FIN ENG I

HM#1

Let $r_p(t) = \beta_p r_m(t) + \theta_p(t)$ $\quad *$

where $\beta_p = \dfrac{Cov(r_p, r_m)}{Var(r_m)}$

Show $cov(r_m, \theta_p) = 0$

Take covariance with r_m for both sides of equation $*$

$$Cov(r_p(t), r_m) = Cov(\beta_p r_m(t), r_m) + Cov(\theta_p(t), r_m$$

β_p is a const.

$$Cov(r_p(t), r_m) = \beta_p Cov(r_m(t), r_m) + Cov(\theta_p, r_m)$$

$$Cov(r_p(t), r_m) = \frac{Cov(r_p, r_m)}{\cancel{Var(r_m)}} \cdot \cancel{Var(r_m)} + Cov(\theta_p, r_m)$$

$$Cov(r_p(t), r_m) = Cov(r_p, r_m) + Cov(\theta_p, r_m)$$

$$\Rightarrow Cov(\theta_p, r_m) = 0$$

HM#2

$$\operatorname{cov}(r_i, r_j) = \operatorname{cov}\left(\sum_{k=1}^{K} \overset{\beta}{X_{ik}}\, \overset{f_m}{b_k} + u_i,\ \sum_{l=1}^{K} X_{jl} b_l + u_j\right)$$

$$\Rightarrow \operatorname{cov}(r_i, r_j) = \operatorname{cov}\left(\sum_{k=1}^{K} X_{ik} b_k,\ \sum_{l=1}^{K} X_{jl} b_l\right) + \operatorname{cov}\left(\sum_{k=1}^{K} X_{ik} b_k,\ u_j\right) + \operatorname{cov}\left(\sum_{l=1}^{K} X_{jl} b_l,\ u_i\right) + \operatorname{cov}(u_i, u_j)$$

$$\Rightarrow \operatorname{cov}(r_i, r_j) = \sum_{k=1}^{K}\sum_{l=1}^{K} \operatorname{cov}(X_{ik} b_k, X_{jl} b_l) + \sum_{k=1}^{K} \cancel{\operatorname{cov}(X_{ik} b_k, u_j)}^{\,0} + \sum_{l=1}^{K} \cancel{\operatorname{cov}(X_{jl} b_l, u_i)}^{\,0} + \operatorname{cov}(u_i, u_j)$$

$$\Rightarrow \operatorname{cov}(r_i, r_j) = \begin{cases} \displaystyle\sum_{k=1}^{K}\sum_{l=1}^{K} \operatorname{cov}(X_{ik} b_k, X_{jl} b_l) & (i \neq j) \\ \displaystyle\sum_{k=1}^{K}\sum_{l=1}^{K} \operatorname{cov}(X_{ik} b_k, X_{il} b_l) + \operatorname{Var}(U_i) & (i=j) \end{cases}$$

$i = 1 \cdots N$

$(j = 1 \cdots N)$

$$\Rightarrow \underbrace{X F X^T}_{\text{how?}} + \Delta$$

3. $U(p) = \bar{r}_p - \lambda \sigma_p^2$

why choose 2nd order of σ_p in Utility function

Reason 1:

As we know when we use Ito's lemma to derive the process followed by $\ln S$. $(G = \ln S)$

we find $dG = (\mu - \frac{\sigma^2}{2})\, dt + dz$.

here μ and σ^2 are linear by combined and should have same meaning.

so in the Utility function, if we want introduce σ_p as a variable, we should use 2nd order

Reason 2: In Financial Economics class, we have learned that people (agents) are trying to maximize their Expected Utility. However, under following assumptions the Mean-Variance solution are same as Expected Utility solution.

① If the utility function is quadratic

or ② Distribution of the Utility function only has two moments such as a normal distribution function.

Specifically, if we assume that the distribution function is normal and utility function is a negative exponential function, we get $E[u(x)] = -\exp[-\lambda(\mu - \lambda\sigma^2)]$

see back

3. $U(p) = f_p - \lambda \sigma_p^2$

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see back

5. Zess

$$f = \beta_{:Q} f_Q$$

If 100% invested in GZ

$$\Rightarrow h_Q = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow GZ$$

$$f = \beta_{:GZ} f_{GZ} + \alpha$$

For GZ is Q $\Rightarrow \alpha = 0$

$$f = \beta_{:GZ} f_{GZ} = \frac{V h_{GZ}}{h_{GZ}^T V h_{GZ}} f_{GZ}$$

$h_{:GZ}$

$$f_p = h_p^T f =$$

$$\vec{\beta}_{\alpha}$$

an

$$\beta_{:GZ}$$

$$f_p = \beta_{p:GZ}$$

6. $f_Q = 6\%$ $\alpha_Q = 0$

$\sigma_Q^2 = \beta_Q^2 \sigma_M^2 + W_Q^2$

$\beta_Q = 1$ $\sigma_Q = \sigma_M$ $\Longrightarrow$ $W_Q = 0$

$SR_Q = \dfrac{f_Q}{\sigma_Q} = \dfrac{6}{15.82} = 0.379$

$IR_Q = 0$

$f_Q = \beta_{QB} f_B + \alpha_{QB}$

$\alpha_{QB} = f_Q - \beta_{QB} f_B$

$= 6\% - 1.004 \cdot 5.79\%$

$= 0.187\%$

$\sigma_Q^2 = \beta_{QB}^2 \sigma_B^2 + W_{QB}^2$

$W_{QB} = [(15.82)^2 - (1.004 \cdot 15.50)^2]^{\frac{1}{2}}$

$= 2.85$

$IR_{QB} = \dfrac{0.187}{2.85} = 0.0656$

$f_B = \beta_{BQ} f_Q = 0.965 \times 6\% = 5.79\%$ $\alpha_{BQ} = 0$

$\alpha_{BB} = 0$

$\sigma_B^2 = \beta_{BQ}^2 \sigma_Q^2 + W_{BQ}^2$

$W_{BQ} = (\sigma_B^2 - \beta_{BQ}^2 \sigma_Q^2)^{\frac{1}{2}} = ((15.50)^2 - (0.965)^2 (15.82)^2)^{\frac{1}{2}}$

$= 2.681$

$B = Q$

$SR_{BQ} = \dfrac{f_B}{\sigma_B} = \dfrac{5.79}{15.50} = 0.374$

$IR_{BQ} = \dfrac{\alpha_{BQ}}{W_{BQ}} = 0$

$f_C = \beta_{CQ} f_Q = 0.831 \times 6\% = 4.986\%$ $\alpha_{CQ} = 0$

$\sigma_C^2 = \beta_{CQ}^2 \sigma_Q^2 + W_{CQ}^2$

$C = Q$

$W_{CQ} = (\sigma_C^2 - \beta_{CQ}^2 \sigma_Q^2)^{\frac{1}{2}} = [14.42^2 - (0.831 \cdot 15.82)^2]^{\frac{1}{2}} = 5.925$

$SR_C = \dfrac{f_C}{\sigma_C} = \dfrac{4.986}{14.42} = 0.346$

$IR_{CQ} = \dfrac{\alpha_{CQ}}{W_{CQ}} = 0$

$f_C = \beta_{CB} f_B + \alpha_{CB}$

$C = B$

$\Rightarrow \alpha_{CB} = f_C - \beta_{CB} f_B = 4.986\% - 0.865 \times 5.79\% = -0.02235\%$

$\sigma_C^2 = \beta_{CB}^2 \sigma_B^2 + W_{CB}^2$

$W_{CB} = (\sigma_C^2 - \beta_{CB}^2 \sigma_B^2)^{\frac{1}{2}} = (14.42^2 - (0.865 \times 15.50)^2)^{\frac{1}{2}} = 5.308$

$IR_{CB} = \dfrac{\alpha_{CB}}{W_{CB}} = \dfrac{-0.02235}{5.308} = -0.00421$

7.

4.3

$$IR_A = IC_A \sqrt{BR_A}$$

$$IR_B = IC_B \sqrt{BR_B}$$

$$IR_A = IR_B$$

$$\Rightarrow \quad IC_B = \sqrt{\frac{BR_A}{BR_B}}\, IC_A$$

$$= \sqrt{\frac{250 \times 4}{4 \times 4}}\; 2\%$$

$$= 15.81\%$$

Problem 8.

MMI Company	ω_n	Signal	Score	α_1	$\frac{\alpha_1}{\omega_n^2}$	Largest positive holding from method 1	$\alpha_2 = IC*\omega_n*Score$	$\frac{\alpha_2}{\omega_n^2}$	Largest positive holding from method 2
American Express	23.26%	SELL	-1	-1.00%	-0.184833454		-2.09%	-0.38693035	
AT&T	15.89%	BUY	1	1.00%	**0.396051995 (largest)**	$	1.43%	**0.566393958 (largest)**	$
Chevron	20.44%	BUY	1	1.00%	0.239352637		1.84%	0.440313112	
Coca-Cola	18.92%	SELL	-1	-1.00%	-0.279355828		-1.70%	-0.4756871	
Disney	19.17%	SELL	-1	-1.00%	-0.272117063		-1.73%	-0.46948357	
Dow Chemicals	16.93%	BUY	1	1.00%	0.348888041		1.52%	0.531600709	
Dupont	17.29%	BUY	1	1.00%	0.334510699		1.56%	0.520532099	
Eastman Kodak	19.20%	BUY	1	1.00%	0.271267361		1.73%	0.46875	
Exxon	21.13%	SELL	-1	-1.00%	-0.223975753		-1.90%	-0.42593469	
General Electrics	14.42%	SELL	-1	-1.00%	-0.48091628		-1.30%	-0.62413315	
General Motors	23.46%	BUY	1	1.00%	0.181695422		2.11%	0.383631714	
IBM	30.32%	BUY	1	1.00%	0.108778134		2.73%	0.296833773	
International Paper	19.83%	SELL	-1	-1.00%	-0.254304808		-1.78%	-0.45385779	
Johnson&Johnson	18.97%	BUY	1	1.00%	0.27788515		1.71%	0.474433316	
McDonalds	20.54%	BUY	1	1.00%	0.23702771		1.85%	0.438169426	
Merck	20.43%	SELL	-1	-1.00%	-0.239587009		-1.84%	-0.44052863	
3M	13.41%	SELL	-1	-1.00%	-0.556086618		-1.21%	-0.67114094	
Procter & Gamble	16.29%	SELL	-1	-1.00%	-0.376840726		-1.47%	-0.55248619	
Phillip Morris	20.17%	BUY	1	1.00%	0.24580358		1.82%	0.446207238	
Sears	22.33%	SELL	-1	-1.00%	-0.200549948		-2.01%	-0.40304523	

+1 $\frac{6}{7}$ + 1 e.c

Howework II

Financial Engineering I

Instructor: Dr. Kercheval

Jiangwen Dong

Due 09/26/2003

prove $\boxed{\nabla h^T V h = 2Vh}$

2.1

$$h^T V h = [h_1 \cdots h_N] \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1N} \\ \vdots & \ddots & & \vdots \\ V_{N1} & & \cdots & V_{NN} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}$$

$$= \left[\sum_{i=1}^{N} h_i V_{i1}, \sum_{i=1}^{N} h_i V_{i2}, \cdots \sum_{i=1}^{N} h_i V_{iN} \right] \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}$$

$$= \sum_{j=1}^{N} \left(h_j \sum_{i=1}^{N} h_i V_{ij} \right) = \sum_{j=1}^{N} \sum_{i=1}^{N} h_i V_{ij} h_j$$

$$\frac{\partial (h^T V h)}{\partial h_k} = \frac{\partial \left(\sum_{j=1}^{N} \sum_{i=1}^{N} h_i V_{ij} h_j \right)}{\partial h_k} = \sum_{i=1}^{N} h_i V_{ik} + \sum_{j=1}^{N} V_{kj} h_j$$

$$= \sum_{i=1}^{N} h_i V_{ik} + \sum_{i=1}^{N} h_i V_{ik} = 2 \sum_{i=1}^{N} h_i V_{ik} =$$

$$\nabla h^T V h = 2 * \begin{bmatrix} \sum_{i=1}^{N} h_i V_{1i} \\ \vdots \\ \sum_{i=1}^{N} h_i V_{Ni} \end{bmatrix} = 2 \begin{bmatrix} V_{11} & \cdots & V_{1N} \\ \vdots & \ddots & \\ V_{N1} & & V_{NN} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}$$

$$= 2Vh$$

2.2. $\vec{a} = K_d \vec{d} + K_f \vec{f}$

prove, $\vec{h_a} = K_d \cdot \frac{\sigma_a^2}{\sigma_d^2} \vec{h_d} + K_f \frac{\sigma_a^2}{\sigma_f^2} \vec{h_f}$

$$\frac{1}{\sigma_a^2} = \frac{K_d a_d}{\sigma_d^2} + \frac{K_f a_f}{\sigma_f^2}$$

$$\vec{a} = K_d \vec{d} + K_f \vec{f}$$

$$\frac{V h_a}{\sigma_a^2} = K_d \frac{V h_d}{\sigma_d^2} + K_f \frac{V h_f}{\sigma_f^2} \quad (*)$$

$$(*) \times \sigma_a^2 V^{-1} \Rightarrow \vec{h_a} = K_d \frac{\sigma_a^2}{\sigma_d^2} \vec{h_d} + K_f \frac{\sigma_a^2}{\sigma_f^2} \vec{h_f}$$

$$(*) \times V^{-1} h_a^{-1} \quad \frac{1}{\sigma_a^2} = K_d \frac{V h_d}{\sigma_d^2 V h_a} + K_f \frac{V h_f}{\sigma_f^2 V h_a}$$

$$= K_d \frac{h_a^T V h_d}{\sigma_d^2 h_a^T V h_a} + K_f \frac{h_a^T V h_f}{\sigma_f^2 h_a^T V h_a}$$

$$= K_d \frac{\mathrm{cov}(r_a, r_d)}{\sigma_d^2 \sigma_a^2} + K_f \frac{\mathrm{cov}(r_a, r_f)}{\sigma_f^2 \sigma_a^2}$$

we know that $\mathrm{cov}(r_a, r_d) = h_a^T V h_d = \sigma_a^2 a_d$

$\mathrm{cov}(r_a, r_f) = \sigma_a^2 a_f$

$$\frac{1}{\sigma_a^2} = K_d \frac{a_d}{\sigma_d^2} + K_f \frac{a_f}{\sigma_f^2}$$

2.3.

$$e_Q = h_Q^T e = \left(\frac{h_q}{e_q}\right)^T e = \frac{h_q^T e}{e_q} = \frac{e_q}{e_q} = 1$$

$\therefore$ Q is a fully invested portofolio.

Also we know that q and any $\frac{q}{k}$ (k is scaling factor) maximize SR.

$Q = \frac{h_q}{e_q}$ where e_q can be looked as k.

$\therefore$ Q also maximize SR.

prove Uniqueness.

From proposition 1 in chapter 2,

we know for any attribute a, there is a unique portofolio h_a that has minimum risk and unit exposure to a. Therefore Q is unique.

2.4. Prove $\vec{\alpha} = IR \left(\frac{\vec{V}\vec{h_A}}{w_A}\right)$

From definition of α we know

$$\vec{\alpha} = \frac{\vec{V}\vec{h_A}}{\sigma_A^2}$$

From proposition 5.3 we also know

$$w_A = \sigma_A = \frac{1}{IR}$$

$$\Rightarrow \vec{\alpha} = \frac{\vec{V}\vec{h_A}}{\sigma_A w_A} = IR\left(\frac{\vec{V}\vec{h_A}}{w_A}\right)$$

$\boxed{\text{Prove } SR_B^2 = SR^2 - IR^2}$

$$\alpha^T V^{-1}\beta = \beta^T V^{-1}\alpha = \beta^T V^{-1}(f - \beta f_B) = \beta^T V^{-1} f - \beta^T V^{-1}\beta f_B$$

$$= \beta^T V^{-1} f - \sigma_B^2 h_B^T f = \beta^T V^{-1} f - \cancel{\sigma_B^2}\frac{(V^{-1}\beta)^T}{\cancel{\sigma_B^2}} f$$

$$= \beta^T V^{-1} f - (V^{-1}\beta)^T f = 0$$

$$SR_B^2 = \left(\frac{f_B}{\sigma_B}\right)^2 = f_B^2 \beta^T V^{-1}\beta$$

$$SR^2 = f^T V^{-1} f = (\beta f_B + \alpha)^T V^{-1}(\beta f_B + \alpha)$$

$$= (f_B\beta^T + \alpha^T) V^{-1}(\beta f_B + \alpha)$$

$$= f_B^2 \beta^T V^{-1}\beta + \cancel{f_B\beta^T V^{-1}\alpha} + \cancel{f_B\alpha^T V^{-1}\beta} + \alpha^T V^{-1}\alpha$$

$$= SR_B^2 + IR^2$$

$$\Rightarrow SR_B^2 = SR^2 - IR^2$$

2.5 (Notes P54 b.1)

$U_R(z) = U_R(0) + \Delta U_R(2z - z^2)$

when $z = 0 \Rightarrow U_R(z) = U_R(0) = 0$.

when $z = 1 \Rightarrow U_R(z) = 0 + \Delta U_R(2 - 1^2) = \Delta U_R = 200 b_1$

when $z = \frac{100}{200} = 0.5 \Rightarrow U_R(z) = 200(1 - (\frac{1}{2})^2) = \frac{3}{4} \times 200 = 150$

2.6 (Notes P54 b.2)

$U_R(z) = 100 = 200(2z - z^2) \Rightarrow z = \frac{2 - \sqrt{2}}{2} = 0.293$

$0.293 \times 200\% = 58.579\%$

2.7 (notes P54. 6.3)

$$\alpha_{cost} = \left|\frac{\Delta P}{P}\right| / \text{holding period}$$

$$= 1\% / 1.5 = 0.667\%$$

$$\alpha = 1.5\% > \alpha_{cost} = 0.667\%$$

$\therefore$ I will trade.

