Financial Engineering 2005, HW 5

Problem: (Baxter and Rennie p 73 Ex 3.9). Prove:

$$E_{\mathbb{Q}}[\exp(\theta(\tilde{W}_{t+s} - \tilde{W}_s))|\mathcal{F}_s] = \exp(\frac{1}{2}\theta^2 t).$$

Solution.

First, recall that $\tilde{W}_t = W_t + \gamma t$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(-\gamma W_T - \gamma^2 T/2)$. We also have the rule

$$E_{\mathbb{Q}}[X_t|\mathcal{F}_s] = \zeta_s^{-1} E_{\mathbb{P}}[\zeta_t X_t|\mathcal{F}_t],$$

where $\zeta_t = E_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]$.

Proposition 1 *For all s, $\zeta_s = \exp(-\gamma W_s - \gamma^2 s/2)$.*

Proof of proposition.

$$\begin{aligned}
E_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_s] &= E_{\mathbb{P}}[\exp(-\gamma W_T - \gamma^2 T/2)|\mathcal{F}_s] \\
&= \exp(-\gamma^2 T/2) E_{\mathbb{P}}[\exp(-\gamma(W_T - W_s))\exp(-\gamma W_s)|\mathcal{F}_s] \\
&= \exp(-\gamma^2 T/2)\exp(-\gamma W_s) E_{\mathbb{P}}[\exp(-\gamma(W_T - W_s)|\mathcal{F}_s] \\
&= \exp(-\gamma^2 T/2)\exp(-\gamma W_s)\exp(\gamma^2(T-s)) \\
&= \exp(-\gamma W_s)\exp(-\gamma^2 s/2).
\end{aligned}$$

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Now, let $LHS = E_{\mathbb{Q}}[\exp(\theta(\tilde{W}_{t+s} - \tilde{W}_s))|\mathcal{F}_s]$. Then

$$\begin{aligned}
LHS &= \zeta_s^{-1} E_{\mathbb{P}}[\zeta_{t+s}\exp(\theta(W_{t+s} - W_s + \gamma t))|\mathcal{F}_s] \\
&= \exp(\gamma W_s)\exp(\gamma^2 s/2) \\
&\quad E_{\mathbb{P}}[\exp(-\gamma W_{t+s})\exp(-\gamma^2(t+s)/2)\exp(\theta W_{t+s})\exp(-\theta W_s)\exp(\theta\gamma t)|\mathcal{F}_s] \\
&= \exp(\theta\gamma t)\exp((\gamma - \theta)W_s)\exp(-\gamma^2 t/2) E_{\mathbb{P}}[\exp((\theta - \gamma)W_{t+s})|\mathcal{F}_s] \\
&= \exp(\theta\gamma t)\exp(-\gamma^2 t/2) E_{\mathbb{P}}[\exp((\theta - \gamma)(W_{t+s} - W_s)|\mathcal{F}_s] \\
&= \exp(\theta\gamma t)\exp(-\gamma^2 t/2)\exp((\theta - \gamma)^2 t/2) \\
&= \exp(\theta^2 t/2).
\end{aligned}$$