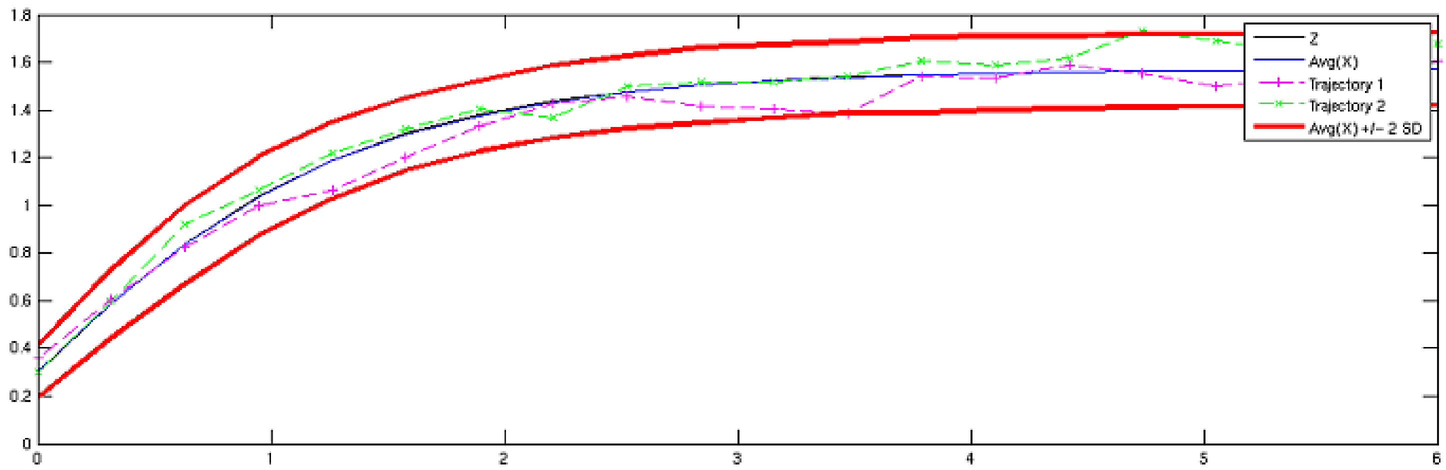
- **Figure one:** Number of time steps: $\mathbf{N = 10}$.



