Slides for MAD 5932-01, SCS-MATH FSU Tallahassee*

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*Based on the lecture notes *Stochastic and Partial Differential Equations with Adapted Numerics,* by J. Goodman, K.-S. Moon, A. Szepessy, R. Tempone, G. Zouraris.

Aug 29, 2006 - Class contents:

1. Course Introduction, Admin details

2. Motivating examples (Chapter 1)

3. Brief Probability review

Admin details

- syllabus

- class location

- student groups, email list, order of groups for assignments

- HMW presentations by groups on Thursdays. /Hand in days are Tuesdays for the group that makes the presentation/

- Matlab access, SCS computer facilities, personal accounts.

- course webpage

Course goal: to understand numerical methods for problems formulated by stochastic or partial differential equations models in science, engineering and mathematical finance.

Motivating examples (Chapter 1)

Example 1 (Noisy Evolution of Stock Values) *Denote stock value by $S(t)$. Assume that $S(t)$ satisfies the differential equation*

$$\frac{dS}{dt} = a(t)S(t),$$

which has the solution

$$S(t) = e^{\int_0^t a(u)du} S(0).$$

Since we do not know precisely how $S(t)$ evolves we would like to generalize the model to a stochastic setting

$$a(t) = r(t) + "noise".$$

For instance, we will consider

$$dS(t) = r(t)S(t)dt + \sigma S(t)dW(t), \qquad (1)$$

where $dW(t)$ *will introduce noise in the evolution.*

What is the meaning of (1)? The answer is not as direct as in the deterministic ode case.

One way to give meaning to (1) is to use the Forward Euler discretization,

$$S_{n+1} - S_n = r_n S_n \Delta t_n + \sigma_n S_n \Delta W_n. \qquad (2)$$

Here ΔW_n *are independent normally distributed random variables with zero mean and variance* Δt_n, *i.e.*

$$E[\Delta W_n] = 0$$

and

$$Var[\Delta W_n] = \Delta t_n = t_{n+1} - t_n.$$

Then (1) *is understood as a limit of* (2) *when* $\max \Delta t \to 0$.

Applications to Option pricing

European call option: is a contract signed at time t which gives the right, but not the obligation, to buy a stock (or other financial instrument) for a fixed price K at a fixed future time $T > t$.

At time t the buyer pays the seller the amount $f(s, t; T)$ for the option contract.

What is a fair price for $f(s, t; T)$?

The Black-Scholes model for the value $f : (0,T) \times (0,\infty) \to \mathbb{R}$ of a European call option is the partial differential equation

$$\partial_t f + rs\partial_s f + \frac{\sigma^2 s^2}{2}\partial_s^2 f = rf, \; 0 < t < T, \\ f(s,T) = \max(s-K,0), \quad (3)$$

where the constants r and σ denote the riskless interest rate and the volatility, respectively.

Stochastic representation of $f(s,t)$ The Feynmann-Kač formula gives the alternative probability representation of the option price

$$f(s,t) = E[e^{-r(T-t)} \max(S(T) - K, 0)) | S(t) = s], \tag{4}$$

where the underlying stock value S is modeled by the stochastic differential equation (1) satisfying $S(t) = s$.
Thus, $f(s,t)$ is both the solution of a PDE (3) with the expected value of the solution of a SDE (4)!

Which one should we choose to discretize?

Example 2 (Porous media flow) *Consider an incompressible flow*

$$div(V) = 0, \tag{5}$$

and Darcy's law

$$V = -K\nabla P. \tag{6}$$

Here V *is the flow velocity and* P *is the pressure field. The function* K*, the so called conductivity of the material, is the source of randomness, since in practical cases, it is not precisely known.*

To study the concentration C of an inert pollutant carried by the flow V, we solve the convection equation

$$\partial_t C + V \cdot \nabla C = 0.$$

Observe: The variation of K is, via Darcy's law (6), determines the concentration C.

One way to determine the flow velocity is to solve the pressure equation

$$div(K\nabla P) = 0, \tag{7}$$

in a domain with given values of the pressure on the boundary of this domain.

Assume now that the flow is two dimensional with $V = (1, \widehat{V})$*, where* $\widehat{V}(x)$ *is stochastic with mean zero, i.e.* $E[\widehat{V}] = 0$*. Thus,*

$$\partial_t C + \partial_x C + \widehat{V} \partial_y C = 0.$$

Let us define $\bar{C}$ *as the solution of*

$$\partial_t \bar{C} + \partial_x \bar{C} = 0.$$

Is $\bar{C}$ *is the expected value of* C*, i.e. is*

$$\bar{C} \stackrel{?}{=} E[C]$$

true?

The answer is in general no. The difference comes from the expected value

$$E[\widehat{V}\partial_y C] \neq E[\widehat{V}]E[\partial_y C] = 0.$$

Can you see why?

The desired averaged quantity $\tilde{C} = E[C]$ *is an example of turbulent diffusion and in the simple case* $\widehat{V}(x)dx = dW(x)$ *(cf. (1)) it will satisfy a convection diffusion equation of the form*

$$\partial_t \tilde{C} + \partial_x \tilde{C} = \frac{1}{2}\partial_{yy}\tilde{C},$$

which is related to the Feynman-Kač formula (4).

Example 3 (Optimal Control of Investments)
Suppose that we invest in a risky asset, whose value $S(t)$ *evolves according to the stochastic differential equation*

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

and in a riskless asset $Q(t)$ *that evolves with*

$$dQ(t) = rQ(t)dt.$$

It is reasonable to assume $r < \mu$*, why?*

Our total wealth is then $X(t) = Q(t) + S(t)$.

Goal: *determine an optimal instantaneous policy of investment to maximize the expected value of our wealth at a given final time* T.

Let the time dependent proportion,

$$\alpha(t) \in [0, 1],$$

be defined by

$$\alpha(t)X(t) = S(t),$$

so that

$$(1 - \alpha(t))X(t) = Q(t).$$

Then our optimal control problem can be stated as

$$\max_{\alpha \in \mathcal{A}} E[g(X(T))|X(t) = x] \equiv u(t,x), \qquad (8)$$

where g is a given function.

How can we determine α?

The solution to (8) can be obtained by means of a Hamilton Jacobi equation, which is in general a nonlinear partial differential equation satisfied by $u(t,x)$ of the form

$$u_t + H(u, u_x, u_{xx}) = 0.$$

Part of our work is to study the theory of Hamilton Jacobi equations and numerical methods for control problems to determine the Hamiltonian H and the control α.