Numerical Methods for Nonlinear Stochastic Differential Equations with Jumps*

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Abstract

We present and analyse two implicit methods for Ito stochastic differential equations (SDEs) with Poisson-driven jumps. The first method, SSBE, is a split-step extension of the backward Euler method. The second method, CSSBE, arises from the introduction of a compensated, martingale, form of the Poisson process. We show that both methods are amenable to rigorous analysis when a one-sided Lipschitz condition, rather than a more restrictive global Lipschitz condition, holds for the drift. Our analysis covers strong convergence and nonlinear stability. We prove that both methods give strong convergence when the drift coefficient is one-sided Lipschitz and the diffusion and jump coefficients are globally Lipschitz. On the way to proving these results, we show that a compensated form of the Euler–Maruyama method converges strongly when the SDE coefficients satisfy a local Lipschitz condition and the pth moment of the exact and numerical solution are bounded for some $p > 2$. Under our assumptions, both SSBE and CSSBE give well-defined, unique solutions for sufficiently small stepsizes, and SSBE has the advantage that the restriction is independent of the jump intensity. We also study the ability of the methods to reproduce exponential mean-square stability in the case where the drift has a negative one-sided Lipschitz constant. This work extends the deterministic nonlinear stability theory in numerical analysis. We find that SSBE preserves stability under a stepsize constraint that is independent of the initial data. CSSBE satisfies an even stronger condition, and gives a generalization of B-stability. Finally, we specialize to a linear test problem and show that CSSBE has a natural extension of deterministic A-stability. The difference in stability properties of the SSBE and CSSBE methods emphasizes that the addition of a jump term has a significant effect that cannot be deduced directly from the non-jump literature.

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1 Introduction

We consider jump-diffusion Ito stochastic differential equations (SDEs) of the form

$$dX(t) = f(X(t^-))\,dt + g(X(t^-))\,dW(t) + h(X(t^-))\,dN(t), \quad t > 0, \quad X(0^-) = X_0, \tag{1}$$

where $X(t^-)$ denotes $\lim_{s\to t^-} X(s)$. Here, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^n$, $W(t)$ is an m-dimensional Brownian motion and $N(t)$ is a scalar Poisson process with intensity λ. Such problems arise in a range of scientific, engineering and financial applications. [3, 6, 18]. We consider the case where

$$f, g, h \in C^1, \tag{2}$$

the drift coefficient f satisfies a one-sided Lipschitz condition

$$\langle x - y, f(x) - f(y)\rangle \leq \mu|x - y|^2, \qquad \text{for all } x, y \in \mathbb{R}^n, \tag{3}$$

and the diffusion and jump coefficients satisfy global Lipschitz conditions

$$|g(x) - g(y)|^2 \leq L_g|x - y|^2, \qquad \text{for all } x, y \in \mathbb{R}^n, \tag{4}$$
$$|h(x) - h(y)|^2 \leq L_h|x - y|^2, \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{5}$$

Here, and throughout, $\langle\cdot,\cdot\rangle$ denotes the scalar product and $|\cdot|$ denotes both the Euclidean vector norm and the Frobenius matrix norm. We note for later use that the following linear growth bounds hold, see for example, [9, Section 3.1]:

$$|\langle f(x), x\rangle| \leq \tfrac{1}{2}|f(0)|^2 + (\mu + \tfrac{1}{2})|x|^2, \tag{6}$$
$$|g(x)|^2 \leq 2|g(0)|^2 + 2L_g|x|^2, \tag{7}$$
$$|h(x)|^2 \leq 2|h(0)|^2 + 2L_h|x|^2. \tag{8}$$

We also assume finite moment bounds for the initial data; that is, for any $p > 0$ there is a finite M_p such that

$$\mathbb{E}|X_0|^p < M_p. \tag{9}$$

For a given, constant, stepsize $\Delta t > 0$, we define the *split-step backward Euler* (SSBE) method for (1) by $Y_0 = X(0^-)$ and

$$Y_n^\star = Y_n + f(Y_n^\star)\Delta t, \tag{10}$$
$$Y_{n+1} = Y_n^\star + g(Y_n^\star)\Delta W_n + h(Y_n^\star)\Delta N_n. \tag{11}$$

Here, Y_n is the approximation to $X(t_n)$ for $t_n = n\Delta t$, with $\Delta W_n := W(t_{n+1}) - W(t_n)$ and $\Delta N_n := N(t_{n+1}) - N(t_n)$ representing the increments of the Brownian motion and the Poisson process, respectively.

A key component in our analysis is the compensated Poisson process

$$\widetilde{N}(t) := N(t) - \lambda t, \tag{12}$$

which is a martingale. Defining

$$f_\lambda(x) := f(x) + \lambda h(x) \tag{13}$$

we may rewrite the jump-diffusion Ito SDE (1) in the form

$$dX(t) = f_\lambda(X(t^-))\,dt + g(X(t^-))\,dW(t) + h(X(t^-))\,d\widetilde{N}(t). \tag{14}$$

We note that f_λ also satisfies a one sided Lipschitz condition with larger constant; that is,

$$\langle x-y, f_\lambda(x) - f_\lambda(y)\rangle \le \left(\mu + \lambda\sqrt{L_h}\right)|x-y|^2, \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{15}$$

The compensated Poisson process motivates an alternative to the SSBE method in (10)–11). We define the *compensated split-step backward Euler* (CSSBE) method for (1) by $Y_0 = X(0^-)$ and

$$Y_n^\star = Y_n + f_\lambda(Y_n^\star)\Delta t, \tag{16}$$

$$Y_{n+1} = Y_n^\star + g(Y_n^\star)\Delta W_n + h(Y_n^\star)\Delta\widetilde{N}_n, \tag{17}$$

where $\Delta\widetilde{N}_n := \widetilde{N}(t_{n+1}) - \widetilde{N}(t_n)$.

Both SSBE and CSSBE are implicit methods, and hence the question of existence and uniqueness arises. Under our one-sided Lipschitz condition (3), the equation (10) for SSBE has a unique solution, with probability one, for all

$$\Delta t\mu < 1, \tag{18}$$

whereas with the one-sided Lipschitz condition (15), the equation (16) for CSSBE has a unique solution, with probability one, for all

$$\Delta t(\mu + \lambda\sqrt{L_h}) < 1, \tag{19}$$

see, for example, [7, Theorem 14.2].

