Exercise Show that if $W(t)$ is a Wiener process, then

$$B(t) \equiv c\,W\left(\frac{t}{c^2}\right)$$

is also a Wiener process. Here $c \neq 0$ is a constant.

Approximation and Definition of Stochastic Integrals

Questions on the definition of a stochastic integral

Remark 1 Problem: *How to define the stochastic integral* $\int_0^T W(t)dW(t)$, *where* $W(t)$ *is the Wiener process.*

Can we use the same approach as with Riemann integrals, taking sums

$$\sum_{n=0}^{N-1} W(\xi_n)(W(t_{n+1}) - W(t_n))$$

with $\xi_n \in [t_n, t_{n+1}]$*?*

As a first step, use the forward Euler *discretization*

$$\sum_{n=0}^{N-1} W(t_n)\underbrace{(W(t_{n+1}) - W(t_n))}_{=\Delta W_n}.$$

Taking expected values we obtain (why?)

$$\begin{aligned} E[\sum_{n=0}^{N-1} W(t_n)\Delta W_n] &= \sum_{n=0}^{N-1} E[W(t_n)\Delta W_n] \\ &= \sum_{n=0}^{N-1} E[W(t_n)]\underbrace{E[\Delta W_n]}_{=0} \\ &= 0. \end{aligned}$$

Now let us use instead the backward Euler *discretization*

$$\sum_{n=0}^{N-1} W(t_{n+1})\Delta W_n.$$

Taking expected values yields a different re-

sult:

$$
\begin{aligned}
\sum_{n=0}^{N-1} E[W(t_{n+1})\Delta W_n] \\
&= \sum_{n=0}^{N-1} E[W(t_n)\Delta W_n] + E[(\Delta W_n)^2] \\
&= \sum_{n=0}^{N-1} \Delta t \\
&= T \neq 0.
\end{aligned}
$$

Moreover, if we use the trapezoidal *method the result is*

$$
\begin{aligned}
&\sum_{n=0}^{N-1} E\left[\frac{W(t_{n+1}) + W(t_n)}{2}\Delta W_n\right] \\
&\quad = \sum_{n=0}^{N-1} E[W(t_n)\Delta W_n] + E[(\Delta W_n)^2/2] \\
&\quad = \sum_{n=0}^{N-1} \frac{\Delta t}{2} = T/2 \neq 0.
\end{aligned}
$$

$\square$

Conclusion: we need more information to define $\int_0^T W(s)dW(s)$ than to define a deterministic integral!

In fact, limits of the forward Euler define the so called *Itô integral*, while the trapezoidal method yields the so called *Stratonovich integral*.

Strong and weak convergence

Depending on the application, we focus either on

- *strong convergence*, where approximation of the outcomes of $X(T)$ is relevant,
- or *weak convergence*, where only the distribution (law) of $X(T)$ needs to be approximated.

Definition. The sequence of random variables $\{Y_n\}_{n\in\mathbb{N}}$ converges *strongly* to the random variable Y if

$$\|Y - Y_n\|_{L^2_P(\Omega)} \equiv \sqrt{E[(Y - Y_n)^2]} \to 0$$

Obs: By Chebychev we have

$$P(|Y - Y_n| \geq \epsilon) \leq \frac{E[(Y - Y_n)^2]}{\epsilon^2} \to 0$$

for ay fixed $\epsilon > 0$.

Definition. The sequence of random variables $\{Y_n\}_{n\in\mathbb{N}}$ converges *weakly* to the random variable Y if $E[g(Y)] - E[g(Y_n)] \to 0$, for all bounded continuous functions g.

Observe: strong convergence $\Rightarrow$ weak convergence, but the converse is in general not true.

Strong and weak convergence

Counterexample. Let random variables $\{Y_n\}_{n\in\mathbb{N}}$ be *iid* in $(\Omega, \mathcal{F}, P)$, and $Y_n \sim N(0,1), n = 1, \ldots$.
Verify that Y_n converges weakly but not strongly!

Proof of ($\Rightarrow$) for Lipschitz functions:

$$
\begin{aligned}
|E[g(X) - g(Y_n)]| &\le E[|g(X) - g(Y_n)|] \\
&\le C_g E[|X - Y_n|] \\
&\le C_g \underbrace{\sqrt{E[|X - Y_n|^2]}}_{=\|X-Y_n\|_{L^2_P(\Omega)}} \to 0.
\end{aligned}
$$

Obs: The previous estimate may not be optimal. There are cases where the weak error goes to zero much faster than the strong one.

