Homework Set 3, due Thursday Sept 21.

Last revised, Aug 27, 2006.

Exercise 1 *Formulate and motivate a forward Euler method for approximation of the Stratonovich SDE*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t).$$

Exercise 2 *Consider the deterministic differential equation*

$$dZ(t) = a(Z(t))dt, \qquad Z(0) = x_0, \qquad 0 \le t \le T,$$

and a perturbation of it, the Ito stochastic differential equation

$$dX(t) = a(X(t))dt + b\, dW(t), \qquad X(0) = x_0, \qquad 0 \le t \le T,$$

where a is a smooth function and $b > 0$ is a positive constant. The aim of this excercise is to compare the solution of both equations. Define then the difference

$$e(t) = X(t) - Z(t).$$

a *Consider $a(x) = ax$ (linear case) and compute $E(e(t))$, and $var(e(t))$. Hint: Use Ito's formula when necessary.*

b *Assume now that $|a(x) - a(y)| \le C_a|x - y|$ with a positive constant C_a. Find bounds for the expectation $E(|e(t)|^2)$ use it to bound the variance $var(e(t))$. Discuss what happens as $b \to 0$.*

c *Implement a uniform time step forward Euler discretization of the above equations taking $a(x) = \cos(x)$, $b = 0.1$ and $T = 6$. Plot the sample estimator for $var(e(t))$ vs. time, and compare it with the bound obtained in part (b). Use $M = 10^3$ sample paths and different number of time steps: $N = 10, 20, 40$.*