Our aims in this work are to analyze the strong convergence and stability of the SSBE and CSSBE methods under the one-sided Lipschtiz condition for the drift. We remark that the need for pathwise solutions to asset models in mathematical finance provides motivation for a strong (as opposed to weak) convergence theory, [2]. In section 2 we give some preliminary analysis for the SDE (1). We show that our assumptions are sufficient to guarantee existence of a unique solution and we develop moment bounds. Section 3 establishes a convergence result, Theorem 1, that is needed in the later analysis. That theorem applies to a compensated version of the explicit Euler–Maruyama method, and proves strong convergence in the case where the SDE coefficients are locally Lipschitz. Section 4 gives our finite-time, strong convergence results, Theorems 2 and 3, for SSBE and CSSBE. In the case where the one-side Lipschitz constant is negative, it is possible for the SDE solution to be mean-square stable, and section 5 deals with the corresponding long-time stability of the numerical methods. We derive a sufficient condition, Theorem 4, for the SDE to have trajectories that are exponentially contractive in mean-square. We then show in Theorem 5 that under this condition SSBE inherits the contractivity property independently of initial data for all stepsizes up to a limit that scales inverse linearly with $L_h\lambda^2$. For CSSBE, we find even better behavior. Theorem 6 shows that the method preserves stability for all stepsizes under the condition for the SDE in Theorem 4. Section 6 focusses on the special case of a linear test equation. Here, we are able to derive a very strong stability result for CSSBE; namely a complete generalization of A-stability.

In summary, both methods offer strong convergence, CSSBE has superior nonlinear stability properties and SSBE has a less restrictive existence condition.

A key aspect of this work is the reliance on a one-sided Lipschitz assumption (3). This makes the results relevant to range of nonlinear drift terms, [4, 7, 19]. Previous convergence theory for jump-diffusion systems has been based on the more restrictive assumption that the drift satisfies a global Lipschitz property, [8, 12, 13]. The one-sided Lipschitz condition has proved to be an extremely useful abstraction in deterministic numerical dynamics, and has led to a large body of important results; see, for example, [4, 7, 19]. The work in [9, 11, 15, 17] has looked at numerical methods for non-jump SDEs under a one-sided Lipschitz assumption on the drift, and our work is a natural extension to the jump case. We remark that the one-sided Lipschitz condition is closely related to the existence of a quadratic Lyapunov function; a Lyapunov function approach to dealing with non-globally Lipschitz non-jump SDEs is taken in [16]. Lyapunov functions are also used in [1] for non-jump SDEs with delay. We found it pleasing that clean and easily interpretable nonlinear stability results can be proved for SSBE and CSSBE, and we found it surprising that incorporating the compensated process into a numerical method can make a significant improvement to nonlinear stability.

The SSBE method (10)–(11) is a straightforward generalization of the non-jump split-step backward Euler method studied in [9, 15]. It was found in [9, 15] that the structure of the method fits in well with the one-sided Lipschitz condition and allows positive results to be derived. We find here that the same is true in the jump case. The idea of developing the CSSBE method (16)–(17) based on the compensated process appears to be new and leads to clear advantages in terms of nonlinear stability.

2 Existence, Uniqueness and Moment Estimates

By extending the non-jump proof of [14, Theorem 2.3.1], it can be shown that a unique solution exists for (1) under our assumptions. The essential change to that proof is the inclusion of the jump term; this can be estimated with the martingale isometry for the compensated Poisson process:

$$\mathbb{E}\left(\int_0^t F(s^-)\, d\widetilde{N}(s)\right)^2 = \lambda \int_0^t \mathbb{E}\,|F(s)|^2\, ds, \tag{20}$$

which holds for appropriate integrand functions F (in particular, nonanticipative, if random). Thereafter such terms are handled in the same way as the Ito integral terms. This leads to a solution on any bounded time interval $[0, T]$ with

$$\mathbb{E}|X(t)|^2 \le C\left(1 + \mathbb{E}|X_0|^2\right), \qquad t \in [0, T],$$

for some constant $C = C(T)$.

For later use, we need the following more general moment bound.

Lemma 1 *Under the assumptions (2), (3), (4), (5) and (9), for each $p > 2$ there is a constant $C = C(p, T)$ such that*

$$\mathbb{E} \sup_{0 \le t \le T} |X(t)|^p \le C\left(1 + \mathbb{E}|X_0|^p\right).$$

Proof. A proof is given in the Appendix. □

3 The Euler Method for Locally Lipschitz Coefficients

In this section we prove a convergence result that will be needed later. Considering the SDE in compensated form, (14), motivates the explicit method

$$Y_{n+1} = Y_n + f_\lambda(Y_n)\Delta t + g(Y_n)\Delta W_n + h(Y_n)\Delta \widetilde{N}_n, \tag{21}$$

which we will refer to as the *compensated Euler–Maruyama* (CEM) method. We denote the piecewise constant interpolant of the CEM solution by $Y(t)$; so $Y(t) = Y_n$ for $t \in [t_n, t_{n+1})$. We then define the "piecewise linear" interpolant by

$$\overline{Y}(t) = X_0 + \int_0^t f_\lambda(Y(s^-))\,ds + \int_0^t g(Y(s^-))\,dW(s) + \int_0^t h(Y(s^-))\,d\widetilde{N}(s).$$

In this section, we suppose that f, g and h satisfy local Lipschitz conditions, that is, for $a = f, g, h$, given any $R > 0$ there exists a constant L_R such that

$$|a(x) - a(y)|^2 \le L_R |x - y|^2, \qquad \text{for all } |x|, |y| \le R. \tag{22}$$

We note that the function f_λ in (13) automatically inherits this condition, with a larger L_R,

The following result generalizes [9, Theorem 2.2] to the case of jumps.