2.8 (notes P54. 64)

(1) $1 \times 10^9 \times 20 \times 10^{-4} = \$2 \times 10^6$

$10 \times 10^9 \times [(100 - 10) \times 10^{-4}] - 2 \times 10^6 = \$7,000,000$

(2) $10 \times 10^9 \times \{(100 - 10) \times 10^{-4}) - 2 \times 10^6 = \$88,000,000$

(3) $100 \times 10^9 \times ((100 - 10) \times 10^{-4}) - 2 \times 10^6 = \$898,000,000$

should ~~be~~ be negative

2.9. (G & K Pr4 2.1)

$$E(\vec{r}_P) = \beta_P E(\vec{r}_M) = 1.05 \times (-5\%) = -5.25\%.$$

2.10 (G & K Pr4 2.3)

$$Cov(r_A, r_B) = Cov(\beta_A r_M + \theta_A, \beta_B r_M + \theta_B)$$

$$= cov(\beta_A r_M; \beta_B r_M) + \underbrace{cov(\beta_A r_M, \theta_B)}_{0} +$$

$$\underbrace{cov(\beta_B r_M, \theta_A)}_{0} + \underbrace{cov(\theta_A, \theta_B)}_{0} \rightarrow$$ assume the residual returns are uncorrelated.

$$= \beta_A \beta_B \cdot Var(r_M)$$

$$= 1.15 \times 0.95 \times (20\%)^2 = 437$$

$$\rho_{AB} = \frac{Cov(r_A, r_B)}{\sigma_A \sigma_B} = \frac{437}{35 \cdot 33} = \boxed{0.378.}$$

$$w_A = \sqrt{\sigma_A^2 - \beta_A^2 \sigma_M^2}$$

$$= \sqrt{35^2 - 1.15^2 20^2}$$

$$= 26.38\%$$

$$w_B = \sqrt{\sigma_B^2 - \beta_B^2 \sigma_M^2}$$

$$= \sqrt{33^2 - 0.95^2 \cdot 20^2} \qquad \therefore \boxed{w_B > w_A.}$$

$$= 26.98\%$$

2.11 (G&K P38 2.1)

From prop. 1 (2A.21) $\Rightarrow \sigma_{BC} = e_B \sigma_C^2 = \beta_C \sigma_B^2$

use Market portfolio as our Benchmark

$$\sigma_{MC} = e_M \sigma_C^2 = \beta_C \sigma_M^2$$

$$\Rightarrow \quad \beta_C = \frac{e_M \sigma_C^2}{\sigma_M^2}$$

But we know in CAPM model

if we assume $Q = M$.

$$\therefore \quad e_M = e_Q = h_Q^T e = 1$$

Since Q is on the E. F. which is form by fully invested Portfolio

2.12 (G&K, P38 2.2)

$$K = \frac{\sigma_Q^2 - \sigma_c^2}{(f_Q - f_c)^2} = \frac{\sigma_c^2}{f_Q - f_c} \cdot \frac{\frac{\sigma_Q^2}{\sigma_c^2} - 1}{f_Q - f_c}$$

$$\sigma_Q^2 = h_Q^T V h_Q = \frac{h_q^T}{e_q} \cdot V \frac{h_q}{e_q} = \frac{\sigma_q^2}{e_q^2}$$

$$\Rightarrow K = \frac{\frac{\sigma_q^2}{\sigma_c^2 e_q^2} - 1}{f_Q - f_c} \cdot \frac{\sigma_c^2}{f_Q - f_c}$$

$$f_c \sigma_q^2 = e_q \sigma_c^2 \Rightarrow \sigma_q^2 = \frac{e_q \sigma_c^2}{f_c}$$

$$f_Q = h_Q^T f = \frac{1}{e_q} h_q^T f = \frac{1}{e_q} \left(\frac{V^{-1} f}{f^T V^{-1} f} \right) f \cdot \frac{1}{e_q}$$

$$\Rightarrow K = \frac{\frac{e_q \sigma_c^2}{e_q^2 f_c \sigma_c^2} - 1}{f_Q - f_c} \cdot \frac{\sigma_c^2}{f_Q - f_c}$$

$$= \frac{\frac{f_Q}{f_c} - 1}{f_Q - f_c} \cdot \frac{\sigma_c^2}{f_Q - f_c}$$

$$= \frac{\sigma_c^2}{(f_Q - f_c) f_c}$$

or $f_Q = f_c + \sigma_c^2 / K f_c$

2.13 G&K P38 (2.7)

$$(\text{Characteristic of MMI}) = \frac{V h_{MMI}}{\sigma^2_{MMI}}$$

We have historical return of all the asset in MMI. Therefore we can calculate V; σ_{MMI}

For $h_{MMI} = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}$ $\quad h_i = \dfrac{(\text{shares outstanding of } i)(\text{Price}_i)}{\sum_{i=1}^{N} (\text{shares outstanding of } i)(\text{Price}_i)}$

(G&K, Pg 2.4)

2.14. $\max_{h_p} (f_p - \lambda \sigma_p^2)$ s.t. $\vec{h_p}^T \vec{e} = 1$

$$\Longleftrightarrow \begin{cases} \nabla_{h_p} \left(\vec{h_p}^T \vec{f} - \lambda \vec{h_p}^T V \vec{h_p} \right) = \mu \nabla (\vec{h_p}^T \vec{e}) \\ \vec{h_p}^T e = 1 \end{cases}$$

$$\Rightarrow \quad \vec{h_p} = \frac{1}{2\lambda} V^{-1} (\vec{f} - \mu \vec{e})$$

$$\vec{h_p}^T e = \left(\frac{1}{2\lambda} V^{-1} \vec{f} - \mu \vec{e} \right)^T e$$

$$\Rightarrow \mu = \frac{\vec{e}^T V^{-1} \vec{f}}{\vec{e}^T V^{-1} \vec{e}} - \frac{2\lambda}{\vec{e}^T V^{-1} \vec{e}}$$

$$\vec{h_p} = \frac{1}{2\lambda} V^{-1} \vec{f} - \frac{1}{2\lambda} \left(\frac{\vec{e}^T V^{-1} \vec{f}}{\vec{e}^T V^{-1} \vec{e}} - \frac{2\lambda}{\vec{e}^T V^{-1} \vec{e}} \right) V^{-1} \vec{e}$$

$$f^* = \vec{h_p}^T f = \frac{1}{2\lambda} \vec{f}^T V^{-1} \vec{f} - \frac{1}{2\lambda} \frac{\vec{e}^T V^{-1} \vec{f}}{\vec{e}^T V^{-1} \vec{e}} \vec{f}^T V^{-1} \vec{e} + \frac{\vec{f}^T V^{-1} \vec{e}}{\vec{e}^T V^{-1} \vec{e}}$$

$$= \frac{\vec{f}^T V^{-1} \vec{e}}{\vec{e}^T V^{-1} \vec{e}} + \frac{1}{2\lambda} \left[\vec{f}^T V^{-1} \vec{f} - \frac{(\vec{f}^T V^{-1} \vec{e})^2}{\vec{e}^T V^{-1} \vec{e}} \right]$$

$$= \vec{f}^T \vec{h_c} + \frac{1}{2\lambda} \left[\frac{1}{\sigma_q^2} - \frac{\vec{f}^T V^{-1} \vec{e}}{\vec{e}^T V^{-1} \vec{e}} (\vec{f}^T V^{-1} \vec{e}) \right]$$

$$= f_c + \frac{1}{2\lambda} \left(\frac{1}{\sigma_q^2} - \frac{f_c}{\sigma_q^2} \right) = f_c + \frac{1}{2\lambda k}$$

2.15 (G & K, P133 5.4)

$$\lambda_R = \frac{IR}{2W^*} \Rightarrow W^* = \frac{IR}{2\lambda_R} = \frac{0.6}{0.24} = 2.5$$

$$VA^* = \frac{W^* \cdot IR}{2} = \frac{2.5 \times 0.6}{2} = 0.75$$

2.16 (G & K, P133 5.5)

$$IR = 0.3 \Rightarrow W^* = \frac{2.5}{2} = 1.25$$

$$VA^* = \frac{1.25 \times 0.3}{2} = 0.1875$$

$$VA = W_P\, IR - \lambda_P \cdot W_P^2 = 2.5 \times 0.3 - 0.12 \times 2.5^2 = 0$$

the lost 18.75% value added

Homework 3 ||| = 3/3

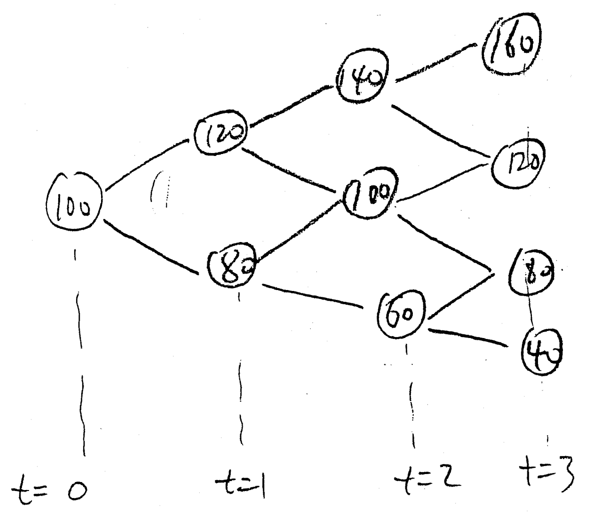
Financial Engineering

Dr. Kercheval

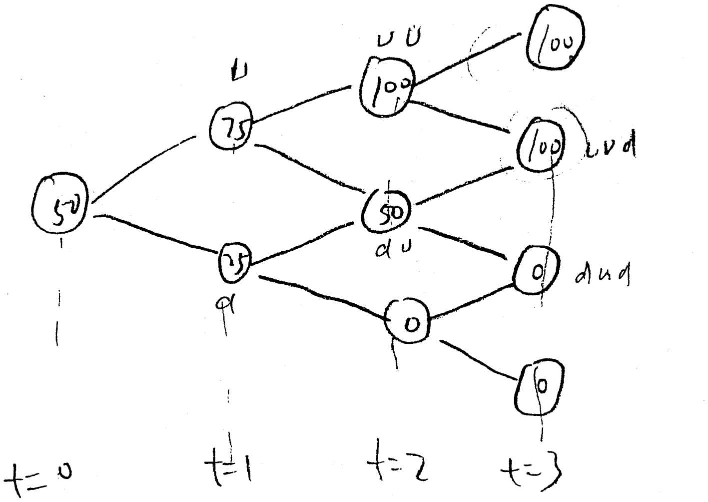
Jiangwen Dong

10/23/03

3.1.



underlying security price tree



Derivative claim tree

Assume $r = 0$

$$q = \frac{S_{now} - S_{down}}{S_{up} - S_{down}} = \frac{140-120}{160-120} = \frac{1}{2}$$

$$V_{u,u} = q\, V_{u,u,u} + (1-q)\, V_{u,u,d}$$

$$= \frac{1}{2} \times 100 + \frac{1}{2} \times 100$$

$$= 100$$

$$V_{u,d} = V_{du} = \frac{1}{2} \times 100 + \frac{1}{2} \times 0 = 50$$

$$V_{d,d} = \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0.$$

$$V_u = \frac{1}{2} \times 50 + \frac{1}{2} \times 100 = 75$$

$$V_d = \frac{1}{2} \times 50 + \frac{1}{2} \times 0 = 25$$

$$V_0 = \frac{1}{2} \times 75 + \frac{1}{2} \times 25 = 50$$

Time i	Last jump	Stock price S_i	Option value V_i	Stock Holding ϕ_i	Bond Holding ψ_i
0	—	100	50	—	—
1	u	120	75	1.25	-75
2	u	140	100	1.25	-75
3	d	120	100	0	100

$$\phi_1 = \frac{75-25}{120-80} = 1.25 \qquad \psi_1 = 50 - 1.25 \times 100 = -75$$

$$\phi_2 = \frac{100-50}{140-100} = 1.25 \qquad \psi_2 = 75 - 1.25 \times 120 = -75$$

$$\phi_3 = \frac{100-100}{160-120} = 0 \qquad \psi_3 = 100 - 0 = 100$$

Time i	Last jump	stock prices	option value V_i	Stock Holding ϕ_i	Bond Holding ψ_i
0	—	100	50	—	—
1	d	80	25	1.25	-75
2	u	100	50	1.25	-75
3	d	80	0	2.5	-200

$$\phi_1 = \frac{75-25}{120-80} = 1.25 \qquad \psi_1 = 50 - 1.25 \times 100 = -75$$

$$\phi_2 = \frac{50-0}{100-60} = 1.25 \qquad \psi_2 = 25 - 1.25 \times 80 = -75$$

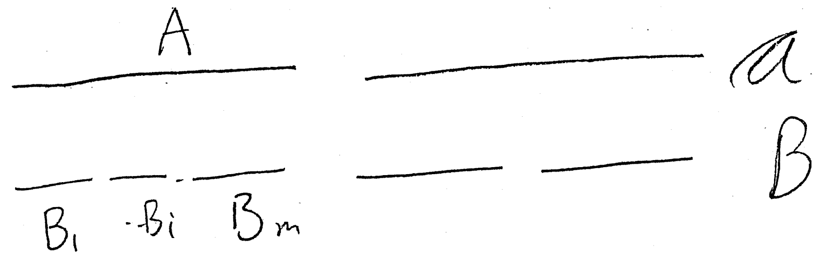
$$\phi_3 = \frac{100-0}{120-80} = 2.5 \qquad \psi_3 = 50 - 2.5 \times 100 = -200$$

Exercise 3.2.

$\mathcal{B}$ is refinement of $\mathcal{A}$

prove $E(E(X|\mathcal{A})|\mathcal{B}) \overset{?}{=} E(E(X|\mathcal{B})|\mathcal{A}) = E(X|\mathcal{A})$

proof



$$A \overset{\text{disjoint}}{=} \bigsqcup_1^m B_i$$

On A $\omega \in A$

def. $E(X|\mathcal{A})(\omega) = a$

$$E(E(X|\mathcal{A})|\mathcal{B})(\omega)$$

$$= E(a|\mathcal{B})(\omega) = \sum_{\eta \in B_i} a\, p(\eta) / P(B_i)$$

$$= a \frac{\sum_{\eta \in B_i} p(\eta)}{P(B_i)} = a \frac{P(B_i)}{P(B_i)} = a = E(X|\mathcal{A})$$

def r.v. $Y = E(X|\mathcal{B})(\omega) = b_i \quad (i = 1 \dots m)$

$$E(E(X|\mathcal{B})|\mathcal{A})(\omega) = E(Y|\mathcal{A})(\omega) = \frac{\sum_{\eta \in A} Y(\eta) p(\eta)}{P(A)}$$

$$\sum_{\eta \in A} Y(\eta) P(\eta) = \sum_{i=1}^m \sum_{\eta \in B_i} Y(\eta) P(\eta) = \sum_{i=1}^m b_i \sum_{\eta \in B_i} p(\eta)$$

$$= \sum_{i=1}^m b_i P(B_i)$$

But $b_i = \dfrac{\sum_{\eta \in B_i} X(\eta) P(\eta)}{P(B_i)}$ $\quad A = \bigcup_1^m B_i$

$$\Rightarrow b_i P(B_i) = \sum_1^m \sum_{\eta \in B_i} X(\eta) P(\eta) = \sum_{\eta \in A} X(\eta) p(\eta)$$

$$E(E(X|\mathcal{B})|\mathcal{A})(\omega)$$

$$= \frac{\sum_{\eta \in A} Y(\eta)\, p(\eta)}{p(A)}$$

$$= \frac{\sum_{\eta \in A} X(\eta)\, p(\eta)}{p(A)}$$

$$= E(X|\mathcal{A})(\omega)$$

3.3

Case 1.

There is an arbitrage but no strong arbitrage

Two securities A, B. (assume 1 periods)

At $t=0$ $S_0^A = S_0^B = 1$.

At $t=T=1$ $S_T^A = (1, 1)$ $S_T^B = (1, 2)$

If $\phi = (-1, +1)$.

$$V_0(\phi) = (-1, +1)\begin{pmatrix}1\\1\end{pmatrix} = 0$$

$$V_T(\phi) = (-1, +1)\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = \begin{pmatrix}0, & 1\end{pmatrix} \geq 0$$

However, we can not find a portfolio ϕ' that

$V_0(\phi') = 0$ $V_T(\phi') > 0$

Case 2

There is a strong arbitrage but no inconsistent pricing.

