Exercise 1: Prove the following identities by taking the limits of the forward Euler method.

(a) $$\int_0^T t\, dW_{(t)} \overset{L^2(P)}{=} \lim_{\Delta t \to 0} \sum_{n=0}^{N-1} t_n (W_{n+1} - W_n)$$

$$= t_0 (W_1 - W_0) + t_1 (W_2 - W_1) + \cdots + t_{N-1} (W_N - W_{N-1})$$

rearranging the terms

$$= \underbrace{-t_0 W_0}_{0} - W_1 (t_1 - t_0) - W_2 (t_2 - t_1) - \cdots - W_{N-1} (t_{N-1} - t_{N-2})$$
$$+ t_{N-1} W_N \underbrace{- t_N W_N + t_N W_N}_{}$$

adding and subtracting the same term.

$$= t_N W_N - \sum_{i=0}^{N-1} W_{i+1} \Delta t_i$$

$$\longrightarrow T W_{(T)} - \int_0^T W(t)\, dt.$$

as $\Delta t \to 0$. $\square$

(b) $$\int_0^T W_t\, dW_t = \frac{1}{2} W_T^2 - \frac{T}{2}$$

or $$T = W_T^2 - 2 \int_0^T W_t\, dW_t$$

r.h.s. can be approximated by

$$\sum_{n=0}^{N-1} \Delta(W_n^2) - 2 \sum_{n=0}^{N-1} W_n \Delta W_n = \sum_{n=0}^{N-1} W_{n+1}^2 - W_n^2 - 2 W_n W_{n+1} + 2 W_n^2$$

$$= \sum_{n=0}^{N-1} W_{n+1}^2 - 2 W_n W_{n+1} + W_n^2 = \sum_{n=0}^{N-1} (W_{n+1} - W_n)^2 \xrightarrow[\text{as } \Delta t \to 0]{L^2(P)} T.$$

Why? Justify please

Exercise 2: Ornstein - Uhlenbeck Process.

$$X(t) = x_\infty + e^{-at}(x_0 - x_\infty) + b\int_0^t e^{-a(t-s)}\,dW(s)$$

$$\mu_t = E(X(t)) = x_\infty + e^{-at}(x_0 - x_\infty) \quad \text{since } E\left[\int_0^t e^{-a(t-s)}\,dW(s)\right] = 0.$$

$$\lim_{t\to\infty} \mu_t = x_\infty .$$

$$\sigma_t^2 = E\left[(X(t) - \mu_t)^2\right] = E\left[b^2\left[\int_0^t e^{-a(t-s)}\,dW(s)\right]^2\right]$$

Ito Isometry.

$$= E\left[\underbrace{b^2\int_0^t e^{-2a(t-s)}\,ds}_{\text{deterministic}}\right]$$

Not needed

$$= \frac{b^2}{2a}\left[1 - e^{-2at}\right]$$

$$\lim_{t\to\infty} \sigma_t^2 = \frac{b^2}{2a}.$$

$X(t)$ is a finite variance, mean reverting process.

OK