Theorem 1 *Suppose that f, g, h satisfy the local Lipschitz condition (22), and that for some $p > 2$ there is a constant A such that*

$$\mathbb{E} \sup_{0\le t\le T} |X(t)|^p \le A, \qquad \mathbb{E} \sup_{0\le t\le T} |\overline{Y}(t)|^p \le A.$$

Then

$$\lim_{\Delta t\to 0} \mathbb{E} \sup_{0\le t\le T} |\overline{Y}(t) - X(t)|^2 = 0.$$

Proof. We mention only the key changes that are needed to the proof in [9, Theorem 2.2] due to the inclusion of the jump integral. In particular, we write $\theta_R = \tau_R \wedge \rho_R$, where

$$\tau_R = \inf\{t \ge 0 \ :\ \left|\overline{Y}(t)\right| \ge R\}, \qquad \rho_R = \inf\{t \ge 0 \ :\ |X(t)| \ge R\},$$

and $E(t) = \overline{Y}(t) - X(t)$. We start from inequality (2.8) in [9], that is

$$\mathbb{E} \sup_{0\le t\le T} |E(t)|^2 \le \mathbb{E} \sup_{0\le t\le T} \left|\overline{Y}(t\wedge\theta_R) - X(t\wedge\theta_R)\right|^2 + \frac{2^{p+1}\delta A}{p} + \frac{2(p-2)A}{p\delta^{2/(p-2)}R^p}, \tag{23}$$

for any $\delta > 0$. Now

$$\begin{aligned}
\left|\overline{Y}(t\wedge\theta_R) - X(t\wedge\theta_R)\right|^2 &= \Big| \int_0^{t\wedge\theta_R} f_\lambda(Y(s^-)) - f_\lambda(X(s^-))\,ds \\
&\quad + \int_0^{t\wedge\theta_R} g(Y(s)) - g(X(s)\,dW(s) \\
&\quad + \int_0^{t\wedge\theta_R} h(Y(s^-)) - h(X(s^-))\,d\widetilde{N}(s)\Big|^2 \\
&\le 4\Big[T\int_0^{t\wedge\theta_R} |f_\lambda(Y(s)) - f_\lambda(X(s))|^2\,ds \\
&\quad + \left|\int_0^{t\wedge\theta_R} g(Y(s)) - g(X(s))\,dW(s)\right|^2 \\
&\quad + \left|\int_0^{t\wedge\theta_R} h(Y(s^-)) - h(X(s^-))\,d\widetilde{N}(s)\right|^2 \Big].
\end{aligned}$$

From the local Lipschitz property and the Doob martingale inequality applied to the two martingale integrals we obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le\tau}\left|\overline{Y}(t\wedge\theta_R)-X(t\wedge\theta_R)\right|^2 &\le 4L_R(T+8)\mathbb{E}\int_0^{\tau\wedge\theta_R}|Y(s)-X(s)|^2\,ds\\ &\le 8L_R(T+8)\left[\mathbb{E}\int_0^{\tau\wedge\theta_R}\left|Y(s)-\bar{Y}(s)\right|^2 ds\right.\\ &\quad\left.+\int_0^{\tau}\mathbb{E}\sup_{0\le r\le s}\left|\overline{Y}(r\wedge\theta_R)-X(r\wedge\theta_R)\right|^2 ds\right]. \quad (24)\end{aligned}$$

To bound the first term inside the parentheses we denote by n_s the integer for which $s \in [t_{n_s}, t_{n_{s+1}})$ and note that

$$Y(s)-\overline{Y}(s) = -f_\lambda(Y_{n_s})(s-t_{n_s}) - g(Y_{n_s})\left(W(s)-W(t_{n_s})\right) - h(Y_{n_s})\left(\widetilde{N}(s)-\widetilde{N}(t_{n_s})\right)$$

and hence that

$$\left|Y(s)-\overline{Y}(s)\right|^2 \le 4\left[|f_\lambda(Y_{n_s})\Delta t|^2 + |g(Y_{n_s})\Delta W_{n_s}|^2 + \left|h(Y_{n_s})\Delta\widetilde{N}_{n_s})\right|^2\right].$$

The local linear growth bounds, the second moments of the martingale increments and the pth moment bound on the numerical solution yield

$$\mathbb{E}\int_0^{\tau\wedge\theta_R}\left|Y(s)-\overline{Y}(s)\right|^2 ds \le C_1\Delta t,$$

for a constant $C_1 = C_1(R,T,A)$. Using this bound in (24) and applying the continuous Gronwall inequality gives

$$\mathbb{E}\sup_{0\le t\le T}\left|\overline{Y}(t\wedge\theta_R)-X(t\wedge\theta_R)\right|^2 \le C_2\Delta t e^{8L_R(T+8)T},$$

for a constant $C_2 = C_2(R,T,A)$. The rest of the proof then follows as in [9], where for arbitrary $\epsilon > 0$, first $\delta > 0$ and then R and finally $\Delta t > 0$ are chosen so that each term on the left side of (23) is less than $\epsilon/3$, giving $\mathbb{E}\sup_{0\le t\le T}|E(t)|^2 \le \epsilon$, from which the assertion of the theorem follows. □

4 Strong Convergence of the Backward Euler Methods

It was shown in [9] that the SSBE method for an Ito SDE without jumps is equivalent to the explicit Euler–Maruyama method applied to a modified SDE. We now show that this extends to the jump case. As in [9, Lemma 3.4] we define $F_{\Delta t} : \mathbb{R}^n \to \mathbb{R}^n$ by $F_{\Delta t}(x) = y$, where y satisfies $y = x + f(x)\Delta t$. Following (18) and (19), we know that such a y exists uniquely for all $\Delta t \in (0, \Delta t^\star)$, where we may take $\Delta t^\star = 1/|\mu|$ for SSBE and $\Delta t^\star = 1/|\mu + \lambda\sqrt{L_h}|$ for CSSBE. We then define

$$f_{\Delta t}(x) = f(F_{\Delta t}(x)), \quad g_{\Delta t}(x) = g(F_{\Delta t}(x)), \quad h_{\Delta t}(x) = h(F_{\Delta t}(x)).$$

Under the assumptions (3) on f, it follows that that $f_{\Delta t}$ satisfies an analogous one-sided Lipschitz condition uniformly in $\Delta t \in (0, \Delta t^\star)$. Similarly, under (4), (5), there are global Lipschitz conditions for $g_{\Delta t}$ and $h_{\Delta t}$ uniformly in $\Delta t \in (0, \Delta t^\star)$.

It follows by construction that SSBE in (10)–(11) is is equivalent to the explicit Euler–Maruyama method

$$Y_{n+1} = Y_n + f_{\Delta t}(Y_n)\Delta t + g_{\Delta t}(Y_n)\Delta W_n + h_{\Delta t}(Y_n)\Delta N_n$$

applied to the SDE

$$dX_{\Delta t}(t) = f_{\Delta t}(X_{\Delta t}(t^-))dt + g_{\Delta t}(X_{\Delta t}(t^-))dW(t) + h_{\Delta t}(X_{\Delta t}(t^-))dN(t), \quad X_{\Delta t}(0^-) = X_0. \tag{25}$$

Lemma 1 applies to this SDE to give the following result.