At $t=0$ $S_0^A = 1$ $S_0^B = 2$

At $t=T=1$ $S_T^A = (1, 1)$ $S_T^B = (\frac{3}{2}, 1)$

If $\phi = (1, -\frac{1}{2})$

$$V_0(\phi) = (1, -\tfrac{1}{2})\begin{pmatrix}1\\2\end{pmatrix} = 0$$

$$V_T(\phi) = (1, -\tfrac{1}{2})\begin{pmatrix}1 & 1\\\frac{3}{2} & 1\end{pmatrix} = \left(\tfrac{1}{4}, \tfrac{1}{2}\right) > 0$$

please note that here ϕ is also a portfolio that admits arbitrage.

However we can not find a portofolio ϕ' that make $V_0(\phi') \neq 0$ $V_T(\phi') \equiv 0$.

Case 3 inconsistent pricing.

At $t=0$ $S_0^A = 1$ $S_0^B = 2$

At $t=T=1$ $S_T^A = (1,1)$ $S_T^B = (0,0)$

If $\phi = (0,1)$

$$V_0(\phi) = (0, 1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2$$

$$V_T(\phi) = (0, 1)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = (0, 0) \equiv 0$$

For this case we also can find ϕ' that admit strong arbitrage.

E.g. $\phi' = (1, -\frac{1}{2})$

$$V_0(\phi') = (1, -\tfrac{1}{2})\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$$

$$V_T(\phi') = (1, -\tfrac{1}{2})\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = (1, 1) > 0$$

3.4

Prove If a market admits a arbitrage then there exists a local arbitrage.

Proof: suppose there exists an arbitrage strgy ϕ consider the set Λ of times $s \leq T$ such that $V_s(\phi)$ is positive for all states. $T \in \Lambda$, so $\Lambda \neq \phi$. Let t be the least element of Λ. Note $t > u$

We now argue that $t-1$ time is a strong local arbitrage. By choice of t, there are only three possibilities: $V_{t-1}(\phi)$ is negative in some state W_1, or positive in some state W_2, or else it is zero in all states

If first case occurs, then W_1 is the desired state. since one needs only to add a positive position in the numeraire to bring the value in that state up to zero and thereby also making the value at time t positive

If second case occurs, then state W_2 is the desired state since one needs only to add a negative position in the numeraire to bring the value in that state down to zero. and thereby also making the value at time t positive.

If third case occurs, then all the time $t-1$ states lead to positive values of $V_t(\phi)$

3.5. prove $E_\lambda(X|\mathcal{F}_t)=0 \iff E_\lambda[XY]=0$ all Y $\mathcal{F}_t$ measable $\iff \lambda[XY]=0$ all Y $\mathcal{F}_t$ meas,

$\iff \lambda(X1_A)=0$, all atom A of $\mathcal{F}_t$.

proof:

Given $E_\lambda(X|\mathcal{F}_t)=0$

$E_\lambda[XY] = E_\lambda[(X-0)Y] = E_\lambda[(X-E_\lambda(X|\mathcal{F}_t))Y]$

$= E_\lambda[XY] - E_\lambda[E_\lambda(X|\mathcal{F}_t)Y]$

$\xrightarrow{Y \text{ is } \mathcal{F}_t \text{ meas.}} = E_\lambda[XY] - E_\lambda[E_\lambda(XY|\mathcal{F}_t)]$

$\xrightarrow{\text{Tower property}} = E_\lambda[XY] - E_\lambda[XY] = 0$

Given $E_\lambda[XY]=0$

we know $\langle X-E_\lambda[X|\mathcal{F}_t], Y\rangle = 0$

$\Rightarrow E_\lambda[(X-E_\lambda[X|\mathcal{F}_t])Y]=0$

$E_\lambda[[X-E_\lambda(X|\mathcal{F}_t)]Y] = E_\lambda[(X-0)Y]$

for all Y $\mathcal{F}_t$ meas.

$\therefore X-E_\lambda(X|\mathcal{F}_t)=X$

$\Rightarrow E_\lambda(X|\mathcal{F}_t)=0$.

Therefore $E_\lambda(X|\mathcal{F}_t)=0 \iff E_\lambda[XY]=0$

From definition of $\lambda(f)$ we know
$$\lambda(f) = \sum_{w\in\Omega} \lambda(w) f(w) = E_\lambda[f]$$
$$\Rightarrow \lambda(XY) = \sum_{w\in\Omega} \lambda(w) X(w) Y(w) = E_\lambda[XY]$$
$$\Rightarrow E_\lambda[XY] = 0 \Leftrightarrow \lambda(XY) = 0$$
$$\lambda(X 1_A) = \sum_{w\in\Omega} \lambda(w) X \cdot 1_A = \sum_{w\in A} \lambda(w) X(w)$$
$$\frac{\lambda(X 1_A)}{P(A)} = \frac{\sum_{w\in A} \lambda(w) X(w)}{P(A)} = E(X|\mathcal{F}_t)$$
If $E(X|\mathcal{F}_t) = 0 \Leftrightarrow \dfrac{\lambda(X 1_A)}{P(A)} = 0 \Leftrightarrow$
$$\lambda(X 1_A) = 0$$
$$\therefore \quad E(X|\mathcal{F}_t) = 0 \Leftrightarrow \lambda(X 1_A) = 0.$$

Therefore:
$$E_\lambda(X|\mathcal{F}_t) = 0 \Leftrightarrow E_\lambda(XY) = 0 \Leftrightarrow \lambda(XY) = 0$$
$$\Leftrightarrow \lambda(X 1_A) = 0$$

Financial Engineering I

Dr. Kercheval

$\frac{4}{4}$

Jiangwen Dong

4.1 Prove $E_\nu[X|\mathcal{F}_t] = \dfrac{E_\lambda[X \frac{d\nu}{d\lambda}|\mathcal{F}_t]}{E_\lambda[\frac{d\nu}{d\lambda}|\mathcal{F}_t]}$

proof:

$A \in \mathcal{F}_t$ (Def. of conditional Expectation)

$$E_\nu[X|\mathcal{F}_t] = \frac{\sum_{\omega\in A} \nu(\omega) X(\omega)}{\sum_{\omega \in A} \mathbb{1}\nu(\omega)}$$

p is kind mea

$0 \le p \le 1.$ $\sum p(\omega) = 1$

$$= \frac{\sum_{\omega\in A} \lambda(\omega) X(\omega) \frac{S_T^0}{\lambda(S_T^0)}(\omega)}{\sum_{\omega\in A} \lambda(\omega) \mathbb{1} \frac{S_T^0}{\lambda(S_T^0)}(\omega)}$$

def of "ν" meas.

$$= \frac{\sum_{\omega\in A} \lambda(\omega) X(\omega) \frac{d\nu}{d\lambda}(\omega)}{\sum_{\omega\in A} \lambda(\omega) \mathbb{1} \frac{d\nu}{d\lambda}(\omega)} \quad \left(\frac{d\nu}{d\lambda} = \frac{S_T^0}{\lambda(S_T^0)}\right)$$

ν

$$= \frac{\dfrac{\sum_{\omega\in A} \lambda(\omega) X(\omega) \frac{d\nu}{d\lambda}(\omega)}{\sum_{\omega\in A}\lambda(\omega)}}{\dfrac{\sum_{\omega\in A} \lambda(\omega) \mathbb{1} \frac{d\nu}{d\lambda}(\omega)}{\sum_{\omega\in A}\lambda(\omega)}}$$

$$= \frac{E_\lambda[X \frac{d\nu}{d\lambda} | \mathcal{F}_t]}{E_\lambda[\mathbb{1} \frac{d\nu}{d\lambda} | \mathcal{F}_t]}$$

$$= \frac{E_\lambda[X \frac{d\nu}{d\lambda} | \mathcal{F}_t]}{E_\lambda[\frac{d\nu}{d\lambda} | \mathcal{F}_t]}$$

4.2 Prove $\dfrac{d\lambda}{d\nu} = \dfrac{1/S_T^\circ}{\nu(1/S_T^\circ)}$.

Given $\dfrac{d\nu}{d\lambda} = \dfrac{S_T^\circ}{\lambda(S_T^\circ)} \Rightarrow \nu(x) = \lambda\left(\dfrac{S_T^\circ}{\lambda(S_T^\circ)}x\right)$

but $\lambda\left(\dfrac{S_T^\circ}{\lambda(S_T^\circ)}x\right) = \nu\left(\underbrace{\dfrac{S_T^\circ}{\lambda(S_T^\circ)}\dfrac{d\lambda}{d\nu}}_{1}x\right)$

$$\frac{S_T^\circ}{\lambda(S_T^\circ)}\cdot\frac{d\lambda}{d\nu} = 1 \Rightarrow \frac{d\lambda}{d\nu} = \frac{\lambda(S_T^\circ)}{S_T^\circ}.$$

$$\nu(1/S_T^\circ)\,\lambda(S_T^\circ) = \lambda\left(1/S_T^\circ\cdot\frac{S_T^\circ}{\lambda(S_T^\circ)}\right)\cdot\lambda(S_T^\circ)$$
$$= \lambda\left(\frac{1}{\lambda(S_T^\circ)}\right)\cdot\lambda(S_T^\circ)$$

since $\lambda(S_T^\circ)$ is a const

$$\Rightarrow \nu(1/S_T^\circ)\,\lambda(S_T^\circ) = \frac{1}{\cancel{\lambda(S_T^\circ)}}\cdot\lambda(1)\,\cancel{\lambda(S_T^\circ)}$$
$$= \lambda(1) = 1$$

$$\nu(1/S_T^\circ)\,\lambda(S_T^\circ) = 1/S_T^\circ\cdot S_T^\circ$$

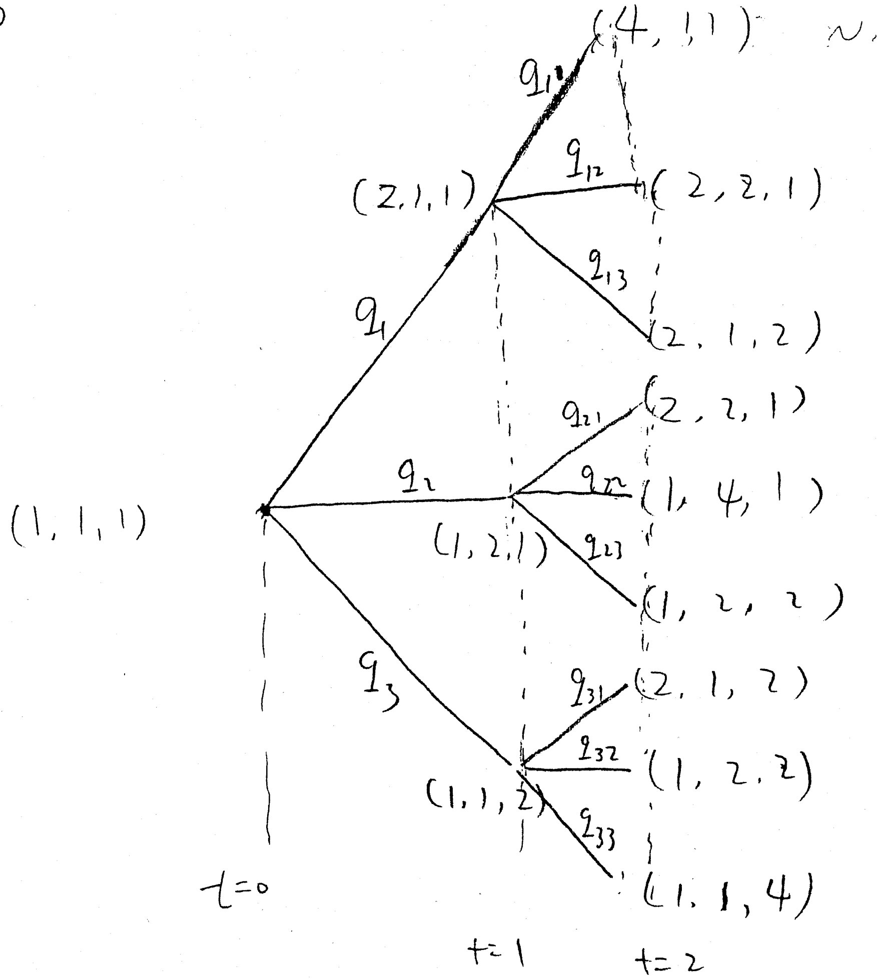
$$\Rightarrow \frac{\lambda(S_T^\circ)}{S_T^\circ} = \frac{1/S_T^\circ}{\nu(1/S_T^\circ)}$$

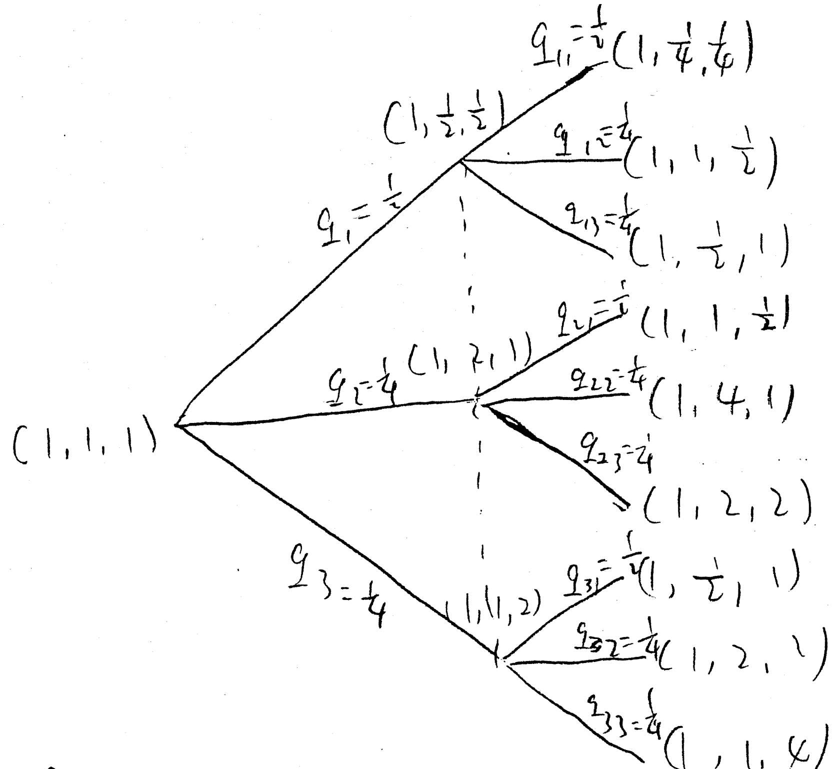
$$\Rightarrow \frac{d\lambda}{d\nu} = \frac{1/S_T^\circ}{\nu(1/S_T^\circ)}$$

$$\lambda(w) = \frac{1/S_T^\circ}{\nu(1/S_T^\circ)}\cdot\nu(w)$$

$\frac{1}{4}$

4.3





$$q_1+q_2+q_3=1$$

$$\begin{cases}\frac{1}{2}q_1+2q_2+q_3=1 \\ \frac{1}{2}q_1+q_2+2q_3=1\end{cases} \Rightarrow \begin{cases} q_1=\frac{1}{2} \\ q_2=\frac{1}{4}\end{cases}$$

$t=0 \longrightarrow 1$

$$\begin{cases} q_{11} + q_{12} + q_{13} = 1 \\ \frac{1}{4} q_{11} + q_{12} + \frac{1}{2} q_{13} = \frac{1}{2} \\ \frac{1}{4} q_{11} + \frac{1}{2} q_{12} + q_{13} = \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} q_{11} = \frac{1}{2} \\ q_{12} = \frac{1}{4} \\ q_{13} = \frac{1}{4} \end{cases}$$

$$\begin{cases} q_{21} + q_{22} + q_{23} = 1 \\ q_{21} + 4 q_{22} + 2 q_{23} = 2 \\ \frac{1}{2} q_{21} + q_{22} + 2 q_{23} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} q_{21} = \frac{1}{2} \\ q_{22} = \frac{1}{4} \\ q_{23} = \frac{1}{4} \end{cases}$$

$$\begin{cases} q_{31} + q_{32} + q_{33} = 1 \\ \frac{1}{2} q_{31} + 2 q_{32} + q_{33} = 1 \\ q_{31} + 2 q_{32} + 4 q_{33} = 2 \end{cases}$$

$$\Rightarrow \begin{cases} q_{31} = \frac{1}{2} \\ q_{32} = \frac{1}{4} \\ q_{33} = \frac{1}{4} \end{cases}$$

$t = 1 \rightarrow 2$

We have found a martingale meas.

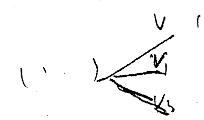
for the process $\dfrac{S_t^i}{S_t^0}$ $\left(\begin{array}{l} i = 0, 1, \dots, K. \\ t = 0, 1, 2. \end{array}\right)$.

Therefore, there is no arbitrage.

b) $v_1 = q_1 \cdot q_{11} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$v_2 = q_1 \cdot q_{12} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$v_3 = q_1 \cdot q_{13} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$



$x = \frac{1}{3}$

$v_4 = q_2 \cdot q_{12} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

$v_5 = q_2 \cdot q_{22} = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$v_6 = q_2 \cdot q_{23} = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$v_7 = q_3 \cdot q_{31} = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

$v_8 = q_3 \cdot q_{32} = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$v_9 = q_3 \cdot q_{33} = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

Since there is only a unique solution

$\therefore$ there is only a unique meas. which

makes $\dfrac{S_t^1}{S_t^0}$ a martingale

$\therefore$ The market is complete.

