1 Definitions

Definition 1.1. ***European Call Option***
It gives holder the right but not the obligation to buy a stock at a fixed (strike) price K at time T. Hence the payoff is

$$\textit{payoff} = \max[0, X_T - K] \tag{1.1}$$

Definition 1.2. ***Barrier Option***
An option which comes into existence or becomes worthless if the underlying asset reaches some prescribed value before expiry.

Down and out *The option expires worthless if the barrier $X_t = b$ is reached from above before expiry. Thus, its domain is $X_t > b$.*

Definition 1.3. ***Lookback Option***
An option whose payoff depends on the maximum or minimum realized asset price in some window $[0, T]$.

$$\textit{payoff for minimum type call} = \max[0, X_T - Y_T] \tag{1.2}$$

where

$$Y_T = \min_{0 \le t \le T} X_t \tag{1.3}$$

2 Pricing Minimum-Type Lookback Options

Given the Black-Scholes partial differential equation

$$\mathcal{L}U = -\frac{\partial U}{\partial t} + r\,U - r\,x\frac{\partial U}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 U}{\partial x^2} = 0 \tag{2.1}$$

and the minimum stock process is defined earlier in Equation 1.3, the price $U(x, y, t)$ in $(x > y,\ t < T)$ of a min-type lookback option satisfies

$$\mathcal{L}U = 0; \quad U(x, y, T) = f(x, y); \quad U'(y, y, t) = 0 \tag{2.2}$$

Remark 2.1. The variable y plays the role of parameter.

Remark 2.2. Equation 2.2 resembles the PDE satisfied by a down and out barrier option with barrier level $x = y$.

3 Pricing Barrier Option

Let $V(x, t)$ denote the price of a down and out barrier option with barrier $x = b$ and expiry T payoff $F(x)$. Then $V(x, t)$ satisfies

$$\mathcal{L}V = 0; \quad V(x, T) = F(x); \quad V(b, t) = 0 \tag{3.1}$$

in the domain $(x > b,\ t < T)$.
This problem can be solved by using method of images.

$$V(x,t) = V_b(x,t) - V_b^*(x,t); \quad x > b \tag{3.2}$$

$$\mathcal{L}V_b = 0; \quad V_b(x,T) = F(x)\mathbf{1}_{x>b}; \quad V(b,t) = 0 \tag{3.3}$$

$$V_b^*(x,t) = (\frac{b}{x})^\beta\ V(\frac{b^2}{x},t), \quad \beta = \frac{2r}{\sigma^2} - 1 \tag{3.4}$$

Remark 3.1. $V_b^*(x,t)$ is the Black-Scholes image of $V_b(x,t)$ relative to $x = b$.
Also, $V_b^*(x,t) = V_b(x,t)$ when $x = b$.

Theorem 3.1. *The equivalent payoff for a minimum-type lookback option with expiry T payoff $U(x,y,T) = f(x,y)$ is given by*

$$U_{eq}(x,y,T) = f(x,y)\boldsymbol{1}_{x>y} + g(x,y)\boldsymbol{1}_{x<y} \tag{3.5}$$

where

$$g(x,y) = f(x,x) - \int_x^y (f^*)'(x,\xi)d\xi \tag{3.6}$$

Proof. If $U(x,y,t)$ solves Equation 2.2 then

$$V(x,y,t) = U'(x,y,t) = \frac{\partial U}{\partial y} \tag{3.7}$$

In fact the PDE satisfied by $V(x,y,t)$ in $(x > y,\ t < T)$ will be given in

$$\mathcal{L}V = 0; \quad V(x,y,T) = f'(x,y); \quad V(y,y,t) = 0 \tag{3.8}$$

This is exactly a PDE for a down and out barrier option with barrier level $x = y$. By solving this PDE by method of images we obtain

$$V_{eq}(x,y,T) = f'(x,y)\mathbf{1}_{x>y} - (f^*)'(x,y)\mathbf{1}_{x<y} \tag{3.9}$$

Thus, integrating $V_{eq}(x,y,T)$ with respect to y we obtain the result we are looking for.

$$U_{eq}(x,y,T) = f(x,y)\mathbf{1}_{x>y} + g(x,y)\mathbf{1}_{x<y}$$

□

4 Binary Option

The analysis for lookback option is further simplified by introducing notation for binary options on which the lookback option prices will ultimately depend.

$$A^+_\xi(x,0) = x\mathbf{1}_{x>\xi} \text{ and } A^-_\xi(x,0) = x\mathbf{1}_{x<\xi} \tag{4.1}$$

- A^+_ξ denotes the price of a contract that pays 1 unit of the underlying asset at expiry T provided the asset price at expiry is above the exercise price ξ.

- vice versa for A^-_ξ.
- $B^s_\xi(x,\tau)$ is the binary bond with the following expiry payoff

$$B^s_\xi(x,0) = x\mathbf{1}_{sx>s\xi} \tag{4.2}$$

By static replication we can express European calls and puts in terms of binaries

$$C_k(x,\tau) = A^+_\xi(x,\tau) - kB^+_\xi(x,\tau) \tag{4.3}$$

$$P_k(x,\tau) = -A^-_\xi(x,\tau) - kB^-_\xi(x,\tau) \tag{4.4}$$

5 Pricing Minimum Lookback Option in terms of Binary Option

Let us recall Theorem 3.1

$$U_{eq}(x,y,T) = f(x,y)\mathbf{1}_{x>y} + g(x,y)\mathbf{1}_{x<y} \tag{5.1}$$

and

$$g(x,y) = f(x,x) - \int_x^y (f^*)'(x,\xi)d\xi \tag{5.2}$$

Setting $f(x,y) = y$ and $f'(x,y) = 1$ in Equations 5.1 and 5.2 we obtain the equivalent price of minimum lookback option.

$$\begin{aligned} m_{eq}(x,y,0) &= y\mathbf{1}_{x>y} + \left[x - \int_x^y (\frac{\xi}{x})^\beta \, d\xi\right] \mathbf{1}_{x<y} \\ &= (1+\alpha)x\mathbf{1}_{x<y} + y[\mathbf{1}_{x<y} - \alpha\mathbf{1}^*_{x>y}] \end{aligned}$$

where

$$\alpha = \frac{\sigma^2}{2r} = \frac{1}{1+\beta}.$$

Using the notation in 4.1 and 4.2for binaries we obtain

$$m(x, y, \tau) = (1+\alpha)A_y^-(x, \tau) + y[B_y^+(x, \tau) - \alpha B_y^{*+}(x, \tau)] \tag{5.3}$$