Corollary 1 *Under the assumptions (2), (3), (4), (5) and (9), for each $p > 2$ there is a constant $C = C(p,T)$ such that, for the SDE (25),*

$$\mathbb{E} \sup_{0\le t\le T} |X_{\Delta t}(t)|^p \le C\left(1 + \mathbb{E}|X_0|^p\right),$$

for all $\Delta t \in (0, \Delta t^\star)$.

In addition we have the following estimate comparing solutions of (1) and (25).

Lemma 2 *Under the assumptions (2), (3), (4), (5) and (9), the solutions $X(t)$ in (1) and $X_{\Delta t}(t)$ in (25) satisfy*

$$\lim_{\Delta t\to 0} \mathbb{E} \sup_{0\le t\le T} |X_{\Delta t}(t) - X(t)|^2 = 0.$$

Proof. The proof follows that of [9, Lemma 3.6] with the addition of the jump process integrals. In particular, we need to include $|h_{\Delta t}(u) - h(u)|^2 \le K_R(\Delta t)$ and $|h(u) - h(v)|^2 \le H_R|u-v|^2$ in conditions (3.20) and (3.21) of [9], respectively. Then in the equation following (3.23) of [9] we add the jump integral term, written as

$$\int_0^{t\wedge\theta_R} \left[\left(h(X(s^-)) - h(X_{\Delta t}(s^-))\right) + \left(h(X_{\Delta t}(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))\right)\right] dN(s).$$

We then split this into two parts corresponding respectively to the martingale compensated Poisson process $\widetilde{N}(t)$ in (12) and the deterministic integrator λt. Using the growth bounds on h and $h_{\Delta t}$, and applying the Doob inequality on the martingale part for $\tau \in [0,T]$, we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{0\le t\le \tau} \left| \int_0^{\tau\wedge\theta_R} \left[\left(h(X(s^-)) - h(X_{\Delta t}(s^-))\right) + \left(h(X_{\Delta t}(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))\right)\right] d\widetilde{N}(s)\right|^2 \\
\le\ & K\left(\int_0^\tau \mathbb{E} \sup_{0\le \tau\le s} |X(t\wedge\theta_R) - X_{\Delta t}(t\wedge\theta_R)|^2\, ds + K_R(\Delta t)\right),
\end{aligned}$$

where, here, and throughout this proof, K is a constant (depending upon T) that may change at each occurrence. Similarly, we may use the Cauchy-Schwarz inequality in the deterministic part to obtain

$$\begin{aligned}
&\mathbb{E} \sup_{0\le t\le \tau} \left| \int_0^{t\wedge\theta_R} \left[(h(X(s)) - h(X_{\Delta t}(s))) + (h(X_{\Delta t}(s)) - h_{\Delta t}(X_{\Delta t}(s)))\right] \lambda ds\right|^2 \\
\le\ & K\left(\int_0^\tau \mathbb{E} \sup_{0\le t\le s} |X(t\wedge\theta_R) - X_{\Delta t}(t\wedge\theta_R)|^2\, ds + K_R(\Delta t)\right).
\end{aligned}$$

These terms and those for the other integrals add up to give a bound of the form

$$\mathbb{E}\sup_{0\le t\le\tau}|X_{\Delta t}(t)-X(t)|^2 \le K\left(K_R(\Delta t)+\int_0^\tau \mathbb{E}\sup_{0\le t\le s}|X(t\wedge\theta_R)-X_{\Delta t}(t\wedge\theta_R)|^2\,ds\right).$$

An application of the continuous Gronwall inequality completes the proof. □

Similarly, we may extend [9, Lemma 3.7] to show that the moments of the SSBE method are bounded on any finite time interval.

Lemma 3 *Under the assumptions (2), (3), (4), (5) and (9), for each $p\ge 2$ there exists a constant $C=C(p,T)$ and a $\Delta t^\star>0$ such that for SSBE in (10)–(11),*

$$\mathbb{E}\sup_{0\le n\Delta t\le T}|Y_n|^{2p}\le C, \qquad \forall \Delta t<\Delta t^\star.$$

Proof. The result may be proved using a similar approach to that in [9, Lemma 3.7] with the additional jump increment ΔN_n being split into $\Delta\widetilde{N}_n+\lambda\Delta t$. □

We now define a continuous time extension $\overline{Y}_{\Delta t}(t)$ of the SSBE method using the fact that it is equivalent to the explicit Euler method applied to the modified SDE (25). Specifically, we define

$$\overline{Y}_{\Delta t}(t_n+s)=Y_n+sf_{\Delta t}(Y_n)+g_{\Delta t}(Y_n)\Delta W_n(s)+h_{\Delta t}(Y_n)\Delta N_n(s), \qquad s\in[0,\Delta t), \tag{26}$$

where

$$\Delta W_n(s)=W(t_n+s)-W(t_n), \qquad \Delta N_n(s)=N(t_n+s)-N(t_n), \qquad s\in[0,\Delta t).$$

The next lemma extends [9, Corollary 3.8] to the jump case.

Lemma 4 *Under the assumptions (2), (3), (4), (5) and (9), for each $p\ge 2$ there exists a constant $C=C(p,T)$ and a $\Delta t^\star>0$ such that*

$$\mathbb{E}\sup_{0\le t\le T}\left|\overline{Y}_{\Delta t}(t)\right|^{2p}\le C, \qquad \forall\Delta t<\Delta t^\star.$$

Proof. For $\theta=s/\Delta t$ we have

$$\overline{Y}_{\Delta t}(t_n+s)=\theta Y_n^\star+(1-\theta)Y_n+g_{\Delta t}(Y_n)\Delta W_n(s)+h_{\Delta t}(Y_n)\Delta N_n(s), \qquad s\in[0,\Delta t).$$