$\Rightarrow$ every claim is attainable

Choose the following claims to find λ.

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
1	0	0	0	0	0	0		0
0	1	0	0					
0		1	0					
0			1					
0			0	1				
0					1			
0						1		
0							1	
0	0	0	0	0	0	0	0	1

we also know

$$\nu(x) = \lambda\left(\frac{S_T^0}{\lambda(S_T^0)} x\right)$$

$$\sum_w \nu(w) x(w) = \sum \lambda(w) \frac{S_T^0(w)}{\lambda(S_T^0)} x(w)$$

For X_1

$$\frac{1}{4}\cdot 1 = \lambda_1 \cdot \frac{4\cdot 1}{\underbrace{4\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9}_{Y}}$$

$$\lambda_1 = \frac{Y}{16}$$

Similarly $\lambda_1 = \lambda_2 = \lambda_3 \cdots = \lambda_9 = \frac{Y}{16}$

but $\lambda_1 + \cdots \lambda_9 = 1$

$\Rightarrow \lambda_1 = \lambda_2 \cdots = \lambda_9 = \frac{1}{9}$

c) $K'_{(0,2)} = E_\lambda[X \mid \mathcal{F}_0]$

$= E_\lambda[X]$

$= \sum_{1}^{9} \lambda_i s_i'$

$= \frac{1}{9} \times (1+2+1+2+4+2+1+2+1)$

$= \frac{16}{9}$

d) $E_\nu[S_T^1 \mid \mathcal{F}_0]$

$= E_\nu[S_T^1]$

$= \sum_{1}^{9} \nu_i s_i'$

$= \frac{1}{4}\times 1 + \frac{1}{8}\times 2 + \frac{1}{8}\times 1 + \frac{1}{8}\times 2 + \frac{1}{16}\times 4 + \frac{1}{16}\times 2$
$+ \frac{1}{8}\times 1 + \frac{1}{16}\times 2 + \frac{1}{16}\times 1)$

$= \frac{25}{16}$

4.4. P49 3.1 (Financial Calculus B. & R)

① $X_0 = \sqrt{0}\, Z = 0$

② $X_{t+s} - X_t = \sqrt{t+s}\, Z - \sqrt{t}\, Z = (\sqrt{t+s} - \sqrt{t})\, Z$

is not independ of $\mathcal{F}_t$

$\therefore$ X is NOT a Brownian motion

4.5. P49 3.2 B & R

① $X_0 = \rho W_0 + \sqrt{1-\rho^2}\, \widetilde{W}_0 = 0$

$X_{t+s} - X_t = (\rho W_{t+s} + \sqrt{1-\rho^2}\, \widetilde{W}_{t+s}) -$

$(\rho W_t + \sqrt{1-\rho^2}\, \widetilde{W}_t)$

$= \rho (W_{t+s} - W_t) + \sqrt{1-\rho^2}\, (\widetilde{W}_{t+s} - \widetilde{W}_t$

Since W_t & $\widetilde{W}_t$ are two independent Brownian motions. $(W_{t+s} - W_t)$ and $\widetilde{W}_{t+s} - \widetilde{W}_t$ are both independ of $\mathcal{F}_t$.

we also know X_t is continuous and has marginal distribution $s\, N(0, t)$

$\therefore$ X_t is a Brownian motion

4.6. Solve $dX_t = X_t(\sigma_t dW_t + \mu_t dt)$

Guess $X_t = X_0 \exp\left(\int_0^t \sigma_s dW_s + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) ds\right)$

Check $dX_t = X_t \frac{\partial \int_0^t \sigma_s dW_s}{\partial W_t} dW_t + X_t \frac{\partial \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) ds}{\partial t} dt$

$+ \frac{1}{2}\left[X_t \left(\frac{\partial \int_0^t \sigma_s dW_s}{\partial W_t}\right)^2 (dW_t)^2\right]$

Since σ_t, μ_t are both intergatable deterministic

$\therefore dX_t = X_t \sigma_t dW_t + X_t(\mu_t - \frac{1}{2}\sigma_t^2) dt$

$+ \frac{1}{2}(X_t \sigma_t^2 dt)$

$dX_t = X_t \sigma_t dW_t + X_t \mu_t dt$ (right)

4.7 Prove $d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + \sigma_t \rho_t\, dt$

where $\underline{dX_t} = \sigma_t\, dW_t + \mu_t\, dt$

$\underline{dY_t} = \rho_t\, dW_t + \nu_t\, dt$

$$d(X_t Y_t) = \frac{1}{2} d\left((X_t + Y_t)^2 - X_t^2 - Y_t^2 \right)$$

$$d(X_t + Y_t) = (\sigma_t + \rho_t)\, dW_t + (\mu_t + \nu_t)\, dt$$

$$d[(X_t + Y_t)^2] = 2(X_t + Y_t)\, d(X_t + Y_t) + \frac{1}{2} 2 [d(X_t + Y_t)]^2$$

$$dX_t^2 = 2 X_t\, dX_t + \frac{1}{2} \cdot 2 (dX_t)^2$$

$$dY_t^2 = 2 Y_t\, dY_t + \frac{1}{2} \cdot 2 (dY_t)^2$$

Ito's formula

$$\Rightarrow d(X_t Y_t) = \frac{1}{2} \Big[2(\cancel{X_t} + Y_t)\, dX_t + 2(X_t + \cancel{Y_t})\, dY_t$$

$$+ \cancel{(dX_t)^2} + \cancel{(dY_t)^2} + 2(dX_t\, dY_t) - 2\cancel{X_t\, dX_t}$$

$$- \cancel{(dX_t)^2} - \cancel{2Y_t\, dY_t} - \cancel{(dY_t)^2} \Big]$$

$$= \frac{1}{2} \{ 2 Y_t\, dX_t + 2 X_t\, dY_t + 2(dX_t\, dY_t)$$

$$= Y_t\, dX_t + X_t\, dY_t + dX_t\, dY_t.$$

$$= Y_t\, dX_t + X_t\, dY_t + (\sigma_t\, dW_t + \mu_t\, dt) \cdot (\rho_t\, dW_t + \nu_t\, dt)$$

$$= Y_t\, dX_t + X_t\, dY_t + (\sigma_t \rho_t (dW_t)^2 + HOT)$$

$$= Y_t\, dX_t + X_t\, dY_t + \sigma_t \rho_t\, dt$$

$\frac{2}{2} + \frac{1}{2}$

Jiang wen Dong

Ex 5.1

Prove $dS_t = S_t(\sigma dW_t + (\mu + \frac{1}{2}\sigma^2) dt)$

$dZ_t = Z_t(\sigma dW_t + (\mu - r + \frac{1}{2}\sigma^2) dt)$

for Black-Scholes model.

$S_t = S_0 \exp(\sigma W_t + \mu t)$

$dS_t = \frac{\partial f}{\partial x} dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dW_t)^2 + \frac{\partial f}{\partial t} dt$

$= S_t(\sigma dW_t + \frac{1}{2}\sigma^2 dt + \mu dt)$

$= S_t(\sigma dW_t + (\mu + \frac{1}{2}\sigma^2) dt)$

✓

$Z_t = B_t^{-1} S_t$

$= S_0 \exp(-rt) \exp(\sigma W_t + \mu t)$

$= S_0 \exp(\sigma W_t + (\mu - r) t)$

$dZ_t = \frac{\partial f}{\partial x} dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dW_t)^2 + \frac{\partial f}{\partial t} dt$

$= Z_t(\sigma dW_t + \frac{1}{2}\sigma^2 dt + (\mu - r) dt)$

$= Z_t(\sigma + (\mu - r + \frac{1}{2}\sigma^2) dt)$

Ex 5.2

Bonds and rates in terms of the Q-Brownian motion $\widehat{W_t}$. The bond prices, forward and short rates are given by

$$P(t,T) = \exp - \left(\sigma(T-t)\widehat{W_t} + \int_t^T f(0,u)du + \tfrac{1}{2}\sigma^2 T(T-t)t\right)$$

$$B_t = \exp\left(\sigma\int_0^t \widehat{W_s}\,ds + \int_0^t f(0,u)\,du + \tfrac{1}{6}\sigma^2 t^3\right)$$

$$f(t,T) = \sigma\widehat{W_t} + f(0,T) + \sigma^2\left(T-\tfrac{1}{2}t\right)t$$

$$r_t = \sigma\widehat{W_t} + f(0,t) + \tfrac{1}{2}\sigma^2 t^2$$

(Assume $\alpha(t,T) = \sigma^2(T-t) + \sigma\gamma_t$)

Proof: $d_t f(t,T) = \sigma\, d\widehat{W_t} + \sigma^2(T-t)\,dt$

$$f(t,T) = f(0,T) + \sigma\widehat{W_t} + \int_0^t \sigma^2(T-t)\,dt$$

$$= f(0,T) + \sigma\widehat{W_t} + \sigma^2 Tt - \tfrac{1}{2}\sigma^2 t^2$$

$$= f(0,T) + \sigma\widehat{W_t} + \sigma^2 t\left(T-\tfrac{1}{2}t\right)$$

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T)$$

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right)$$

$$= \exp - \int_t^T \left(f(0,u) + \sigma\widehat{W_t} + \sigma^2 t\left(u-\tfrac{1}{2}t\right)\right)du$$

$$= \exp - \Big[\int_t^T f(0,u)\,du + \sigma\widehat{W_t}(T-t) + \tfrac{1}{2}\sigma^2 t(T^2-t^2)$$
$$- \tfrac{1}{2}\sigma^2 t^2(T-t)\Big]$$

$$= \exp - \left(\int_t^T f(0,u)\,du + \sigma\widehat{W_t}(T-t) + \tfrac{1}{2}\sigma^2(T-t)tT\right)$$

$$r_t = f(t,t) = f(0,t) + \sigma\widehat{W_t} + \sigma^2 t\left(t-\tfrac{1}{2}t\right)$$

$$B_t = \exp\left(\int_0^t r_s\,ds\right) = \exp\left(\int_0^t \left(f(0,s) + \sigma\widehat{W_s} + \tfrac{1}{2}\sigma^2 s^2\right)ds\right)$$

$$= \exp\left(\int_0^t f(0,u)\,du + \sigma\int_0^t \widehat{W_s}\,ds + \tfrac{1}{6}\sigma^2 t^3\right)$$

Ex 5.3

Prove $d_t P(t,T) = P(t,T)\,(\Sigma(t,T)\,d\hat{W}_t + r_t\,dt)$

where $\Sigma(t,T) = -\int_t^T \sigma(t,u)\,du$

$r_t = f(t,t) = f(0,t) + \int_0^t \sigma(s,t)\,d\hat{W}_s - \int_0^t \sigma(s,t)\Sigma(s,t)\,ds$

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right)$$

$$f(t,u) = f(0,u) + \int_0^t \sigma(s,u)\,d\hat{W}_s - \int_0^t \sigma(s,u)\Sigma(s,u)\,ds$$

$$P(t,T) = \exp -\left(\int_t^T f(0,u)\,du + \int_t^T \int_0^t \sigma(s,u)\,d\hat{W}_s\,du - \int_t^T\int_0^t \sigma(s,u)\Sigma(s,u)\,ds\,du\right.$$

$$d_t P(t,T) = P(t,T)\left(\left(-\int_t^T \sigma(t,u)\,du\right)d\hat{W}_t + \left(f(0,t) + \int_0^t \sigma(s,t)\,d\hat{W}_s - \int_0^t \sigma(s,t)\Sigma(s,t)\,ds\right)dt\right)$$

explain

$$d_t P(t,T) = P(t,T)\,(\Sigma(t,T)\,d\hat{W}_t + r_t\,dt)$$

$\frac{1}{2}$

- futures contract always has value zero (b/c marked to mkt daily)
- $F_T(X) = X$
- need S^0 = money mkt account
  predictable (previsible) and positive (always has to be positive if the numeraire)

Thm if S^0 is a money mkt account and $(S/S^0, \nu, \mathcal{F})$ is a m'gale then the futures price of X is $F_t(X) = E_\nu[X \mid \mathcal{F}_t]$

ν instead of λ in forward price

Proof: Strategy - wait until time t, go long ϕ_t in futures contracts

- at each $\tau > t$ receive margin payment (pos. or neg.) → invest in S^0, close account, then go long ϕ_τ contracts
- this strategy is s.f. b/c it costs nothing enter/leave these contracts
- at an arbitrary time τ receive margin

$\phi_{\tau-1}(F_\tau - F_{\tau-1})$ dollars $\longrightarrow$ profit over one-day period

or $\dfrac{\phi_{\tau-1}}{S^0_\tau}(F_\tau - F_{\tau-1})$ shares of S^0

true for $\tau = t+1, \dots, T$

total # shares S^0 $= \displaystyle\sum_{\tau=t+1}^{T} \frac{\phi_{\tau-1}}{S^0_\tau}(F_\tau - F_{\tau-1})$

choose $\phi_{\tau-1} = S^0_\tau$ ← this is ok b/c S^0_τ is predictable (known at $\tau-$

get $\underbrace{\displaystyle\sum_{\tau=t+1}^{T}(F_\tau - F_{\tau-1})}_{\text{(telescoping sum)}} = F_T - F_t$ # of shares of S^0

$(F_T(X) - F_t(X))$

$\text{value}_T = \underbrace{(F_T - F_t)\, S^0_T}_{\text{claim}}$ → has to have fair price of zero (at time t) (b/c no buy-in pr

$F_T(X) = X$

$$0 = E_\nu\left[(X - F_t(X))\, S^0_T \underbrace{\frac{S^0_t}{S^0_T}}_{\text{discount factor}} \,\Big|\, \mathcal{F}_t\right]$$

S^0_t: $\mathcal{F}_t$-meas.

$$0 = E_\nu[X - F_t(X) \mid \mathcal{F}_t]$$

$F_t(X)$: $\mathcal{F}_t$-meas (can look it up in the paper)

$$F_t(X) = E_\nu[X \mid \mathcal{F}_t]$$

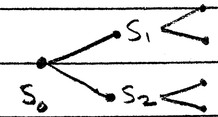
λ = forward measure b/c it values forward contracts

ν = "futures measure" " " futures "

(m'gale measure)

Corollary: if S^0 is deterministic then $S^0_T = \lambda(S^0_T)$ then $\nu = \lambda$

forward ~~agree~~ up front to pay a given price
futures - uncertainty built into the futures contract



if predictable then at time 0 we know what $S_1 + S_2$ are
if deterministic then $S_1 = S_2$

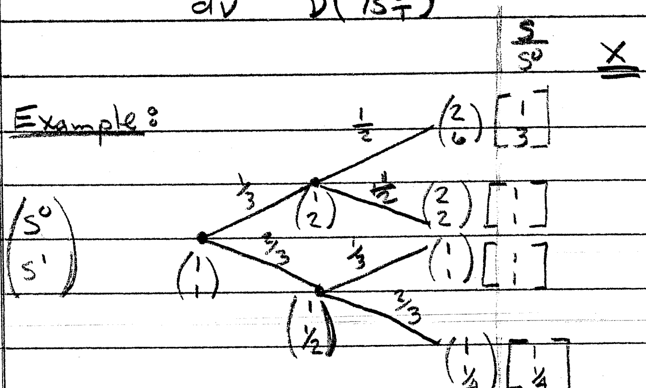
H.W. 6.3 If you want to use this stuff
① need a model / → full price process w/ filtration : $S, \mathcal{F}$
② compute the measures ν, λ

$$\text{R.N derivative } \frac{d\nu}{d\lambda} = \frac{S^0_T}{\lambda(S^0_T)} \longleftrightarrow \nu(X) = \lambda\left(X \frac{S^0_T}{\lambda(S^0_T)}\right)$$

↘ assume this then invert

$$* \ \underline{\text{Check}}: \frac{d\lambda}{d\nu} = \frac{1/S^0_T}{\nu(1/S^0_T)}$$

10/29 Example:



in order to calculate prices, we need a m'gale measure ν ↗ risk neutral meas.

