

Exercise 1: Prove the following identities by taking the limits of the Forward Euler method.

$$(a) \int_0^T t \, d\omega_t \stackrel{L^2(p)}{=} \lim_{\Delta t \rightarrow 0} \sum_{n=0}^{N-1} t_n (\omega_{n+1} - \omega_n)$$

$$= t_0 (\omega_1 - \omega_0) + t_1 (\omega_2 - \omega_1) + \dots + t_{N-1} (\omega_N - \omega_{N-1})$$

rearranging the terms

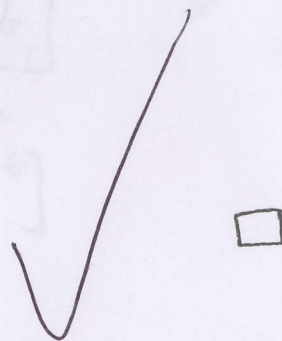
$$= \cancel{-t_0 \omega_0} - \omega_1 (t_1 - t_0) - \omega_2 (t_2 - t_1) - \dots - \omega_{N-1} (t_{N-1} - t_{N-2})$$

0.

$$+ t_{N-1} \omega_N - \underbrace{t_N \omega_N + t_N \omega_N}_{\text{adding and subtracting the same term.}}$$

$$= t_N \omega_N - \sum_{i=0}^{N-1} \omega_{i+1} \Delta t_i$$

$$\xrightarrow{\text{as } \Delta t \rightarrow 0} T \omega(T) - \int_0^T \omega(t) \, dt.$$



$$(b) \int_0^T \omega_t \, d\omega_t = \frac{1}{2} \omega_T^2 - \frac{T}{2}$$

$$\text{or } T = \omega_T^2 - 2 \int_0^T \omega_t \, d\omega_t$$

r.h.s. can be approximated by

$$\sum_{n=0}^{N-1} \Delta(\omega_n^2) - 2 \sum_{n=0}^{N-1} \omega_n \Delta \omega_n = \sum_{n=0}^{N-1} (\omega_{n+1}^2 - \omega_n^2 - 2\omega_n \omega_{n+1} + 2\omega_n^2)$$

$$= \sum_{n=0}^{N-1} \omega_{n+1}^2 - 2\omega_n \omega_{n+1} + \omega_n^2 = \sum_{n=0}^{N-1} (\omega_{n+1} - \omega_n)^2 \xrightarrow[\text{or } \Delta t \rightarrow 0]{L^2(p)} T$$

Why? Justify please

Exercise 2: Ornstein - Uhlenbeck Process.

$$X(t) = X_{\infty} + e^{-at}(X_0 - X_{\infty}) + b \int_0^t e^{-a(t-s)} dW(s)$$

$$M_t = E(X(t)) = X_{\infty} + e^{-at}(X_0 - X_{\infty}) \quad \text{since} \quad E\left[\int_0^t e^{-a(t-s)} dW(s)\right] = 0.$$

$$\lim_{t \rightarrow \infty} M_t = X_{\infty}.$$

$$\begin{aligned} \sigma_t^2 &= E[(X(t) - M_t)^2] = E\left[b^2 \left(\int_0^t e^{-a(t-s)} dW(s)\right)^2\right] \\ &= E\left[b^2 \int_0^t e^{-2a(t-s)} ds\right] \quad \text{Ito Isometry.} \\ &\quad \text{deterministic.} \\ &= \frac{b^2}{2a} [1 - e^{-2at}] \end{aligned}$$

Not needed

$$\lim_{t \rightarrow \infty} \sigma_t^2 = \frac{b^2}{2a}.$$

$X(t)$ is a finite variance, mean reverting process.

(OK)