Using the bound [9, (3.25)] for $|Y_n^\star|^2$, it follows that for some constant C that may change from line to line,

$$\begin{aligned}
\sup_{0\le t\le T}\left|\overline{Y}_{\Delta t}(t)\right|^{2p} &\le \sup_{0\le n\Delta t\le T}\sup_{0\le s\le\Delta t}\left|\overline{Y}_{\Delta t}(t_n+s)\right|^{2p}\\
&\le \sup_{0\le n\Delta t\le T}\sup_{0\le s\le\Delta t} C\left[1+|Y_n|^{2p}+|g_{\Delta t}(Y_n)\Delta W_n(s)|^{2p}+|h_{\Delta t}(Y_n)\Delta N_n(s)|^{2p}\right]\\
&\le C\Bigg[1+\sup_{0\le n\Delta t\le T}|Y_n|^{2p}+\sup_{0\le s\le\Delta t}\sum_{j=0}^{N}|g_{\Delta t}(Y_j)\Delta W_j(s)|^{2p}\\
&\qquad +\sup_{0\le s\le\Delta t}\sum_{j=0}^{N}\left|h_{\Delta t}(Y_j)\Delta\widetilde{N}_j(s)\right|^{2p}+\sup_{0\le s\le\Delta t}\sum_{j=0}^{N}|h_{\Delta t}(Y_j)s\lambda|^{2p}\Bigg],
\end{aligned} \tag{27}$$

where N is the largest integer with $N\Delta t \leq T$. We then take expectations and apply the Doob martingale inequality and linear growth bounds to the martingale terms to obtain

$$\begin{aligned}\mathbb{E} \sup_{0\leq s\leq \Delta t} \left| h_{\Delta t}(Y_j)\Delta t \widetilde{N}_j(s)\right|^{2p} &\leq C\mathbb{E}\left| h_{\Delta t}(Y_j)\Delta \widetilde{N}_j(\Delta t)\right|^{2p} \\ &\leq C\mathbb{E}\left| h_{\Delta t}(Y_j)\right|^{2p} \mathbb{E}\left|\Delta \widetilde{N}_j(\Delta t)\right|^{2p} \\ &\leq C\left(1+\mathbb{E}\left|Y_j\right|^{2p}\right)\Delta t^p \leq C\Delta t,\end{aligned}$$

and similarly for the Gaussian increment terms. In addition

$$\mathbb{E} \sup_{0\leq s\leq \Delta t} \left| h_{\Delta t}(Y_j)s\lambda\right|^{2p} \leq \mathbb{E}\left| h_{\Delta t}(Y_j)\right|^{2p} \lambda^{2p}\Delta t^{2p} \leq C\left(1+\mathbb{E}\left|Y_j\right|^{2p}\right)\Delta t^{2p} \leq C\Delta t.$$

Using Lemma 3 in (27), taking expectations and then summing, we obtain the desired result. □

We can now prove a strong convergence result for SSBE.

Theorem 2 *Under the assumptions (2), (3), (4), (5) and (9), the continuous time extension $\overline{Y}_{\Delta t}(t)$ in (26) of the SSBE method (10)–(11) satisfies*

$$\lim_{\Delta t\to 0} \mathbb{E} \sup_{0\leq t\leq T} |\overline{Y}_{\Delta t}(t) - X(t)|^2 = 0.$$

Proof. Corollary 1 and Lemma 4 allow us to invoke Theorem 1 in order to control the difference $\lim_{\Delta t\to 0}\mathbb{E}\sup_{0\leq t\leq T}|\overline{Y}_{\Delta t}(t) - X_{\Delta t}(t)|^2$. Lemma 2 and the triangle inequality complete the proof. □

The same strong convergence result holds for CSSBE.

Theorem 3 *Under the assumptions (2), (3), (4), (5) and (9), the CSSBE method (16)–(17) has a continuous time extension $\overline{Y}_{\Delta t}(t)$ such that*

$$\lim_{\Delta t\to 0} \mathbb{E} \sup_{0\leq t\leq T} |\overline{Y}_{\Delta t}(t) - X(t)|^2 = 0.$$

Proof. The continuous time extension for CSSBE may be defined in the same way as for SSBE in (26). A convergence proof goes through in an analogous way. □

5 Mean-Square Stability

In the case where we have a negative one-sided Lipschtiz constant μ in (3), it is possible for the SDE to exhibit mean-square contractivity. This phenomenon has been well studied in the deterministic and non-jump cases, both for the continuous problem and its discretizations, [4, 7, 10, 15]. Here we consider the effect of jumps. We begin by giving a sufficient condition for mean-square contractivity of SDE solutions.

Theorem 4 *Under the conditions (3), (4) and (5), any two solutions $X(t)$ and $Y(t)$ of the SDE (1) with $\mathbb{E}|X_0|^2 < \infty$ and $\mathbb{E}|Y_0|^2 < \infty$ satisfy*

$$\mathbb{E}|X(t) - Y(t)|^2 \leq \mathbb{E}|X_0 - Y_0|^2 e^{\alpha t},$$

where

$$\alpha := 2\mu + L_g + \lambda\sqrt{L_h}\left(\sqrt{L_h}+2\right). \tag{28}$$

Hence, $\alpha < 0$ is a sufficient condition for exponential mean-square contraction of trajectories.

Proof. The two solutions satisfy the larger system

$$\begin{pmatrix} dX(t) \\ dY(t) \end{pmatrix} = \begin{pmatrix} f\left(X(t^-)\right) \\ f\left(Y(t^-)\right) \end{pmatrix} dt + \begin{pmatrix} g\left(X(t^-)\right) \\ g\left(Y(t^-)\right) \end{pmatrix} dW_t + \begin{pmatrix} h\left(X(t^-)\right) \\ h\left(Y(t^-)\right) \end{pmatrix} dN_t.$$

Applying Ito's Lemma [5] to $Z(t) = |X(t^-) - Y(t^-)|^2$, we have

$$\begin{aligned} dZ(t) \;=\; & \big(2\langle X(t^-) - Y(t^-), f\left(X(t^-)\right) - f\left(Y(t^-)\right)\rangle + |g\left(X(t^-)\right) - g\left(Y(t^-)\right)|^2 \\ & + 2\lambda\langle X(t^-) - Y(t^-), h\left(X(t^-)\right) - h\left(Y(t^-)\right)\rangle + \lambda|h\left(X(t^-)\right) - h\left(Y(t^-)\right)|^2\big)\, dt + M(t), \end{aligned}$$

where $M(t)$ is a martingale. It follows that

$$\begin{aligned} d|X(t) - Y(t)|^2 \;\leq\; & \left(2\mu|X(t^-) - Y(t^-)|^2 + L_g|X(t^-) - Y(t^-)|^2\right. \\ & \left. +\lambda\left(2\sqrt{L_h}|X(t^-) - Y(t^-)|^2 + L_h|X(t^-) - Y(t^-)|^2\right)\right) dt + M(t), \end{aligned}$$

so that

$$\mathbb{E}|X(t) - Y(t)|^2 \leq \mathbb{E}|X_0 - Y_0|^2 e^{\left(2\mu + L_g + \sqrt{L_h}\lambda(2+\sqrt{L_h})\right)t}.$$

□

For brevity, we will use the phrase "mean-square stability" in place of "exponential mean-square contraction of trajectories", and we note that under the additional assumption $f(0) = g(0) = h(0) = 0$ this property implies mean-square stability of the zero solution. The following corollary shows that when the sufficient condition $\alpha < 0$ for mean-square stability of the SDE holds, the SSBE and CSSBE method have unique solutions for all Δt.