- for ν we need a numeraire → S^0 $\qquad \frac{S}{S^0} = \left(\frac{S^0}{S^0}, \frac{S^1}{S^0}\right) = \left(1, \frac{S^1}{S^0}\right)$

- in this case S^0 is a money mkt → predictable

- $q3 + (1-q)1 = 2$

$q = \frac{1}{2}$

- $q1 + (1-q)\frac{1}{4} = \frac{1}{2}$ $\quad \to \frac{3}{4}q = \frac{1}{4}$

$q + \frac{1}{4} - \frac{1}{4}q = \frac{1}{2}$ $\qquad q = \frac{1}{3}$

$$\nu = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{9}, \frac{4}{9}\right)$$

given a claim $X: V_0(\phi) = E_\nu\left[\frac{X}{S^0_T}\right] = \nu\left(\frac{X}{S^0_T}\right) = \sum_{\omega \in \Omega} \nu(\omega) \frac{X(\omega)}{S^0_T(\omega)}$

Futures price of S^1 at $t=0$ maturity $T=2$, $E_\nu\left[S^1_2\right] = \frac{1}{6}\cdot 6 + \frac{1}{6}\cdot 2 + \frac{2}{9}\cdot 1 + \frac{4}{9}\cdot\frac{1}{4}$

$= 1 + \frac{1}{3} + \frac{2}{9} + \frac{1}{9}$

$= \frac{5}{3}$

Forward price - need 2-forward meas. λ

we know ν and R-N derivative $\frac{d\lambda}{d\nu} = \frac{\frac{1}{S^0_2}}{\nu\left(\frac{1}{S^0_2}\right)}$

$\nu\left(\frac{1}{S^0_2}\right) = \frac{1}{6}\cdot\frac{1}{2} + \frac{1}{6}\cdot\frac{1}{2} + \frac{2}{9}\cdot 1 + \frac{4}{9}\cdot 1$

$= \frac{1}{6} + \frac{6}{9}$

$= \frac{5}{6}$

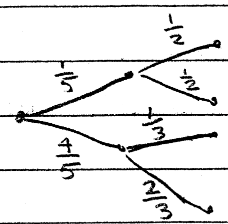
$\lambda(\omega) = \left(\frac{6}{5}\right)\nu\left(\omega\,\frac{1}{S^0_2}\right) = \left(\frac{6}{5}\right)\frac{1}{S^0_2(\omega)}\,\nu(\omega)$

$\lambda(\text{top}) = \frac{6}{5}\cdot\frac{1}{2}\cdot\frac{1}{6} = \frac{1}{10}$

$\lambda(\text{2nd}) = \frac{6}{5}\cdot\frac{1}{2}\cdot\frac{1}{6} = \frac{1}{10}$

$\lambda(\text{3rd}) = \frac{6}{5}\cdot 1\cdot\frac{2}{9} = \frac{4}{15}$

$\lambda(\text{bottom}) = \frac{6}{5}\cdot 1\cdot\frac{4}{9} = \frac{8}{15}$

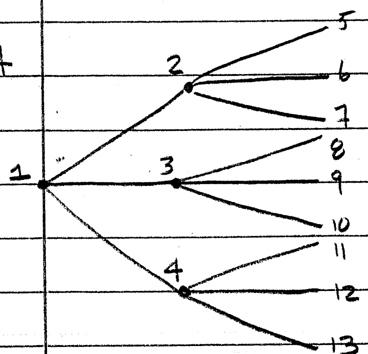


$E_\lambda\left[S^1_2\right] = \frac{1}{10}\cdot 6 + \frac{1}{10}\cdot 2 + \frac{4}{15}\cdot 1 + \frac{8}{15}\cdot\frac{1}{4}$

$= \frac{6}{5}$

W. 6.4

$\begin{pmatrix} S^0 \\ S^1 \\ S^2 \end{pmatrix}$



$1 = (1,1,1)$	$7 = (4,2,2)$
$2 = (1,2,1)$	$8 = (2,2,6)$
$3 = (1,1,2)$	$9 = (2,4,2)$
$4 = (1,.5,.5)$	$10 = (2,1,1)$
$5 = (4,4,16)$	$11 = (1,.5,1)$
$6 = (4,12,4)$	$12 = (1,1,.5)$
	$13 = (1,.25,.25)$

a) Show this mkt admits no arbitrage

→ find a m'gale meas.

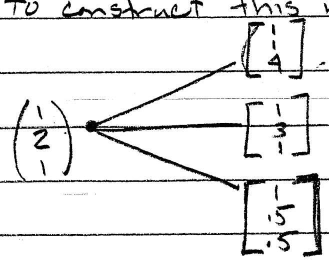
→ numeraire is previsible so we can find futures + forwards

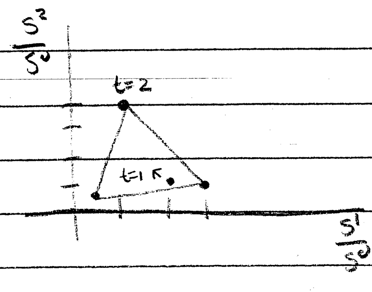
b) find ν, λ m'gale and 2-fwd measures

c) find the 2-fwd price of S^1 at $t=0$

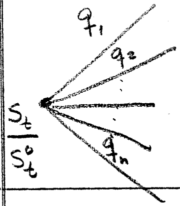
d) find the futures prices of S^1 at $t=0$

to construct this model





any point w/in triangle will be attainable



$$\frac{S_{t+1}}{S_{t+1}^0} = \left(1, \frac{S_{t+1}^1}{S_{t+1}^0}, \dots, \frac{S_{t+1}^k}{S_{t+1}^0}\right)$$

no arb. means there is a meas which makes this a m'gale

$$\sum_{j=1}^{n} q_j \left(1, \frac{S_{t+1}^1(j)}{S_{t+1}^0(j)}, \dots, \frac{S_{t+1}^k(j)}{S_{t+1}^0(j)}\right) = \left(1, \frac{S_t^1}{S_t^0}, \dots, \frac{S_t^k}{S_t^0}\right)$$

$k+1$ equations in n unknowns $(q_1, \dots, q_n)$

if $k+1 < n$ - expect ∞-many solutions, incomplete mkt (by thm 3)

$k+1 = n$ - expect complete mkt if no arb.

$k+1 > n$ - expect arbitrage or redundancy

(assets) (branches)

B+R Ch 3 current hw due Wed 11/3

p. 49 #1, 2 exam 2 11/8

Computer:

>fin-matlab

Next: Asset Pricing Theory.
- Review Binomial - Baxter & Rennie
- General Discrete Models - notes
- B & R - continuous case
  Ito's formula
- BS
- interest rates

Sept. 20
MON

w^3. commonwealthclub.org/archive/index.html
B. Mandelbrot
"Fractals of Financial Markets"
Aug. 19, 2004

WED HW!!!

Baxter and Rennie

Asset Pricing:

Forward Contract on stock S_t

Q: What strike K gives contract (maturity T) value of zero.

Long contract: pays K receives stock at maturity.

K = forward price

Expectation Argument of pricing Forward Contract

Reasonable: $E_p[e^{-rT}(S_T - K)] = 0$.

$\Leftrightarrow E[S_T] = K$.

depends on Probability Measure.

Arbitrage Argument of pricing Forward contract

$t=0$

- Short 1 forward contract at K — costs nothing to short a forward contract.
- Borrow S_0 dollars.
- Buy 1 share

$t=T$

- Receive K dollars, handover share worth S_T
- Repay loan $S_0 e^{rT}$
- ∴ net profit: $\underbrace{K - S_0 e^{rT}}_{\text{CASH}}$ riskless

if $K > S_0 e^{rT}$ then ∃ "arbitrage"

$K < S_0 e^{rT}$, same argument if stock can be shorted.

no arbitrage condition $\Rightarrow$ $\boxed{K = S_0 e^{rT}}$

no arbitrage forward price

→ 1973 all derivs were priced via $\mathbb{E}_P$ except fwd price

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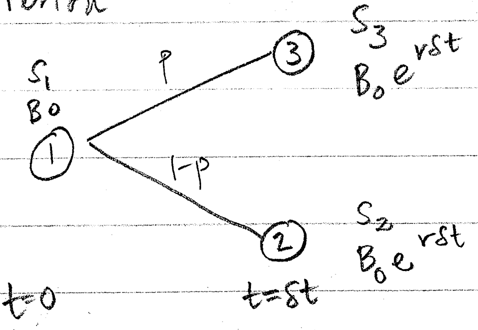
No Arbitrage Pricing

Market : 2 instruments (securities)

stock S

bond B

1-time Period

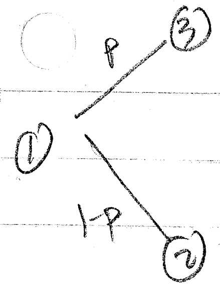


subscripts

S_1 ← node

B_0 ← time

claim X (some "value") — uncertain payoffs



$X(3)$

$X(2)$

the arbitrage here is:

If we can form a portfolio of S & B that perfectly replicates (hedges) X, then $t=0$ price of X must be the $t=0$ " " portfolio

Holdings : (ϕ, ψ)

ϕ: # shares S ; ψ: # shares B

$$V_0 = \phi S_1 + \psi B_0$$

$$V_{\delta t} = \begin{cases} \phi S_2 + \psi B_0 e^{r\delta t} & (\text{node } 2) \\ \phi S_3 + \psi B_0 e^{r\delta t} & (\text{node } 3) \end{cases} \quad \text{set} = \begin{cases} X(2) \\ X(3) \end{cases}$$

Solve for ϕ, ψ:

$$\phi = \frac{X(3) - X(2)}{S_3 - S_2}, \qquad \psi = B_0^{-1} e^{-r\delta t}\left(X(3) - \left(\frac{X(3) - X(2)}{S_3 - S_2}\right) S_3 \right)$$

$$\Rightarrow V_0 = \left(\frac{X(3) - X(2)}{S_3 - S_2}\right) S_1 + e^{-r\delta t}\left(X(3) - \left(\frac{X(3) - X(2)}{S_3 - S_2}\right) S_3\right)$$

no-arb. price of X.

no probability

Δ

rearrange above as:

$$V_0 = e^{-r\delta t}\left[\left(\frac{S_3 - S_1 e^{r\delta t}}{S_3 - S_2}\right) X(2) + \left(\frac{S_1 e^{r\delta t} - S_2}{S_3 - S_2}\right) X(3)\right]$$

$$= e^{-r\delta t}\left(q\, X(2) + (1-q)\, X(3)\right)$$

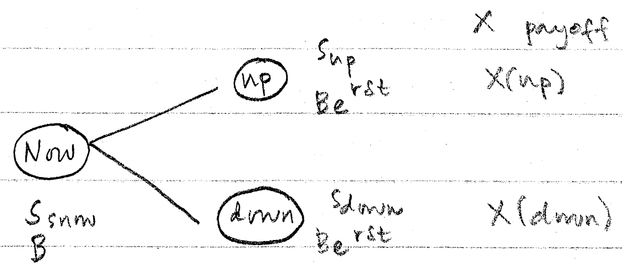
note: $0 \le q \le 1$

$$= E_q\left[e^{-r\delta t} X\right]$$ pricing formula

HW P133 #4,5 } Monday
145 #1

Exam Friday

Sept. 24
FRI



$$\phi S + \psi B = X$$

$$V(now) = e^{-r\delta t}\left(q\, X(up) + (1-q)\, X(down)\right)$$

$$q = \frac{e^{r\delta t} S_{now} - S_{down}}{S_{up} - S_{down}}$$

$$V(\text{now}) = \phi S_{\text{now}} + \psi B_{\text{now}}$$

$$\phi = \frac{X(\text{up}) - X(\text{down})}{S_{\text{up}} - S_{\text{down}}}$$

$$\psi = B_{\text{now}}^{-1}\left(X(\text{up}) - \frac{X(\text{up}) - X(\text{down})}{S_{\text{up}} - S_{\text{down}}} \right).$$

<u>Ex.</u> Forward contract strike K.

$$X(\underset{3}{2}) = \underset{3}{S_2} - K$$

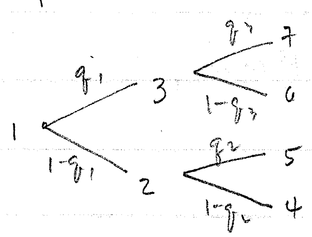
$$0 = V = e^{-r\delta t}\left[q(S_3 - K) + (1-q)(S_2 - K) \right]$$

$$qS_3 + (1-q)S_2 = K$$

$$K = \left(\frac{S_1 e^{r\delta t} - S_2}{S_3 - S_2} \right) S_3 + \left(\frac{S_3 - S_1 e^{r\delta t}}{S_3 - S_2} \right) S_2$$

$$= S_1 e^{r\delta t}$$

Two-period model (p. 20)

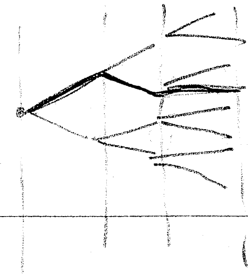


$$V(1) = e^{-r\delta t}\left[q_1 V(3) + (1-q_1) V(2) \right]$$

$$= e^{-2r\delta t}\left[q_1 \left(q_3 V(7) + (1-q_3) V(6) \right) + (1-q_1)\left(q_2 V(5) + (1-q_2) V(4) \right) \right].$$

$$= e^{-2r\delta t}\left[q_1 q_3 X(7) + q_1 (1-q_3) X(6) + (1-q_1) q_2 X(5) + (1-q_1)(1-q_2) X(4) \right].$$

Binary tree



Ω = set of paths on tree

Process $\rightarrow$ S_t = for each t, a random variable

Prob. measure on Ω.

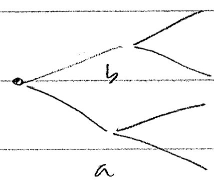
Filtration $\{\mathcal{F}_t\}$

$\equiv$ sequence of σ-algebras on Ω, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ $\forall t$.

$\mathcal{F}_T = 2^{\Omega}$

$\mathcal{F}_0 = \{\phi, \Omega\}$

$\mathcal{F}_t$ = σ-algebra generated by partition whose atoms are of the form {all paths through some t node}.



$\mathcal{F}_1 = \langle$ {paths through a}, {paths through b} $\rangle$

$(S_t)_t$ is "adapted to $\mathcal{F}_t$" if $\forall t$, "S_t is $\mathcal{F}_t$-(m)"

$\Leftrightarrow$ S_t is constant on $\mathcal{F}_t$-atoms.

$S_t : \Omega \rightarrow \mathbb{R}$ constant on $\mathcal{F}_t$-atoms.

Conditional Expectation:

$X : \Omega \longrightarrow \mathbb{R}$

claim: r.v.

$E_Q[X \mid \mathcal{F}_t]$ for each fixed t, a new r.v.

= Q avg value of X

over atoms of gen. partition for $\mathcal{F}_t$

constant on each atom of $\mathcal{F}_t$

$\therefore E[X \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-adapted.

Sept. 27

MON

- $\{E_Q[X \mid \mathcal{F}_t]\}$ = random process
- $E_Q[X \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-measurable

  constant on $\mathcal{F}_t$-atoms.

- Prop: If X is $\mathcal{F}_t$-ⓜ

  then $E_Q[XY \mid \mathcal{F}_t] = X E_Q[Y \mid \mathcal{F}_t]$.

- Prop:

  $E[X \mid \mathcal{F}_t] = X$, if $X \in \mathcal{F}_t$

- Notation:

  Set of all claims : $\mathbb{R}^{\Omega} = \{X : \Omega \longrightarrow \mathbb{R}\}$

  $|\Omega| < \infty \Rightarrow \mathbb{R}^{\Omega} \cong \mathbb{R}^n, \; n = |\Omega|$

- inner product

$$\langle f, g \rangle = \int_{\Omega} fg \, dQ$$

$$= \sum_{\omega \in \Omega} f(\omega) g(\omega) Q(\omega)$$ weighted dot product

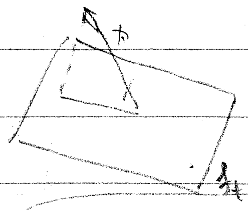
$$= E_Q[fg]$$

Hilbert space = complete inner product space.

Prop: $E[\cdot \mid \mathcal{F}_t] : \mathbb{R}^{\Omega} \to \mathbb{R}^{\Omega}$

is orthogonal projection onto $\mathcal{F}_t$

Pf: WTS: $\langle X - E_Q[X \mid \mathcal{F}_t], Y \rangle = 0$ all $Y \in \mathcal{F}_t$



$E(X \mid \mathcal{F}_t) = X$

WTS $\langle X - E_Q[X \mid \mathcal{F}_t], Y \rangle$

$$= E_Q[XY] - \underbrace{E_Q[Y E_Q[X \mid \mathcal{F}_t]]}_{\underbrace{E_Q[E_Q[XY \mid \mathcal{F}_t]]}_{\text{tower property}}}$$

$E_Q[X \mid \mathcal{F}_0] = E_Q[X]$. $(s = 0)$

$$= E_Q[X - Y]$$

$$= 0.$$

Tower Property.

$$E_Q[E_Q[X \mid \mathcal{F}_t] \mid \mathcal{F}_s] = E_Q[X \mid \mathcal{F}_s] \quad \forall s \le t.$$

Def Predictable Process: $\{\phi_t\}$ ϕ_t is $\mathcal{F}_{t-1}$-m ← Won't make sense in continuous sense.

Def: Q-martingale $\{S_t\}$ a process

$((S_t), \mathcal{F}_t, Q)$ is a m'gale

iff $E[S_t \mid \mathcal{F}_s] = S_s$ all $s \le t$.

Ex of m'gale.

$E_Q[X \mid \mathcal{F}_t]$ since $E[E_Q[X \mid \mathcal{F}_t] \mid \mathcal{F}_s] = E_Q(X \mid \mathcal{F}_s)$.

Binomial Representation theorem

Given (S_t) on a binomial tree

Given Q st S_t is a Q-martingale.

If N_t is any other Q-martingale,

then $\exists$ predictable ϕ_t s.t.

$$N_i = N_0 + \sum_{k=1}^{i} \phi_k \Delta S_k \quad \text{where } \Delta S_k = S_k - S_{k-1}$$

or $\Delta N_i = \phi_i \Delta S_i$

Also:

★ All Q-martingales are "generated" from a given martingale (S_i) via choice of predictable process ϕ_i.

Pf

$$E_Q[B_t^{-1} S_t \mid \mathcal{F}_s] = B_s^{-1} S_s \iff S_s = E_Q[B_s B_t^{-1} S_t \mid \mathcal{F}_s].$$

$$(\underbrace{\cancel{S_0/B_0}}_{B_0^{-1} S_0} = E_Q[B_t^{-1} S_t]).$$

Sept. 29 WED

Sample Test.