Ito Integrals

Theorem 1 *Suppose that there exists* $C > 0$ *s.t.* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|f(t+\Delta t, W+\Delta W) - f(t, W)| \leq C(\Delta t + |\Delta W|)$$

then the forward Euler (left point quadrature) approximations

$$I_{\Delta t} = \sum_{n=0}^{N-1} f(t_n, W(t_n))\Delta W_n,$$

with $0 = t_0 < t_1 < \ldots < t_N = T$, *satisfy*

$$\|I_{\Delta t} - I_{\Delta t'}\|_{L^2_P(\Omega)} = E[I_{\Delta t} - I_{\Delta t'}]^{1/2} \leq \mathcal{O}\left(\sqrt{\Delta t_{\mathrm{max}}}\right) \quad (12)$$

Ito integrals

Remark 2 *The previous theorem implies that $I_{\Delta t}$ is Cauchy in $L^2_P(\Omega)$ and its limit defines the Ito integral*

$$\sum_{n=0}^{N-1} f(t_n, W(t_n))\Delta W_n$$
$$= \int_0^T f(s, W(s))dW(s) + \mathcal{O}\left(\sqrt{\Delta t_{\text{max}}}\right)$$

The previous estimate should be understood *as*

$$E\Big[\Big(\sum_{n=0}^{N-1} f(t_n, W(t_n))\Delta W_n - \int_0^T f(s, W(s))dW(s)\Big)^2\Big]$$
$$= \mathcal{O}\left(\Delta t_{\text{max}}\right)$$

Question: What is the computational work to reach an accuracy ϵ in $L^2_P(\Omega)$ sense using uniform time steps?

Information generated by a process.

Definition 9 *The symbol* $\mathcal{F}_t^W$ *denotes* the information generated by W on the interval $[0, t]$. *If, based on the observation of the trajectory* $\{W(s), 0 \le s \le t\}$ *it is possible to decide if an event* $A \in \mathcal{F}$ *has occurred or not, then we write* $A \in \mathcal{F}_t^W$*.*

If the value of a random variable Z can be completely determined by the observations $\{W(s), 0 \le s \le t\}$ then we write $Z \in \mathcal{F}_t^W$.

A stochastic process g is called **adapted to the filtration** $\{\mathcal{F}_t^W\}_{t\geq 0}$ if $g(t) \in \mathcal{F}_t^W$ for all $t \geq 0$.

Obs Math Grads: The filtration $\{\mathcal{F}_t^W\}_{t\geq 0}$ is actually an increasing family of σ-algebras. See Øksendal's book, Chapter 3, for precise definition.

Examples

1. $A = \{W(10) < 5\}$

2. $Z = \int_0^1 W(s)ds$

3. $f(t) = \sup_{s \le t} W(s)$

4. $g(t) = \sup_{s \le t+1} W(s)$

Remark 3 (Extension to adapted Itô integration
Itô integrals can be extended to adapted processes. Assume $f : [0, T] \times \Omega \to \mathbb{R}$ *is* adapted *to the* filtration $\{\mathcal{F}_t^W\}_{t \geq 0}$ *and that there is a constant* C *such that*

$$\sqrt{E[|f(t + \Delta t, \omega) - f(t, \omega)|^2]} \leq C\sqrt{\Delta t}. \quad (13)$$

Then the proof of Theorem 1 shows that (12) still holds.

Application Let $M(t) \equiv \max_{[0,t]} W(s)$. Then $M(t)$ is adapted to $\{\mathcal{F}_t^W\}_{t\geq 0}$ and

$$0 \leq M(t+\Delta t) - M(t) \leq \max_{[t,t+\Delta t]} (W(s) - W(t))$$

from where*

$$E[(M(t+\Delta t) - M(t))^2] \leq \Delta t \, \underbrace{E[(\max_{[0,1]} W(s))^2]}_{E[W(1)^2]}$$
$$\leq \Delta t$$

* Here we use that $P(M(t) \in db) = P(|W(t)| \in db$, see [K-S] p. 96.

Therefore, (13) is verified for

$$\int_0^T M(t)dW(t)$$

and the rate of strong approximation to it by the F. Euler method is still 1/2.