Corollary 2 *If $\mu < 0$ then the SSBE method (10)–(11) produces a well-defined, unique solution. If $\mu + \lambda\sqrt{L_h} < 0$ then the CSSBE method (16)–(17) produces a well-defined, unique solution. In particular, if $\alpha < 0$ in (28) then both SSBE and CSSBE produce a well-defined, unique solution.*

Proof. The results follow directly from (18) and (19). □

Next, we give sufficient conditions for mean-square stability of SSBE and CSSBE.

Theorem 5 *Under the conditions (3), (4) and (5), if $\alpha < 0$ in (28) then for*

$$\Delta t < -\frac{\alpha}{L_h\lambda^2} \tag{29}$$

any two solutions from the SSBE method with $\mathbb{E}|X_0|^2 < \infty$ and $\mathbb{E}|Y_0|^2 < \infty$ satisfy

$$\mathbb{E}|X_k - Y_k|^2 \leq \mathbb{E}|X_0 - Y_0|^2 e^{\beta(\Delta t)k\Delta t},$$

where

$$\beta(\Delta t) := \frac{1}{\Delta t}\log\left(\frac{1 + L_g\Delta t + L_h\lambda\Delta t(1+\lambda\Delta t) + 2\sqrt{L_h}\lambda\Delta t}{1 - 2\Delta t}\right) < 0. \tag{30}$$

Further,

$$\beta(\Delta t) = \alpha + O(\Delta t), \quad \text{as } \Delta t \to 0.$$

Proof. Consider two numerical solutions X_n and Y_n from SSBE with with different starting values. From [10, Lemma 4.3] we have

$$(1-2\Delta t\mu)|X_n^\star - Y_n^\star|^2 \leq |X_n - Y_n|^2. \tag{31}$$

Now, from (10)–(11),

$$\begin{aligned} |X_{n+1}-Y_{n+1}|^2 &= |\, X_n^\star - Y_n^\star + (g(X_n^\star)-g(Y_n^\star))\,\Delta W_n + (h(X_n^\star)-h(Y_n^\star))\,\Delta N_n\,|^2 \\ &= |\, X_n^\star - Y_n^\star\,|^2 + |[g(X_n^\star)-g(Y_n^\star)]\Delta W_n|^2 + |[h(X_n^\star)-h(Y_n^\star)]\Delta N_n|^2 \\ &\quad + 2\langle X_n^\star - Y_n^\star, [g(X_n^\star)-g(Y_n^\star)]\Delta W_n\rangle + 2\langle X_n^\star - Y_n^\star, [h(X_n^\star)-h(Y_n^\star)]\Delta N_n\rangle \\ &\quad + 2\langle [g(X_n^\star)-g(Y_n^\star)]\Delta W_n, [h(X_n^\star)-h(Y_n^\star)]\Delta N_n\rangle. \end{aligned}$$

Hence, using (31),

$$\begin{aligned} \mathbb{E}|X_{n+1}-Y_{n+1}|^2 &\leq \left(1 + L_g\Delta t + L_h\lambda\Delta t(1+\lambda\Delta t) + 2\sqrt{L_h}\lambda\Delta t\right)\mathbb{E}|X_n^\star - Y_n^\star|^2 \\ &\leq \left(\frac{1 + L_g\Delta t + L_h\lambda\Delta t(1+\lambda\Delta t) + 2\sqrt{L_h}\lambda\Delta t}{1-2\Delta t\mu}\right)\mathbb{E}|X_n - Y_n|^2. \end{aligned}$$

The result now follows. □

Theorem 6 *Under the conditions (3), (4) and (5), if $\alpha < 0$ in (28) then for all $\Delta t > 0$ any two solutions from the CSSBE method with $\mathbb{E}|X_0|^2 < \infty$ and $\mathbb{E}|Y_0|^2 < \infty$ satisfy*

$$\mathbb{E}|X_k - Y_k|^2 \leq \mathbb{E}|X_0 - Y_0|^2 e^{\widehat{\beta}(\Delta t)k\Delta t},$$

where

$$\widehat{\beta}(\Delta t) := \frac{1}{\Delta t}\log\left(\frac{1+\Delta t(L_g + \lambda L_h)}{1-2\Delta t(\mu + \lambda\sqrt{L_h})}\right) < 0. \tag{32}$$

Further,

$$\widehat{\beta}(\Delta t) = \alpha + O(\Delta t), \quad \text{as } \Delta t \to 0.$$

Proof. Since f_λ has one-sided Lipschitz constant $\mu + \lambda\sqrt{L_h}$, in place of (31) we have

$$\left(1-2\Delta t(\mu+\lambda\sqrt{L_h})\right)|X_n^\star - Y_n^\star|^2 \leq |X_n - Y_n|^2. \tag{33}$$

Now, from (16)–(17),

$$|X_{n+1}-Y_{n+1}|^2 = |X_n^\star - Y_n^\star|^2 + |(g(X_n^\star)-g(Y_n^\star))\Delta W_n|^2 + |(h(X_n^\star)-h(Y_n^\star))\Delta\widetilde{N}_n|^2 + M_n, \tag{34}$$

where M_n is a martingale. Using $\Delta\widetilde{N}_n = \Delta N_n - \lambda\Delta t$, we have

$$\begin{aligned} \mathbb{E}\Delta\widetilde{N}_n^2 &= \mathbb{E}\Delta N_n^2 + 2\lambda\Delta t\mathbb{E}\Delta N_n + \lambda^2\Delta t^2 \\ &= \lambda\Delta t(1+\lambda\Delta t) - 2\lambda\Delta t\lambda\Delta t + \lambda^2\Delta t^2 \\ &= \lambda\Delta t. \end{aligned} \tag{35}$$