#3b $\min h^T V h$ efficient frontier portfolio

s.t. $h^T e = 1$ is a l.c. of a and c.

$h^T f = f_p$

Soln: Lagrange Multiplier

$$\nabla h^T V h = \mu_1 \nabla h^T e + \mu_2 \nabla h^T f$$

$$2Vh = \frac{\mu_1}{2} V^{-1} e + \frac{\mu_2}{2} V^{-1} f$$

mult. of h_c mult of h_a.

#4

$(V_R = V - \beta\beta^T\sigma_B^2)$

$h^T\alpha - \lambda h^T V_R h$

s.t. $h^T\beta = 0$.

FOC.

$\alpha - 2\lambda V_R h = \mu\beta$,,0..

$\Rightarrow \alpha - 2\lambda(V - \sigma_B^2\beta\beta^T)h = \mu\beta$

$\Rightarrow \alpha - 2\lambda V h = \mu\beta$

$h = \frac{1}{2\lambda}\left[V^{-1}\alpha - \mu V^{-1}\beta\right]$

$= \frac{1}{2\lambda}\left[\frac{h_A}{\sigma_A^2} - \mu\frac{h_B}{\sigma_B^2}\right]$

$h^T\beta = 0 \Rightarrow \mu = 0$.

since $h_A^T\beta = 0$, $h_B^T\beta = 1$

$h = \frac{h_A}{2\lambda\sigma_A^2}$

Proof of Binomial Representation Theorem. (p.35)

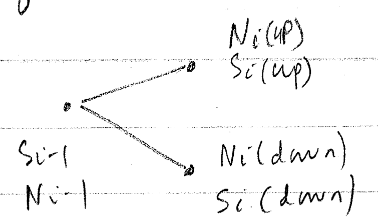
$N_i = N_0 + \sum_{k=1}^{i}\phi_k \Delta S_k$

(p.35)

given S, N are martingales

$\exists\ \phi$ predictable

equiv. $\Delta N_i = \phi_i \Delta S_i$



defin $\phi_i = \frac{Ni(up) - Ni(down)}{Si(p) - Si(down)}$ is predictable since known at time i-1.

clarify ε_i

$$\Delta N_i(\text{up}) = N_i(\text{up}) - N_{i-1}$$
$$= \phi_i(S_i(\text{up}) - S_{i-1}) + \underline{\varepsilon(\text{up})}$$
$$\Delta N_i(\text{down}) = N_i(\text{down}) - N_{i-1}$$
$$= \phi_i(S_i(\text{down}) - S_{i-1}) + \varepsilon(\text{down})$$

$$\Rightarrow N_i(\text{up}) - N_i(\text{down})$$
$$= \phi_i(S_i(\text{up}) - S_i(\text{down})) + \underbrace{(\varepsilon(\text{up}) - \varepsilon(\text{down}))}_{=0 \text{ by def of } \phi_i}$$

$\Rightarrow \varepsilon =$ predictable (same value up/down!)

$$\Delta N_i = \phi_i \Delta S_i + \varepsilon$$

$$E_Q[\Delta N_i \mid \mathcal{F}_{i-1}] = E_Q[\phi_i \Delta S_i \mid \mathcal{F}_{i-1}] + E[\varepsilon \mid \mathcal{F}_{i-1}]$$
$$= E[N_i - N_{i-1} \mid \mathcal{F}_{i-1}] = \phi_i \underbrace{E_Q[\Delta S_i \mid \mathcal{F}_{i-1}]}_{0 \text{ since } S_i \text{ is martingale}} + \varepsilon$$
$$= \underbrace{\underbrace{E[N_i \mid \mathcal{F}_{i-1}]}_{N_{i-1} \text{ since martingale}} - N_{i-1}}_{=0} =$$

$\therefore \varepsilon = 0$.

#.

Binomial Model For Evolution of Stock Prices.

- S_i, B_i
- let $Z_i = B_i^{-1} S_i$
- Sps have mgale measure Q s.t. Z is a mgale.
- X, let $E_i = E_Q[B_T^{-1} X \mid \mathcal{F}_i]$

$t=T$ 2^T nodes claim X

Bin. Rep thm $\Rightarrow \exists\, \phi$ predictable s.t. $\Delta E_i = \phi_i \Delta Z_i$

investing R_X: at time i, hold (ϕ_{i+1}, ψ_{i+1}) shares

ϕ_{i+1}: stock, ψ_{i+1}: bond.

predictable.

$$\psi_{i+1} = E_i - \phi_{i+1} B_i^{-1} S_i$$

<u>Facts</u>: (1) this strategy is "self-financing".

(2) at time T, the strategy is worth X.

no arbitrage $\Rightarrow$ time zero value of X must be equal to the time zero value of this strategy.

$$
\begin{aligned}
V_0 &= \phi_1 S_0 + \underbrace{\psi_1} B_0, \quad (B_0 = 1) \\
&= \phi_1 S_0 + E_0 - \phi_1 B_0^{-1} S_0 \qquad = \phi_1 S_0 + \left[E_0 - \phi_1 B_0^{-1} S_0 \right] B_0 \\
&= E_0 \\
&= E_Q [B_T^{-1} X].
\end{aligned}
$$

Oct 4, 2004

MON

★ Extra Credit (5 Points) ← Transaction Cost.

- Read Chap X =16 in Grinold - Kahn.
  - \+ one paper from the references (journal article).
- Write a report 4-5 pages
  - don't do technical appendix.

DUE: Before thanksgiving.

TEST Solutions:

1. a) CAPM $\Rightarrow \alpha = 0$

   b) α in terms of . . .

$$\alpha = f - \beta f_B = f - \frac{Vh_B}{h_B^T V h_B} h_B^T f$$

$$\beta = \frac{Vh_B}{h_B^T V h_B}, \quad f_B = h_B^T f$$

   c) portfolio C

   d) Fundamental Law of Active Mgt: $IR = IC\sqrt{BR}$

   e) Max V.A 60b to 20%

   10% to 45bp.

---

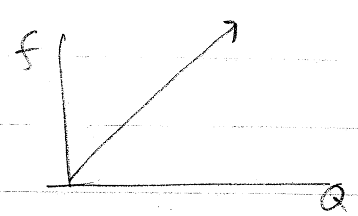
2. invest in arbitrarily large...

YES. can go short or r.f. portfolio F

longer in Q

$h_P = c h_Q, \quad c > 0$

$f_P - \lambda \sigma_P^2$.

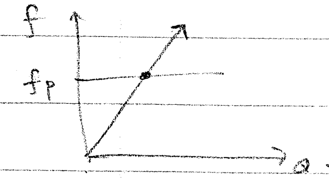


#3 a) $h^T e \neq 1$ means not fully invested

fully invested means no cash position.

b) max SR portfolio is a l.c. of h_F and h_Q.

max SR portfolio $= c_1 h_F^{"0"} + c_2 h_Q$



Pf: 1. every point on capital market line is ~~obtained~~ attained by ch_Q.

$$\frac{cf_Q}{c\sigma_Q} = \frac{f_Q}{\sigma_Q}$$

So max SR $\longleftrightarrow$ solve $\min h^T V h$

s.t. $h^T f = f_P$

claim: all portfolios h maximizing SR with $h^T f = f_P$ are solutions of this problem.

LM: $\begin{cases} 2Vh = \mu f \\ h^T f = f_P \end{cases}$

$\Rightarrow h = \frac{\mu}{2} V^{-1} f = ch_Q$ } unique solution

hence (b)

2(c)

Problem #4 . $\alpha_P - \lambda \omega_P^2$

(a) max $\underbrace{h^T \alpha}_{=1} - \lambda h^T V_R h$ s.t. $h^T \alpha = 1$, $\underbrace{h^T \beta = 0}$.

$\Rightarrow \omega_P = \sigma_P$

obj $1 - \lambda \omega_P^2$

$1 - \lambda \sigma_P^2$

$\Leftrightarrow \min \sigma_P^2$ sol to $\min \sigma_P^2$ s.t. $h^T \alpha = 1$.

$\boxed{h_A}$

LM:

$$\alpha - 2\lambda(V - \sigma_B^2 \beta\beta^T)h = \mu_1 \alpha + \mu_2 \beta$$

$$\alpha - 2\lambda V h = \mu_1 \alpha + \mu_2 \beta$$

$$h = c_1 V^{-1}\alpha + c_2 V^{-1}\beta$$

$$= c_1' h_A + c_2' h_B$$

$$= h_A$$

b) $\max\ h^T\alpha - \lambda h^T V_R h$

s.t. $h^T\beta = 1$

$$\alpha - 2\lambda(V - \sigma_B^2 \beta\beta^T)h = \mu\beta$$

$$\Rightarrow \alpha + 2\lambda\sigma_B^2\beta - 2\lambda V h = \mu\beta$$

$$h = \frac{1}{2\lambda}\Big(\underbrace{V^{-1}\alpha}_{\frac{h_A}{\sigma_A^2}} + (2\lambda\sigma_B - \mu)V^{-1}\beta\Big)$$

$$= \frac{h_A}{2\lambda\sigma_A^2} + c h_B \Rightarrow c = 1$$

to get $\beta = 1$.

$$h^T\beta = 1 \Rightarrow h = \frac{h_A}{2\lambda\sigma_A^2} + h_B.$$

HW

due Monday:

3.1 do exercise 2.2. of B&R after working through the example on pp. 23-27

3.2 Prove Tower property (Binomial Tree ok) both tree and time

Finite Binomial Tree $\quad t = 0, 1, \ldots, T$

claim X $\quad X : \{\text{paths}\} \longrightarrow \mathbb{R}$

Bond
Stock

— framework for pricing

Oct. 6
WED.

- $B_0 = 1$

$B_t = e^{rt}$

found $\mathbb{Q}$: $\quad B_t^{-1} S_t$ is $\mathbb{Q}$-martingale.

discounted stock price

$Z_t = B_t^{-1} S_t$ a $\mathbb{Q}$-mart.

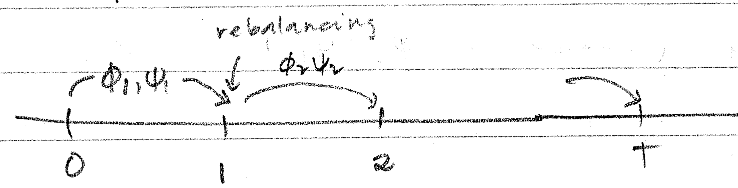
let $E_t = E[B_T^{-1} X \mid f_t]$ martingale for any measure

Martingale Representation Theorem:

$\exists$ predictable ϕ_t s.t. $\quad E_t = E_0 + \sum_{i=0}^{t} \phi_i \Delta Z_i$

strategy at time i, buy (ϕ_{i+1}, ψ_{i+1}) of (S, B)

where $\psi_{i+1} = E_i - \phi_{i+1} B_i^{-1} S_i$



At time k, before rebalancing

portfolio value is $\quad \phi_k S_k + \psi_k B_k$

$$= \phi_k S_k + B_k (E_{k-1} - \phi_k B_{k-1}^{-1} S_{k-1})$$

$$= B_k (E_{k-1} + \phi_k \underbrace{(B_k^{-1} S_k - B_{k-1}^{-1} S_{k-1})}_{\Delta Z_k})$$

$$= B_k (E_{k-1} + \phi_k \Delta Z_k)$$

$$= B_k E_k$$

Set $k = T$

$$V_T = B_T E_T$$

$$= B_T E[B_T^{-1} X \mid f_T]$$

$$= B_T (B_T^{-1} X)$$

$$= X.$$

So, (ϕ_t, ψ_t) is a replicating portfolio

<u>time 0 value</u>

$$\phi_1 S_0 + \psi_1 B_0$$
$$= \phi_1 S_0 + (E_0 - \phi_1 B_0^{-1} S_0) B_0$$
$$= B_0 S_0$$
$$= E_0$$
$$= E_Q[B_T^{-1} X]$$

<u>need</u>: strategy is self-financing ← arbitrage

Value of portfolio at time k after rebalancing.

$$= \phi_{k+1} S_k + (E_k - \phi_{k+1} B_k^{-1} S_k) B_k$$
$$= B_k E_k.$$

<u>Conclusion</u>: price of X is : $E_Q[B_T^{-1} X]$.

hedging strategy : (ϕ_k, ψ_k)

<u>DEF</u>. <u>Self-financing</u>:

need for all i $\quad \phi_i S_i + \psi_i B_i = \phi_{i+1} S_i + \psi_{i+1} B_i$

$$\Delta \phi_i = \phi_i - \phi_{i-1}$$

$$\boxed{\Delta \phi_{i+1} S_i + \Delta \psi_{i+1} B_i = 0.}$$

self-financing condition

V_i = value of portfolio at time i

$$\Delta V_i = \Delta(\phi_{i+1} S_i) + \Delta(\psi_{i+1} B_i)$$
$$= (\Delta \phi_{i+1} S_i + \Delta \psi_{i+1} B_i) \to 0.$$
$$+ (\phi_i \Delta S_i + \psi_i \Delta B_i)$$

$$\boxed{\Delta V_i = \phi_i \Delta S_i + \psi_i \Delta B_i}$$ — self-financing condition

≡ equivalent in discrete

price vs value :

price of a derivative. ←

value :

Notes on Foundations of Financial Mathematics -- Draft

Oct. 8 FRI

Monday - 2 HWK due

Wed - no class

Fri - normal.

Discrete Model

finite state, finite time.

$t = 0, 1, 2, \dots, T$

Finite Ω = state space.

$(\Omega, \mu, \mathcal{F} = 2^{\Omega})$

model increasing information with the passage of time.

filtration $(\mathcal{F}_t)_{t=0}^{T}$

$\mathcal{F}_t$ = σ-field

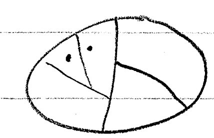
$\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ all t.

$\mathcal{F}_0 = \{\phi, \Omega\}$ no information

$\mathcal{F}_T = \mathcal{F}$

atomic σ-fields

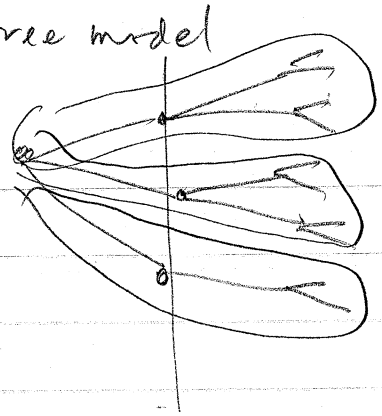
→ generated by partitions.



Ω

$\Longleftrightarrow$ tree model

path with



Ω = set of paths in finite tree

$\mathcal{F}_0$ gen. by Ω

$\mathcal{F}_1$ gen. by 3 subsets

3 sets of paths

through of time 1 nodes.

<u>stoch process</u>

$$S = (S_t)_{t=0}^{T}$$

<u>adapted</u> to $\{\mathcal{F}_t\}$

$\Longleftrightarrow$ S_t is $\mathcal{F}_t$-ⓜ

$\Longleftrightarrow$ S_t is constant on $\mathcal{F}_t$-atoms

$$S_t : \Omega \longrightarrow \mathbb{R}^{k+1}$$

$$S_t = (\boxed{S_t^0}, \ldots, S_t^k) \quad \text{(all positive)}$$

numeraire

"discounting purpose"

S_t^0 is the discounting factor

$$(S_t^0)^{-1} S_t^k$$

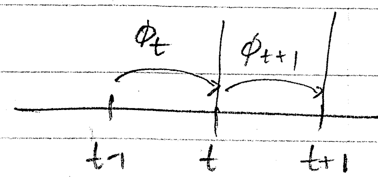
Trading Strategy:

Predictable $\mathbb{R}^{k+1}$ – valued process

$$\phi = \{(\phi_t^0, \dots, \phi_t^k)\}_{t=0}^{T}$$

ϕ_t $\mathcal{F}_{t-1}$-ⓜ

ϕ_t = holdings of stocks from $t-1$ to t



"strategy"

\# shares

dot product

Given S_t, ϕ_t

get value process $V_t = \phi_t \cdot S_t$ ← value at t just before rebalancing

$$V_t = V_t(\phi)$$

$$V_0 = \phi_1 \cdot S_0$$

Self-financing:

$$\phi_{t-1} \cdot S_{t-1} = \phi_t \cdot S_{t-1}$$

$$\iff (\Delta\phi_t) \cdot S_{t-1} = 0 \qquad \Delta\phi_t = \phi_t - \phi_{t-1}$$

Def **Claim**: a rv $X : \Omega \to \mathbb{R}$ $\mathcal{F}_T$-ⓜ

attainable claim: X is attainable claim if $\exists$ self-financing strategy ϕ s.t. $V_T(\phi)^{(\omega)} = X^{(\omega)}$ $\forall \omega$

Ex. Strategy: buy $1^{\$}$ of j^{th} stock at time t_0 via borrow $\$1$ of bond S^0 then hold to time T.