Theorem 2 (Basic properties of Itô integrals)

Suppose that $f, g : [0, T] \times \Omega \to \mathbb{R}$ *are Itô integrable, e.g.* $\mathcal{F}_t^W$*-adapted and satifying (13), and that* c_1, c_2 *are constants in* $\mathbb{R}$*. Then:*

(1)

$$\int_0^T (c_1 f(s,\cdot) + c_2 g(s,\cdot))dW(s) \\ = c_1 \int_0^T f(s,\cdot)dW(s) + c_2 \int_0^T g(s,\cdot)dW(s).$$

(2) $E\left[\int_0^T f(s,\cdot)dW(s)\right] = 0.$

(3)

$$E\left[(\int_0^T f(s,\cdot)dW(s))(\int_0^T g(s,\cdot)dW(s))\right]$$
$$= \int_0^T E\left[f(s,\cdot)g(s,\cdot)\right] ds.$$

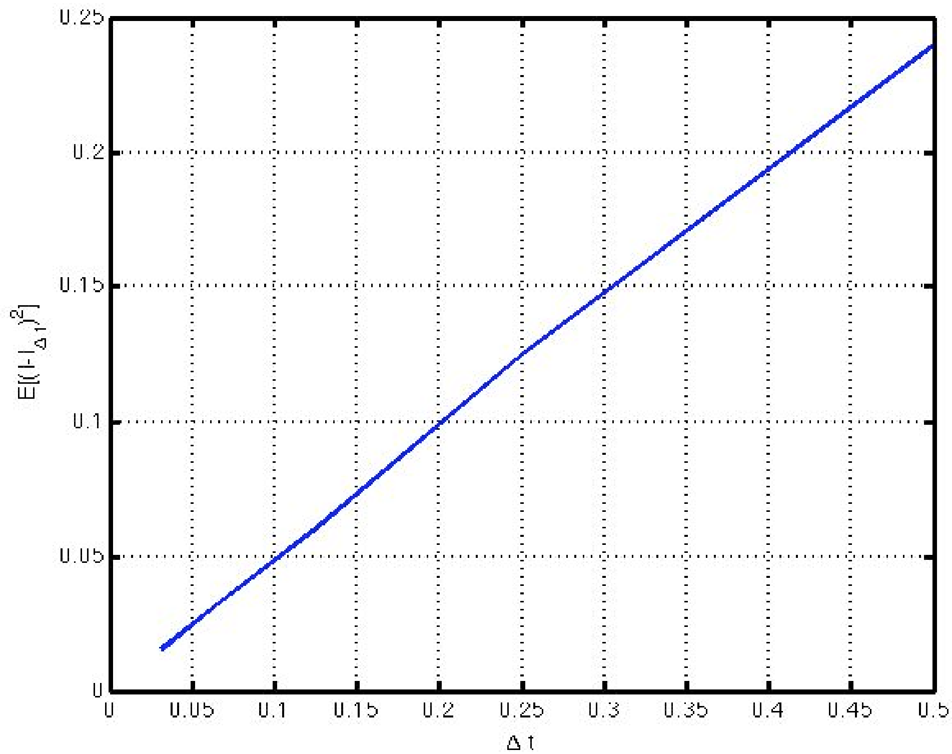
$\boxed{\textbf{Problem}}$ How can we approximate numerically the object $\int_0^T f(s, W(s))dW(s)$?

Approximation of $\int_0^1 W(s)dW(s)$:

```
dt = 1/N;
 t = linspace(0,1,N+1);
 dW = sqrt(dt)*randn(N,M);
 W = cumsum(dW);
 W = [zeros(1,M);W];
 I = 0.5*(W(N+1,:).^2-t(N+1));
 IFE = sum(W(1:N,:).*dW);
 Error(J,:) = I-IFE;
 mean_Square_Error = mean((Error.^2)')';
```

Question: Which changes are needed to extend the previous code to variable step size, i.e. to approximate I based on given points, not necessarily evenly distributed,

$$0 = t_0 < \ldots < t_N?$$



Strong approximation for $\int_0^1 W(s)dW(s) = \frac{W^2(1)-1}{2}$ using F. Euler with uniform time steps, $M = 10^3$.

Problem: Write a code to reproduce the previous results.

Question: Does the Wiener process really exist?

Answer: yes, see Example 2.18 in the notes, where we construct it as a limit of piecewise linear stochastic processes.