It is convenient that $\mathbb{E}\Delta\widetilde{N}_n^2$ does not involve an $O(\Delta t^2)$ term. Taking expectations in (34), using (35) and (33), we find that

$$\begin{aligned} \mathbb{E}|X_{n+1}-Y_{n+1}|^2 &\leq \mathbb{E}|X_n^\star - Y_n^\star|^2 + \Delta t L_g\mathbb{E}|X_n^\star - Y_n^\star|^2 + \lambda\Delta t L_h\mathbb{E}|X_n^\star - Y_n^\star|^2 \\ &= (1+\Delta t(L_g+\lambda L_h))\,\mathbb{E}|X_n^\star - Y_n^\star|^2 \\ &\leq \frac{1+\Delta t(L_g+\lambda L_h)}{1-2\Delta t(\mu+\lambda\sqrt{L_h})}\mathbb{E}|X_n - Y_n|^2. \end{aligned}$$

The result now follows. □

6 Mean-Square Linear Stability

Although the main focus of this work is on nonlinear SDEs, in this section we show that CSSBE has a very desirable linear stability property. Hence, we consider the scalar, linear test equation where $f(x) = ax$, $g(x) = bx$ and $h(x) = cx$ in (1); that is,

$$dX(t) = aX(t^-)dt + bX(t^-)dW(t) + cX(t^-)dN(t). \tag{36}$$

Here, a, b, c are real constants and we assume that $\mathbb{E}X_0^2 < \infty$ and $X_0 \neq 0$ with probability one. The mean-square stability of a class of implicit methods for this SDE was studied in [8]. That class includes natural generalizations of the trapezoidal and backward Euler methods. In particular, it was found that no methods in the class were able to reproduce completely the natural extension of deterministic A-stability.

On the SDE (36), the SSBE method reduces to the standard backward Euer method; that is, the method of [8, (1.2)] with $\theta = 1$. Hence, the linear mean-square stability properties of SSBE are described in [8, Theorems 3.2–3.4]. In particular, when $c < 0$ in (36), SSBE is not guaranteed to preserve stability for all $\Delta t > 0$.

To analyze the properties of CSSBE, we first recall from [8, (3.3)] that for (36)

$$\lim_{t\to\infty} \mathbb{E}X(t)^2 = 0 \qquad \Leftrightarrow \qquad 2a + b^2 + \lambda c(2 + c) < 0. \tag{37}$$

We also note in passing that the sufficient condition for mean-square stability derived in Theorem 4 for the general case matches this necessary and sufficient condition in the linear case when we take the optimal values $\mu = a$, $L_g = b^2$ and $L_h = c^2$.

We have the following characterization and corollary.

Theorem 7 *If $1 - (a + \lambda c)\Delta t \neq 0$ then for CSSBE applied to (36),*

$$\lim_{n\to\infty} \mathbb{E}Y_n^2 = 0 \qquad \Leftrightarrow \qquad (a + \lambda c)^2 \Delta t > 2a + b^2 + \lambda c(2 + c). \tag{38}$$

Proof. Applying CSSBE to (36) gives

$$Y_{n+1} = \frac{1 + b\Delta W_n + c\Delta \widetilde{N}_n}{1 - (a + \lambda c)\Delta t} Y_n.$$

Hence,

$$(1 - (a + \lambda c)\Delta t)^2 Y_{n+1}^2 = \left(1 + b^2 \Delta W_n^2 + c^2 \widetilde{N}_n^2 + M_n\right) Y_n^2,$$

where M_n is a martingale that is independent of Y_n. So, using (35)

$$(1 - (a + \lambda c)\Delta t)^2 \mathbb{E}Y_{n+1}^2 = \left(1 + b^2\Delta t + c^2 \lambda \Delta t\right) \mathbb{E}Y_n^2.$$

It follows that the linear mean-square stability property, $\lim_{n\to\infty} \mathbb{E}Y_n^2 = 0$, is characterized by

$$(1 - (a + \lambda c)\Delta t)^2 > 1 + b^2\Delta t + c^2\lambda\Delta t,$$

which simplifies to the inequality in (38). □

Corollary 3 shows that CSSBE has the natural extension of A-stability.

Corollary 3 *For the SDE (36), if $\lim_{t\to\infty} \mathbb{E}X(t)^2 = 0$ then for all $\Delta t > 0$ CSSBE produces a well-defined solution satisfying*

$$\lim_{n\to\infty} \mathbb{E}Y_n^2 = 0.$$

Proof. The SDE stability property requires a, b, c to satisfy the inequality in (37). It follows that $a + \lambda c < 0$, and hence $1 - (a + \lambda c)\Delta t \neq 0$. So CSSBE produces a well-defined solution. Since $(a + \lambda c)^2 \Delta t > 0$ and $2a + b^2 + \lambda c(2 + c) < 0$, the stability condition for CSSBE in (38) is satisfied. $\square$

APPENDIX: Proof of Lemma 1

The proof of Lemma 1 involves the Burkholder-Davis-Gundy inequality for martingales, which in turn requires the following estimate.

Lemma 5 *Under the assumptions (2), (3), (4), (5) and (9), for each* $p \geq 2$ *there is a constant* $\widehat{C} = \widehat{C}(p, t)$ *such that*

$$\mathbb{E}|X(t)|^p \leq \widehat{C}\left(1 + \mathbb{E}|X_0|^p\right), \qquad t \in [0, T].$$

Proof. Our proof is an adaptation of the proof for the case without jumps in [14, Theorem 2.4.1]. Throughout, we use K to denote a generic constant that may change between occurrences. We apply the Ito formula for the jump-diffusion SDE (1) to the function $U(t, x) = (1 + |x|^2)^{p/2}$, see [5], to obtain

$$\begin{aligned}
\left(1 + |X(t)|^2\right)^{p/2} &= \left(1 + |X_0|^2\right)^{p/2} + p\int_0^t \left(1 + |X(s^-)|^2\right)^{(p-2)/2} \left\langle X(s^-), f(X(s^-))\right\rangle ds \\
&\quad + \frac{p}{2}\int_0^t \left(1 + |X(s^-)|^2\right)^{(p-2)/2} \left|g(X(s^-))\right|^2 ds \\
&\quad + \frac{p(p-2)}{2}\int_0^t \left(1 + |X(s^-)|^2\right)^{(p-4)/2} \left|\left\langle X(s^-), g(X(s^-))\right\rangle\right|^2 ds \\
&\quad + p\int_0^t \left(1 + |X(s^-)|^2\right)^{(p-2)/2} \left\langle X(s^-), g(X(s^-))\right\rangle dW(s) \\
&\quad + \int_0^t \left(\left(1 + |X(s^-) + h(X(s^-))|^2\right)^{p/2} - \left(1 + |X(s^-)|^2\right)^{p/2}\right) dN(s).
\end{aligned}$$