"instantaneous" zero value long-short position.

$$\Phi^{t_0,j} \doteq \phi_1 = 0, \dots, \phi_{t_0} = 0$$

\# shares $\longrightarrow$ $\phi_{t_0+1} = \left(-\frac{1}{S^0_{t_0}}, 0, 0, \frac{1}{S^j_{t_0}}, 0, 0 \right)$

$$\phi_{t_0+2} = \phi_{t_0+1}, \text{ etc.}$$

Claim $\phi^{t_0,j}$ is self-financing.

$$\Delta \phi_{t_0+1} \cdot S_{t_0} \overset{?}{=} 0$$

$$\left(-\frac{1}{S^0_{t_0}}, 0, \dots, 0, \frac{1}{S^j_{t_0}}, 0, \dots, 0 \right) \cdot S_{t_0} = -1 + 1 = 0.$$

$$V_0(\Phi^{t_0,j}) = 0, \quad V_T(\Phi^{t_0,j})$$

$$V_T(\Phi^{t_0,j}) = -\frac{S^0_T}{S^0_{t_0}} + \frac{S^j_T}{S^j_{t_0}}$$

$$\Phi^{t_0,j}_A = \begin{cases} \Phi^{t_0,j} & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$A \in \mathcal{F}_{t_0}$

$$V_T(\Phi^{t_0,j}) = \left(-\frac{S^0_T}{S^0_{t_0}} + \frac{S^j_T}{S^j_{t_0}} \right) 1_A.$$

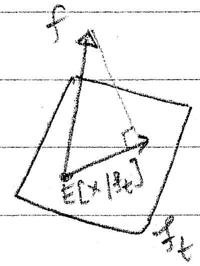
(1) HW: Show: let λ be a prob. measure on Ω

$E_\lambda[X \mid \mathcal{F}_t]$ = orthogonal projection of X into the subspace of $\mathcal{F}_t$-(m) functions.

Oct. 11 MON

(L²-inner product) $\langle f, g \rangle = E_\lambda[fg] = \int_\Omega fg\,d\lambda = \sum_{w \in \Omega} f(w) g(w) \lambda(w)$

$\langle f, f \rangle = \|f\|^2$



(2) Verify: $E_\lambda[X \mid \mathcal{F}_t] = 0 \iff E_\lambda[XY] = 0 \quad \forall Y\ \mathcal{F}_t$-(m)

$\iff \lambda(XY) = 0 \quad \forall Y$

$\iff \lambda(X 1_A) = 0 \quad \forall A\ \mathcal{F}_t$-atoms

$f: \Omega \to \mathbb{R}$

$\lambda(f) = E_\lambda[f] = \int_\Omega f\,d\lambda = \sum_{w \in \Omega} f(w) \lambda(w)$

linear functional on $\{ \text{fns } f: \Omega \to \mathbb{R} \}$

Arbitrage.

3 kinds:

Def. a (weak) arbitrage

= is a self-financing strategy ϕ

s.t. $V_0(\phi) = 0$. and $\underbrace{V_T(\phi) \geq 0}_{\text{no downside!}}$ & $\mu(V_T(\phi) > 0) > 0$

"free lottery ticket".

Def: a market is "viable" iff it does not admit a (weak) arbitrage.

Def. a <u>strong arbitrage</u>
is a self-financing strategy ϕ
s.t. $V_0(\phi) = 0$ and $V_T(\phi) > 0$ "free lunch"

Def. a <u>market admits inconsistent pricing</u>
if $\exists$ a self-financing strategy ϕ s.t. $V_T(\phi) = 0$ and $V_0(\phi) \neq 0$.

Prop 1: (i) no weak arb $\Rightarrow$ no strong arb.
(ii) no strong arb $\Rightarrow$ no inconsistent pricing.

Pf of (ii)

Suppose $\exists \phi$ s.t. $V_T(\phi) = 0$, $V_0(\phi) \neq 0$. (Inconsistent pricing)
WLOG $V_0(\phi) < 0$.

$\phi' = \phi +$ pos amount of $\underset{\neq 0}{S^0}$ chosen so $V_0(\phi') = 0$

$$V_T(\phi') = \underbrace{V_T(\phi)}_{=0} + \underbrace{V_T(S^0_{\text{holding}})}_{>0} > 0$$

HW

e.g. ϕ stock price tree.

③ Give examples showing

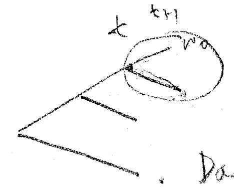
no strong arb $\not\Rightarrow$ no weak arb
no inconsistent pricing $\not\Rightarrow$ no strong arb.

Equivalent

<u>Note</u>: no inconsistent pricing
$\Rightarrow$ attainable claims have unique prices at time 0.

Pf: if not, $V_0(\phi) \neq V_0(\psi)$, $V_T(\phi) = X = V_T(\psi)$
$V_0(\phi - \psi) \neq 0$ but $V_T(\phi - \psi) = 0$ X

Def. A local arbitrage



$\exists$ local arb. if $\exists$ time t, path w_0, holdings vector h

st $h \cdot S_t(w_0) = 0$

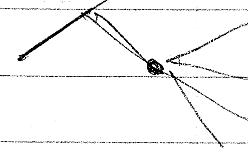
$h \cdot S_{t+1}(w) \geq 0 \quad \forall w \in \underbrace{w_0(t)}_{\mathcal{F}_t\text{-atom containing } w_0}$

& $\mu(h \cdot S_{t+1}(w) > 0) > 0$.

· Does the investor know the stock price tree

· Historical? Probability

· Realization.



Prop. 2 A mkt admits arbitrage iff it admits local arbitrage.

"arbitrage is a local property."

Oct. 15 FRI

Note: HW due Monday.

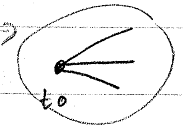
HW #3 Equivalently:

· Find a market that admits a weak arb., but not a strong arb.

· ... strong arb but not inconsistent pricing.

Pf of Prop 2.

($\Leftarrow$) Suppose we have a local arbitrage



Make arb. strategy ϕ.

Idea: do nothing until time t_0

[enter local arbitrage holding] if you're at that node t_0

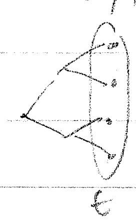
· At t_0+1, cash out into S^0, hold to T.

($\Rightarrow$) Suppose we have a global weak arb. strategy (self-finance) ϕ.

$V_t(\phi)$ = value at time t.

Let $\Lambda = \{ s \leq T : V_s(\phi) \geq 0$ all states, > 0 in at least one state $\}$

diff states of S at time



$V_T(\phi)$ satisfies, $T \in \Lambda \neq \emptyset$

$0 \notin \Lambda$ since ($V_0(\phi) = 0$ self-financing) let $t = \min \Lambda$, $t > 0$

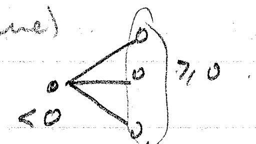
claim: at $t-1$ there is a local arbitrage

case 1: $V_{t-1}(\phi) < 0$ in some state (by defn of Λ and $t = \min \Lambda$).

or

case 2: $V_{t-1}(\phi) = 0$ in all states.

in case 1. (node with negative value)



Take this state (node)

where $V_{t-1}(\phi) < 0$

Add to ϕ a little δ^0 — also raises a little

to raise $V_{t-1}(\phi)$ to zero

get holdings that give local (strong) arbitrage.

$\delta^0 \geq 0$ numeraire

see pf of no strong $\Rightarrow$ no weak

$$V_{t-1}(\phi') = \underbrace{V_{t-1}(\phi)}_{\geq 0} + \underbrace{V_{t-1}(\overset{\geq 0}{\text{add}})}_{\geq 0} = 0$$

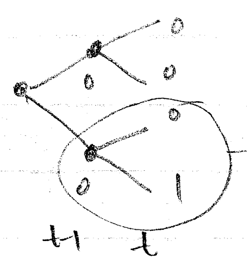
in case 2. $V_{t-1}(\phi) = 0$

Find a $(t-1)$ node that leads to $V_T(\phi)$ in some state ϕ

will be a local arb there.

e.g.

value of strat at nodes



HW Prop. 3

§ Discrete Arbitrage Pricing theorem $\quad S = (S^0, \dots, S^k)$

3 theorems $\quad$ $S^j_t(\omega)$

Theorem 1. If $\left(S/S_0 , \nu, \mathcal{F} \right)$ is a martingale
discounted stock price — gives the tree structure

for some probability measure ν, then for any attainable claim X.
($\exists$ self-fin. replicating strat. ϕ st $V_T(\phi) = X$), the value of
the replicating strategy $V_t(\phi) = E_\nu \left[X \frac{S_t^0}{S_T^0} \mid \mathcal{F}_t \right]$
(note: if no arb, $V_t(\phi)$ will be the correct price)

note: martingale measure allows you to price the strategy.

Oct. 18 MON $\quad$ Solution Day.

Oct. 20 WED

Recall:

equivalent " " used for $\equiv$ function — $V_T(\phi)$ is a fn for T

inconsistent pricing ($\exists$ s.f ϕ s.t. $V_T(\phi) \equiv 0$ & $V_0(\phi) \neq 0$)
$\Rightarrow$ $\exists$ strong arbitrage ($\exists$ sf ϕ st $V_T(\phi) > 0$ & $V_0(\phi) = 0$)
$\Rightarrow$ $\exists$ weak arb ($\exists$ s.f. st $V_T(\phi) \geq 0$, $\not\equiv 0$, & $V_0(\phi) = 0$).

Converses do not hold.

Law of One Price $\quad$ Lo1P
If X is attainable then all replicating strategies for X
have the same initial value. (at $t=0$)
(i.e. if $V_T(\phi) \equiv V_T(\psi)$, then $V_0(\phi) = V_0(\psi)$).

Prop: Law of One Price
is equivalent to no inconsistent pricing.

Pf Suppose Lo1P. WTS no inconsistent pricing.
Suppose $V_T(\phi) \equiv 0$.
let z be the zero strategy $V_T(z) = 0$.
$\overset{\text{Lo1P}}{\Longrightarrow}$ $V_0(\phi) = V_0(z) = 0$.
$\therefore$ no inconsistent pricing.

($\Leftarrow$) Suppose no inconsistent pricing:

Suppose $V_T(\phi) \equiv V_T(\psi)$.

$\Longrightarrow V_T(\phi - \psi) \equiv 0$

$\overset{\text{no incon}}{\Longrightarrow} V_0(\phi - \psi) = 0$

$\Longrightarrow V_0(\phi) = V_0(\psi)$

$\therefore$ Lo1P

$(\Omega, \mathcal{F}, \mu)$

Theorem 1. numeraire S^0

If $(S/S^0, \nu, \mathcal{F})$ is a martingale for some ν

(i.e. $E_\nu\left(\frac{S_T}{S_T^0} \middle| \mathcal{F}_t\right) = \frac{S_t}{S_t^0}$),

$\tilde{S}$ discounted stock price.

S^0 is like e^{rT}

then, for any attainable X with replicating strategy ϕ, $V_T(\phi) \equiv X$, we have

$$V_t(\phi) = E_\nu\left[X \frac{S_t^0}{S_T^0} \middle| \mathcal{F}_t\right]$$

note: $V_t(\phi)$ same for any ν

$V_t(\phi_1) = V_t(\phi_2)$ for any ϕ_1, ϕ_2

Pf: consider:

discounted value process

$$\tilde{V}_t(\phi) = \frac{V_t(\phi)}{S_t^0}$$

$$= \frac{\phi_t S_t}{S_t^0} = \phi_t \cdot \tilde{S}_t$$

ϕ s.f $\Rightarrow \Delta\phi_{t+1} \cdot S_t = 0$

$\Longrightarrow \Delta\phi_{t+1} \tilde{S}_t = 0$

[ϕ is s.f. wrt S iff ϕ is s.f wrt $\tilde{S}$]

· $\Delta \tilde{V}_{t+1} = \tilde{V}_{t+1} - \tilde{V}_t$

$= \phi_{t+1} \Delta\tilde{S}_{t+1} + \underbrace{\tilde{S}_t \Delta\phi_{t+1}}_{=0 \text{ bec. s.f.}}$ (Product Rule)

$= \phi_{t+1} \Delta \tilde{S}_{t+1}$,

$$\Rightarrow E_\nu[\Delta \tilde{V}_{t+1} \mid \mathcal{F}_t]$$
$$= E_\nu[\phi_{t+1} \Delta \tilde{S}_{t+1} \mid \mathcal{F}_t]$$
$$= \phi_{t+1} \cdot \underbrace{E_\nu[\Delta \tilde{S}_{t+1} \mid \mathcal{F}_t]}_{=0 \text{ since } \tilde{S}_t \text{ is a martingale}} \qquad \phi_{t+1} \text{ predictable}$$

$$\underbrace{E_\nu[\tilde{S}_{t+1} - \tilde{S}_t \mid \mathcal{F}_t]}$$

$$E_\nu[\tilde{S}_{t+1} \mid \mathcal{F}_t] - \tilde{S}_t = 0.$$

$\Leftrightarrow$ $\tilde{V}_{t+1}$ is a ν-martingale.

$$\text{So, } \frac{V_t}{S_t^0} = E_\nu\left[\frac{V_T}{S_T^0} \,\middle|\, \mathcal{F}_t\right] \qquad V_T(\phi) = X.$$

note: Having a martingale measure is a good thing.
$\Rightarrow$ we can price claims.

Theorem 2. ① No (weak) arbitrage $\Longleftrightarrow$ $\exists\, \nu \sim \mu$ s.t. $(S/S^0, \nu, \mathcal{F})$ is a martingale.

means [ν and μ have the same sets of measure 0
$\nu(\text{any path}) \neq 0$)

note: original μ gives positive prob. to all paths.

② no strong arb $\Longleftrightarrow$ $\exists\, \nu$ st $(S/S^0, \nu, \mathcal{F})$ is a martingale.

// off-the-record:

③ no inconsistent pricing $\Longleftrightarrow$ $\exists\, \nu$ (possibly signed) st $(S/S^0, \nu, \mathcal{F})$ is a m'gale.
//

Pf: ($\Leftarrow$) Suppose $\nu \sim \mu$ makes S/S^0 a martingale.
WB: no arbitrage.

Suppose ϕ is s.f., and $V_T(\phi) \geq 0$, $\not\equiv 0$.
To prove $V_0(\phi) \neq 0$.

By thm 1. $V_0(\phi) = E_\nu\left[\frac{V_T(\phi)}{S_T^0} \middle| \mathcal{F}_0 \right]$

$$= E_\nu\left[\frac{V_T(\phi)}{S_T^0} \right] > 0$$

$V_T(\phi) > 0$ in at least one path.

$\nu \sim \mu \Rightarrow \nu(\text{any path}) \neq 0$.

wts:

($\Rightarrow$) No arb $\Rightarrow \exists\, \nu \sim \mu$ s.t. S/S^0 is a ν-mgale.

$\nu(\text{any path}) > 0$

Pf:

Oct. 22

FRI

Notations:

Ω = set of paths in tree

$\mathbb{R}^\Omega$ = set of claims (payoffs at time T)

$= \{ X : \Omega \to \mathbb{R} \}$

$|\Omega|$ = finite $\Rightarrow$ makes $\mathbb{R}^\Omega = \mathbb{R}^{|\Omega|}$ Euclidean space.

vector

$\mathcal{X}^+ = \{ X \in \mathbb{R}^\Omega : X \geq 0 \text{ and } \underbrace{\neq 0}_{\mu(X>0)>0} \}$

$\mathcal{X}^{++} = \{ X \in \mathbb{R}^\Omega : \underbrace{X > 0}_{\text{each component} > 0} \}$

$\mathcal{X}^0 = \{ X \in \mathbb{R}^\Omega : \exists$ self-financing strategy ϕ s.t. $X = V_T(\phi)$ and $\underbrace{V_0(\phi)}_{=0} \}$

attainable claims $\rightarrow$ $\mathcal{A} = \{ X \in \mathbb{R}^\Omega : \exists$ s.f. str ϕ s.t. $X = V_T(\phi) \}$.

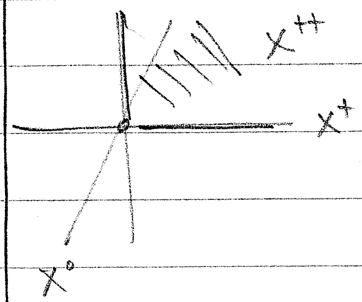
note: • $\mathcal{X}^0 \subseteq \mathcal{A} \subseteq \mathbb{R}^\Omega$

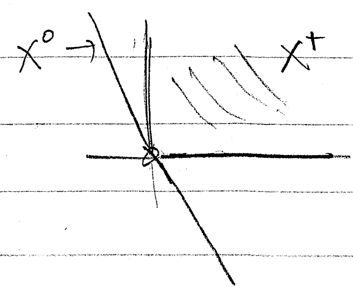
• $\mathcal{X}^0$ = linear subspace of $\mathbb{R}^n$

• $\mathcal{A}$ = linear subspace of $\mathbb{R}^n$

But
- $\mathcal{X}^{++}$ = open positive orthant in $\mathbb{R}^{\Omega} = \mathbb{R}^{|\Omega|}$
- $\mathcal{X}^{+}$ = closed positive orthant $\setminus \{0\}$ ← origin

not Linear





note: $X^0 \cap X^+ = \phi$

this implies: (claim: ?)
that $\exists\, v \neq 0$ s.t. $v \perp X^0$ and $v \in \mathcal{X}^{++}$

Pf: See Appendix.