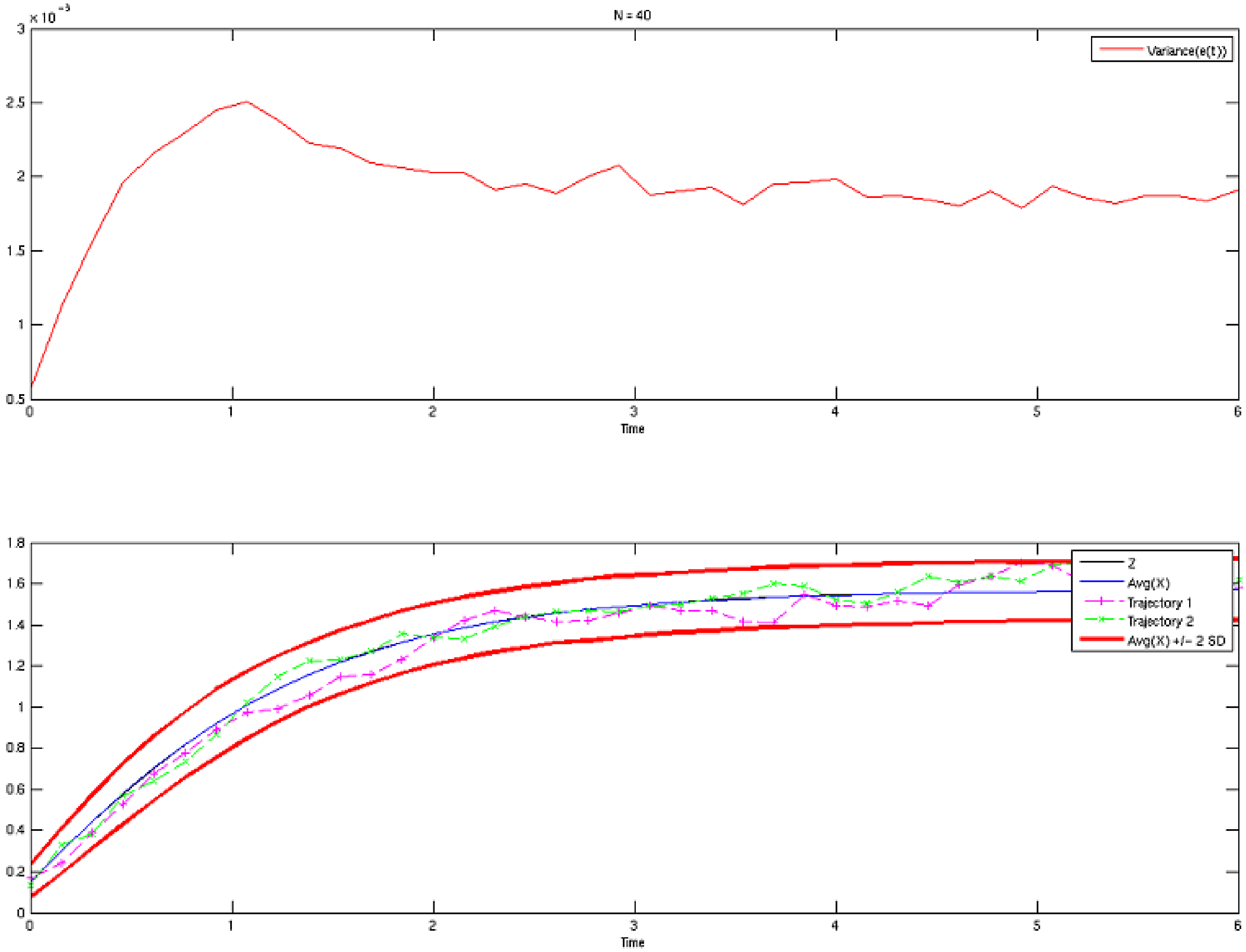


- **Figure two:** Number of time steps: $\mathbf{N = 20}$.





- **Figure three:** Number of time steps: $\mathbf{N} = \mathbf{40}$.



APPENDIX: Problem 2(c) Mathlab code

```
clf randn(round(clock),1); T = 6; N = 40; Delta_t = T/N; b=0.1; X0=0;
C1=2*(b^2); C2=8; v_error_mean=zeros(N,1); v_error_square=zeros(N,1);
v_variance=zeros(N,1); v_xdet=zeros(N,1); v_xdet2=zeros(N,1); M = 1e+3;
for s = 1:M
    X=X0;
    Z=X0;
    for j = 1:N
        Winc1 = sqrt(Delta_t)*randn;
        X=X+cos(X)*Delta_t + b*Winc1;
        Z=Z+cos(Z)*Delta_t;
        v_xdet(j,1)=v_xdet(j,1)+X;
        v_xdet2(j,1)=v_xdet2(j,1)+X^2;
        e=abs(X-Z);
        v_error_mean(j,1)=v_error_mean(j,1)+e;
        e2=(X-Z)^2;
        v_error_square(j,1)=v_error_square(j,1)+e2;
    end end X=X0; X1=X0; Z=X0; for j=1:N
        v_xdet(j,1)=v_xdet(j,1)/M;
        v_xdet2(j,1)=v_xdet2(j,1)/M;
        v_variance_det(j,1)=v_xdet2(j,1)-((v_xdet(j,1))^2);
        Winc1 = sqrt(Delta_t)*randn;
        X=X+cos(X)*Delta_t + b*Winc1;
        v_notmean(j,1) = X;
        Winc2 = sqrt(Delta_t)*randn;
        X1=X1+cos(X1)*Delta_t + b*Winc2;
        v_notmean2(j,1) = X1;
        v_error_mean(j,1)=v_error_mean(j,1)/M;
        v_error_square(j,1)=v_error_square(j,1)/M;
        v_variance(j,1)=v_error_square(j,1)-((v_error_mean(j,1))^2);
        Z=Z+cos(Z)*Delta_t;
        v_deterministic(j,1)=Z;
        v_time=linspace(0,T,N);
        v_minus_2d(j,1)=v_xdet(j,1)-2*sqrt(v_variance_det(j,1));
        v_plus_2d(j,1)=v_xdet(j,1)+2*sqrt(v_variance_det(j,1)); end
title('N = 10'); subplot(2,1,1); plot(v_time, v_variance,'r');
legend('Variance(e(t))'); xlabel('Time (in seconds)'); subplot(2,1,2);
plot(v_time, v_deterministic, 'k', v_time, v_xdet,'b', v_time, v_notmean,
'm+--', v_time, v_notmean2,'gx--'); hold on; plot(v_time, v_minus_2d,'r',
v_time, v_plus_2d,'r','LineWidth',3); legend('Z','Avg(X)','Trajectory
1','Trajectory 2','Avg(X) +/- 2 SD');
xlabel('Time (in seconds)');
```