Now, using the linear growth bound (8) for h, we have

$$\begin{aligned}
\left(1 + |x + h(x)|^2\right)^{p/2} - \left(1 + |x|^2\right)^{p/2} &\leq \left(1 + 2|x|^2 + 2|h(x)|^2\right)^{p/2} - \left(1 + |x|^2\right)^{p/2} \\
&\leq K\left(1 + |x|^2\right)^{p/2},
\end{aligned}$$

which can be used in estimates of the deterministic integral part of the jump integral. In addition,

$$\left(1 + |x|^2\right)^{p/2} \leq 2^{(p-2)/2}\left(1 + |x|^p\right),$$

which we will use to estimate the initial value.

We need to use stopping times $\tau_N := T \wedge \inf\{t \in [0, T] : X(t) \geq N\}$, for which $\tau_N \to T$ as $N \to \infty$ since $X(t)$ is cadlag. We take expectations over the interval $[0, \tau_N \wedge t]$ and then the limit as $N \to \infty$, using the fact that the expectations of the Ito and compensated

Poisson integrals vanish (page 60, [14]) as well as the linear growth bounds (6)–(8) and the Cauchy-Schwarz inequality to obtain

$$\mathbb{E}\left(1+|X(t)|^2\right)^{p/2} \le 2^{(p-2)/2}\left(1+\mathbb{E}|X_0|^p\right) + K\int_0^t \mathbb{E}\left(1+|X(s)|^2\right)^{p/2}\, ds.$$

We omit the details, as similar estimates arise in the next proof, apart from noting that $|\langle x, g(x)\rangle|^2 \le \frac{1}{2}\left(|x|^2+|g(x)|^2\right) \le |g(0)|^2 + (\frac{1}{2}+L_g)|x|^2$, and similarly for h.

The continuous Gronwall inequality then gives

$$\mathbb{E}\left(1+|X(t)|^2\right)^{p/2} \le 2^{(p-2)/2}\left(1+\mathbb{E}|X_0|^p\right) e^{KT},$$

from which follows

$$\mathbb{E}|X(t)|^p \le \mathbb{E}\left(1+|X(t)|^2\right)^{p/2} \le K\left(1+\mathbb{E}|X_0|^p\right),$$

as required. □

We may now give a proof of Lemma 1.

Proof. (Of Lemma 1) Our proof follows that of [9, Lemma 3.2] with the inclusion of the jump terms. We apply the Ito formula [5] for the jump-diffusion SDE (1) to the function $U(t,x)=|x|^2$ and after splitting the jump term into its martingale and deterministic integral parts and using the linear growth bounds (6)–(8) we obtain

$$\begin{aligned}
|X(t)|^2 &= |X_0|^2 + 2\int_0^t \langle X(s), f(X(s))\rangle\, ds \\
&\quad + \int_0^t |g(X(s))|^2\, ds + 2\int_0^t \left\langle X(s^-), g(X(s^-))\right\rangle dW(s) \\
&\quad + \int_0^t \left(\left\langle X(s^-), h(X(s^-))\right\rangle + |h(X(s^-))|^2\right) d\widetilde{N}(s) \\
&\quad + \lambda\int_0^t \left(\langle X(s), h(X(s))\rangle + |h(X(s))|^2\right) ds \\
&\le |X_0|^2 + K\int_0^t \left(1+|X(s)|^2\right) ds + 2\int_0^t \left\langle X(s^-), g(X(s^-))\right\rangle dW(s) \\
&\quad + \int_0^t \left\langle X(s^-), h(X(s^-))\right\rangle d\widetilde{N}(s) + \int_0^t |h(X(s^-))|^2\, d\widetilde{N}(s),
\end{aligned}$$

where K denotes a generic constant that may change between occurrences. There then exists a constant $C=C(p)$ such that

$$\begin{aligned}
C^{-1}\sup_{0\le\tau\le t}|X(\tau)|^p &\le |X_0|^p + \left(\int_0^t \left(1+|X(s)|^2\right) ds\right)^{p/2} \\
&\quad + \sup_{0\le\tau\le t}\left|\int_0^\tau \left\langle X(s^-), g(X(s^-))\right\rangle dW(s)\right|^{p/2} \\
&\quad + \sup_{0\le\tau\le t}\left|\int_0^\tau \left\langle X(s^-), h(X(s^-))\right\rangle d\widetilde{N}(s)\right|^{p/2} \\
&\quad + \sup_{0\le\tau\le t}\left|\int_0^\tau |h(X(s^-))|^2\, d\widetilde{N}(s)\right|^{p/2}.
\end{aligned}$$

Since $\mathbb{E}|X(t)|^p < \infty$ on the interval $[0, T]$, we can take expectations and apply the Burkholder-Davis-Gundy inequality to each of the martingale integral terms. Using the linear growth bound we obtain

$$\begin{aligned}\mathbb{E} \sup_{0\le\tau\le t} \left| \int_0^\tau \left\langle X(s^-), g(X(s^-)) \right\rangle dW(s) \right|^{p/2} &\le K \int_0^t \mathbb{E} \left| \langle X(s), g(X(s)) \rangle \right|^{p/2} ds \\ &\le K \int_0^t \mathbb{E} \left(1 + |X(s)|^2\right)^{p/2} ds \\ &\le K \int_0^t \left(1 + \mathbb{E}|X(s)|^p\right) ds \\ &\le K \int_0^t \left(1 + \mathbb{E} \sup_{0\le\tau\le s} |X(\tau)|^p\right) ds,\end{aligned}$$

and similarly for the other two integrals.

Combining all of the above estimates we obtain

$$\mathbb{E} \sup_{0\le\tau\le t} |X(\tau)|^p \le K\left(1 + \mathbb{E}|X_0|^p\right) + K \int_0^t \left(1 + \mathbb{E} \sup_{0\le\tau\le s} |X(\tau)|^p\right) ds.$$

An application of the continuous Gronwall inequality completes the proof. □

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