Essential questions for the exam 2006

1. Formulate two models based on stochastic differential equations. Discuss the choise of noise, numerical method and Ito or Stratonovich form.

2. Formulate the basic properties a Wienerprocess.

3. Define the Ito integral by the limit of the forward Euler method and show that the limit is independent of the partition used in the time discretization.

4. Show by an example that piecewise linear approximation of the Wienerprocess in a SDE approximates Stratonovich integrals.

5. Show by Itos formula that if u solves Kolmogorov's backward equation with data $u(T, x) = g(x)$, then

$$u(t, x) = E[g(X(T))|\ X(t) = x],$$

where X is the solution of a certain SDE (which?).

6. Formulate and prove a theorem on error estimates for weak convergence of the forward Euler method for SDE's.

7. Motivate the use of Monte-Carlo methods to compute Europian options based on a basket of several stocks and discuss some possibilities of methods of variance reduction.

8. State and derive Ito's formula.

9. State and derive Feynman-Kac formula.

10. State and derive the central limit theorem.

11. Show how to apply the Monte Carlo method to compute an integral and discuss the error.

12. Formulate a finite difference method for a heat equation and use Lax equivalence theorem to motivate stability, convergence and consistency of the approximation.

13. Present shortly an application of Hamilton-Jacobi equations.

14. Show by an example that the standard FEM Galerkin method does not work for the differential equation

$$u' - \epsilon u'' = 0,$$

when ϵ is much smaller than the mesh size.

15. Formulate a cure of the problem in problem 14.

16. Show by an example non uniqueness of weak solutions of

$$
\begin{aligned}
&u_t + (\frac{u^2}{2})_x = 0,\\
&u(\cdot, 0) = u_0.
\end{aligned}
\tag{2}
$$

17. Describe a formulation that gives uniqueness for (2).

18. Relate problem 17 and the Lax-Friedrich method.

19. Relate the problem of non uniqueness of weak solutions of the conservation law

$$
u_t + (\frac{u^2}{2})_x = 0, \tag{4a}
$$

and the Hamilton-Jacobi equation

$$
v_t + \frac{(v_x)^2}{2} = 0. \tag{4b}
$$

20. Present related formulations that give uniqueness for the two problems (4a,b).

21. Formulate numerical methods for (4a,b) that work and relate it to problem 20.

22. Formulate a numerical method for the SDE

$$
\begin{aligned}
&dX(t) = a(t, X(t))dt + b(t, X(t))dW,\\
&X(0) = x_0,
\end{aligned}
$$

and discuss its accuracy in case of weak and strong approximation.

23. Give a motivation for the Black-Scholes equation.

24. Discuss the alternatives to determine the option prize in problem 23 numerically in case of contracts based on one and many stocks.

25. Consider a time discrete Markov chain $X(t)$ with discrete state space and Markov control variable α. Formulate and derive a recursive relation to determine $\min_{\alpha\in\mathcal{A}} E_\alpha[g(X(T))]$. ∎

26. Motivate mathematically the Hamilton-Jacobi-Bellman equation for solving a optimal control problem of a stochastic differential equation and relate it to Problem 25.

27. Present something you find mathematical interesting related to stochastic partial differential equations.