Define λ (measure): $\lambda(X) = v \cdot X$

(i.e. $\omega \in \Omega$: $\lambda(\omega) = \omega^{th}$ coordinate of v)

$\mu(X) = \int_\Omega X \, d\mu = \sum_{\omega \in \Omega} X(\omega)\mu(\omega)$

by normalizing, assume WLOG λ is a prob measure.
$\sum_{\omega \in \Omega} \lambda(\omega) = 1$.

note: $v > 0$: $\lambda \sim \mu$

λ is positive on X^+

λ is zero on X^0

Strategy $\Phi^{t,n}_A$: A = atom of $\mathcal{F}_t$ (i.e. A known at t)

n = index of stock

→ means do nothing until time t
If A occurs, borrow \$1 of S^0 and buy \$1 of S^n. hold. } s.f!

$$V_T(\Phi_A^{t,n}) = \left(\frac{-S_T^0}{S_t^0} + \frac{S_T^n}{S_t^n} \right) 1_A \quad \in \mathcal{X}_0$$

$$V_0(\Phi_A^{t,n}) = 0.$$

note: $\lambda\left[\left(-\frac{S_T^0}{S_t^0}+\frac{S_T^n}{S_t^n}\right)1_A\right]=0$ since λ is zero on X^0.

True for all t, $\forall A\in \mathcal{F}_t$, all n.

$$\overset{HW}{\iff} E_\lambda\left[\left(-\frac{S_T^0}{S_t^0}+\frac{S_T^n}{S_t^n}\right)\Big|\,\mathcal{F}_t\right]=0$$

$$\iff E_\lambda\left[\frac{S_T^0}{S_t^0}\,\Big|\,\mathcal{F}_t\right]=E_\lambda\left[\frac{S_T^n}{S_t^n}\,\Big|\,\mathcal{F}_t\right]$$

"λ-conditional expected returns of all stocks are equal"

$$\iff E_\lambda[S_T^n\,|\,\mathcal{F}_t]=E_\lambda\left[\frac{S_t^n}{S_t^0}S_T^0\,\Big|\,\mathcal{F}_t\right] \quad \text{since } S_t^n \text{ is } \mathcal{F}_t \text{ (m)}$$

$$[*] \iff \lambda\left(\left(S_T^n-\frac{S_t^n}{S_t^0}S_T^0\right)1_A\right)=0 \quad \forall \text{ atoms of } \mathcal{F}_t$$

Define ν:

by $\frac{d\nu}{d\lambda}=\frac{S_T^0}{\lambda(S_T^0)}$

i.e. $\nu(f)=\lambda\left(f\frac{S_T^0}{\lambda(S_T^0)}\right)$, $f\in\mathbb{R}^\Omega$ (with $\lambda(S_T^0)>0$)

$$\nu(\mathbb{1})=\lambda\left(\mathbb{1}\frac{S_T^0}{\lambda(S_T^0)}\right)=\frac{\lambda(S_T^0)}{\lambda(S_T^0)}=1.$$

constant 1 in each state

need this martingale and: $\frac{S_t^n}{S_t^0}=E_\nu\left[\frac{S_T^n}{S_T^0}\,\Big|\,\mathcal{F}_t\right]$

$$\iff E_\nu\left[\frac{S_T^n}{S_T^0}-\frac{S_t^n}{S_t^0}\,\Big|\,\mathcal{F}_t\right]=0$$

$$\Leftrightarrow \quad \nu\left(\left(\frac{S_T^n}{S_T^0} - \frac{S_t^n}{S_t^0}\right) 1_A\right) = 0 \qquad \forall A \text{ atoms of } \mathcal{F}_t$$

$$\Leftrightarrow \quad \lambda\left(\frac{S_T^0}{\lambda(S_T^0)}\left(\frac{S_T^n}{S_T^0} - \frac{S_t^n}{S_t^0}\right) 1_A\right) = 0$$

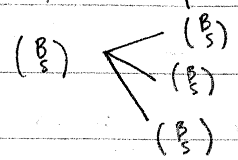
$$\Leftrightarrow \quad \lambda\left(\left(S_T^n - \frac{S_t^n}{S_t^0} S_T^0\right) 1_A\right) = 0 \qquad \leftarrow (*)$$

∎

Oct. 25
MON

<u>Def</u>: A market is <u>complete</u> iff every claim is attainable.

e.g. of not complete



<u>Theorem 3</u> a viable market is complete iff the equiv. m'gale measure is unique.

<u>Pf</u>:

($\Rightarrow$) Suppose market is complete.

let ν_1, ν_2 are m'gale measures.

<u>thm 1</u> $\Rightarrow$ $E_{\nu_1}\left[\frac{X}{S_T^0}\right] = E_{\nu_2}\left[\frac{X}{S_T^0}\right] \quad \forall X.$

$\Rightarrow$ $\nu_1 = \nu_2$.

ν_1, ν_2 will disagree only on unattainable claim.

($\Leftarrow$) Suppose market not complete.

$\mathcal{A}$ = set of attainable claims

$\mathcal{A} \neq \mathbb{R}^\Omega$

$X^0 \subsetneq \mathcal{A} \subsetneq \mathbb{R}^\Omega$

attainable claim t=0 value

$\dim X^0 \leq \dim \mathbb{R}^\Omega - 2$

$\Rightarrow$ get $\lambda \perp X^0$, λ in X^{++}

∞-many possible choices for λ

hence ν [see pf. of thm 2].

Remark: Suppose incomplete mkt.

instr < states of the world

let Σ be the set of all equiv. m'gale measures (convex $\Rightarrow$ infinite)

$\Rightarrow$ can't replicate all claims.

. $\mu_1, \mu_2 \in \Sigma \Rightarrow t\mu_1 + (1-t)\mu_2 \in \Sigma$

and $\nu \in \Sigma$ gives a no-arb pricing scheme for all claims agrees with any other on A.

. So, given X, $\{ E_\nu [\frac{X}{S_T^0}] : \nu \in \Sigma \}$, let $S_0^0 = 1$

= set of allowable prices for X.

= an interval since Σ is convex.

$\Rightarrow$ below/above the interval gives arb. prices.

Forwards & Forward Measures

Recall $\frac{d\nu}{d\lambda} = \frac{S_T^0}{\lambda(S_T^0)}$

So, if S^0 is a T-bond (i.e. $S_T^0 = 1$)

$= \frac{d\nu}{d\lambda} = 1$

$\Rightarrow \nu = \lambda$. $\leftarrow$ aha FORWARD MEASURE

note:

λ defined before choice of numeraire (Pf. 2 cec)

$=$ T- forward measure

$=$ m'gale measure when $S^0 =$ T-bond

HW #6

6.1 Prop 3 notes ($\Rightarrow$, and $\Leftarrow$ counterexample)

6.2. Verify: $E_\nu [X | \mathcal{F}_t] = \dfrac{E_\lambda [X \frac{d\nu}{d\lambda} | \mathcal{F}_t]}{E_\lambda [\frac{d\nu}{d\lambda} | \mathcal{F}_t]}$

change of measure

$V_T(\phi) = X$

$$V_t(\phi) = E_\nu\left[X \frac{S_t^0}{S_T^0} \,\middle|\, \mathcal{F}_t\right]$$

$$= \frac{E_\lambda\left[X \frac{S_t^0}{S_T^0} \frac{d\nu}{d\lambda} \,\middle|\, \mathcal{F}_t\right]}{E_\lambda\left[\frac{d\nu}{d\lambda} \,\middle|\, \mathcal{F}_t\right]}$$

$$= \frac{E_\lambda\left[\frac{S_t^0}{S_T^0} S_T^0 \,\middle|\, \mathcal{F}_t\right]}{E_\lambda\left[S_T^0 \,\middle|\, \mathcal{F}_t\right]}$$

$$V_t(\phi) = \frac{1}{E_\lambda\left[\frac{S_T^0}{S_t^0} \,\middle|\, \mathcal{F}_t\right]} \cdot E_\lambda[X \mid \mathcal{F}_t]$$

discounting factor.

see Pf 2 S^0 can be S^k for any k. So, indept of choice of stock hence

$$\boxed{V_t(\phi) = d(t,T)\, E_\lambda[X \mid \mathcal{F}_t]}$$

an alternate pricing formula.

depends only on t and T, not on stock.

doesn't depend on numeraire but depends on T-bond.

Suppose, T-bond is attainable. $\mathbb{1}$

$\Rightarrow$ price of T-bond $= d(t,T)\, E_\lambda[\mathbb{1} \mid \mathcal{F}_t]$
$= d(t,T)$.

$(\approx e^{-r(T-t)}$ for constant $r)$

interpretation of $d(t,T)$.

DEF. let X be a claim paying off at T, ~~K~~

For $t<T$, let K be an $\mathcal{F}_t$- measurable

The <u>forward contract</u> on X struck at K

is the claim $X-K$.

DEF. The <u>time t T-forward price of X</u> is the strike K

such that $X-K$ has value 0 at time t.

Theorem. In a viable market, the time t T-forward price of X is

$$K = E_\lambda[X \mid \mathcal{F}_t].$$

Pf.

$$0 = d(t,T)\, E_\lambda[X - K \mid \mathcal{F}_t]. \quad ∎$$

$$V_0(\phi) = E_\nu\left[\frac{X}{S^\circ_T}\right]$$

$$= d(0,T)\, E_\lambda[X]$$

$t=0$: forward price of X $= E_\lambda[X]$

FUTURES price of X. $= E_\nu[X]$

note:

fwd = futures

If S° is deterministic

$$\frac{d\nu}{d\lambda} = \frac{S^\circ_T}{\lambda(S^\circ_T)} = 1 \quad \Rightarrow \nu = \lambda.$$

Futures $\longrightarrow$ next time

Oct. 27 WED

. HW due a week from now:

· Handout 2 then back to Baxter & Rennie

Set-up: paths

. finite Ω, $t=0,1,2,\dots,T$

. filtration $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$

$\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_T = 2^{\Omega}$

. $(S=(S^0,\dots,S^n), \mu, \mathcal{F})$

. self-financing strategy

no arb $\longleftrightarrow$ $\exists\, \nu \sim \mu$ s.t. $\left(\frac{S}{S^0}, \nu, \mathcal{F}\right)$ is a m'gale

---

"T- forward measure"

= m'gale measure when S^0 = T-bond

$$\frac{d\nu}{d\lambda} = \frac{S_T^0}{\lambda(S_T^0)}$$

Let's Price attainable claim X

$$V_0(X) = E_\nu\left[\frac{X}{S_T^0} \,\Big|\, \mathcal{F}_0\right]$$

$$= E_\nu\left[\frac{X}{S_T^0}\right]$$

$$= d(0,T)\, E_\lambda[X], \qquad d(t,T) = \frac{1}{\lambda\left(\frac{S_T^0}{S_t^0}\right)}$$

Forward prices.

- time t forward price for a contract maturity at T is $E_\lambda[X \mid \mathcal{F}_t]$

Futures

- "marked-to-market"

i.e. $\exists$ "futures price" $F_t(X)$ at time t

underlying value at T

- daily, the holder (long position) is paid $F_t(X) - F_{t-1}(X)$

· Futures contract has value zero.

· $F_T(X) = X$ (i.e. at time T, $F_t(X) \to X$

DEF money mkt acct

· need S^0 = $\underbrace{\text{money mkt account}}_{\text{predictable, } >0}$

Theorem. If S^0 is a money mkt acct.

and $(S/S^0, \nu, \mathcal{F})$ is a mgale

then

futures price of X

$$F_t(X) = E_\nu[X \mid \mathcal{F}_t]$$

Pf:

Strategy: · wait until time t

· Go long ϕ_t futures contract. (costs nothing)

· At each $\tau > t$: receive the margin payment, $\approx$ invest in S^0

close futures acct

go long ϕ_τ contracts

· self-financing!

· At time τ, receive

$$\phi_{\tau-1}(F_\tau - F_{\tau-1}) = \$$$

or $\dfrac{\phi_{\tau-1}}{S^0_\tau}(F_\tau - F_{\tau-1})$ shares of S^0

true for $\tau = t+1, \ldots, T$

accumulate total # shares of S^0 : $\displaystyle\sum_{\tau=t+1}^{T} \frac{\phi_{\tau-1}}{S^0_\tau}(F_\tau - F_{\tau-1})$

Choose $\phi_{\tau-1} = S^0_\tau$ (~~[illegible]~~

predictable, so ok.

get $\sum_{\tau=t+1}^{T} (F_\tau - F_{\tau-1})$

$= F_T - F_t$ # shares of S^0

r.v.
payoff on strategy; must have fair price because it cost nothing to create.

$$\text{value}_T = \underbrace{(F_T(X) - F_t(X))\, S^0_T}_{\text{think of this as a claim.}} \neq ~~\phi~~$$

~~p/f of ... futures~~

X

discounting

fair price of the claim.

$$0 = E_\nu\left[(X - F_t(X)) S^0_T \cdot \frac{S^0_t}{S^0_T} \,\middle|\, \mathcal{F}_t \right]$$

$$0 = E_\nu\left[X - F_t(X) \,\middle|\, \mathcal{F}_t \right]$$

$$\Rightarrow \quad F_t(X) = E_\nu\left[X \mid \mathcal{F}_t \right]. \quad \blacksquare$$

λ = forward measure

ν = "futures" measure

Corollary. If S^0 is deterministic, then $S^0_T = \lambda(S^0_T)$, then $\nu = \lambda$. So forwards = futures.

~~HW~~ Use this stuff:

1. Model: S, $\mathcal{F}$

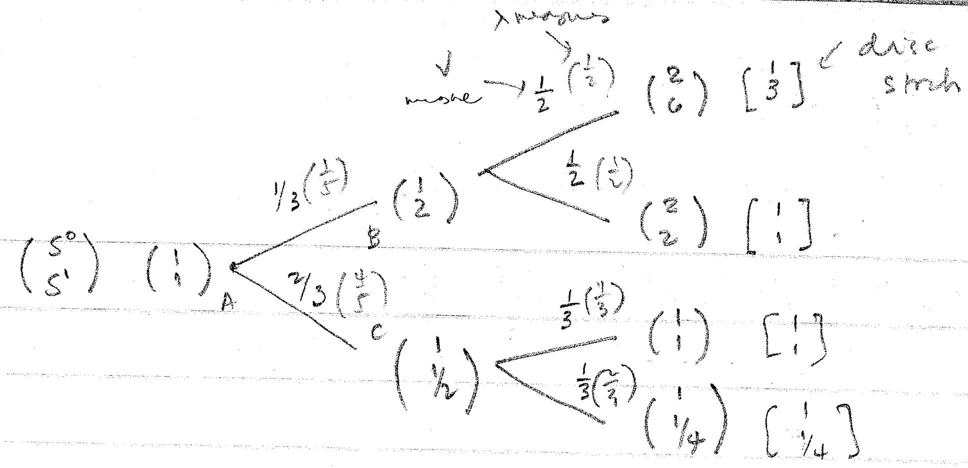
compute ν, λ

$$\frac{d\nu}{d\lambda} = \frac{S^0_T}{\lambda(S^0_T)} \longleftrightarrow \nu(X) = \lambda\left(X \frac{S^0_T}{\lambda(S^0_T)} \right)$$

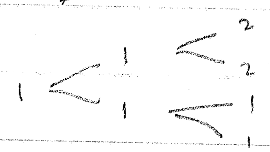
$\boxed{\text{HW}}$ #3 check $\dfrac{d\lambda}{d\nu} = \dfrac{1/S^0_T}{\nu(1/S^0_T)}$

Oct 29

FRI



- need mgale measure
- S^0 = numeraire
- $\frac{S}{S^0} = \left(\frac{S^0}{S^0}, \frac{S^1}{S^0}\right) = \left(1, \frac{S^1}{S^0}\right]$ disc. stock price brackets
- S^0 is predictable



- money market :

to get mgale measure;

$$q(3) + (1-q)\,1 = 2 \quad \text{at } \underline{\text{node D}}$$

$$q = \frac{1}{2}$$

node E : $q = \frac{1}{3}, \frac{2}{3}$

etc !

prob. meas.

$$V = \left(\tfrac{1}{6} \;\; \tfrac{1}{6} \;\; \tfrac{2}{9} \;\; \tfrac{4}{9} \right) \leftarrow \text{probabilities of paths}$$

up up   up down

given claim X ; $\quad V_0(\phi) = E_V \left[\frac{X}{S_T^0} \right]$

$$= V\left(\frac{X}{S_T^0}\right) = \sum_{\omega \in \Omega} V(\omega) \frac{X(\omega)}{S_T^0(\omega)}$$

· Futures price of S^1 at $t=0$
maturity $t=2$

$$E_\nu[S_2^1] = \frac{1}{6}\cdot 6 + \frac{1}{6}\cdot 2 + \frac{2}{9}1 + \frac{4}{9}\cdot 4$$
$$= \frac{5}{3}$$

· Forward price

Need 2-forward meas λ

$$\frac{d\lambda}{d\nu} = \frac{1/S_2^0}{\nu(1/S_2^0)}$$

$$\nu\left(\frac{1}{S_2^0}\right) = \frac{1}{6}\cdot\frac{1}{2} + \frac{1}{6}\frac{1}{2} + \frac{2}{9}\cdot 1 + \frac{4}{9}\cdot 1$$
$$= 5/6$$

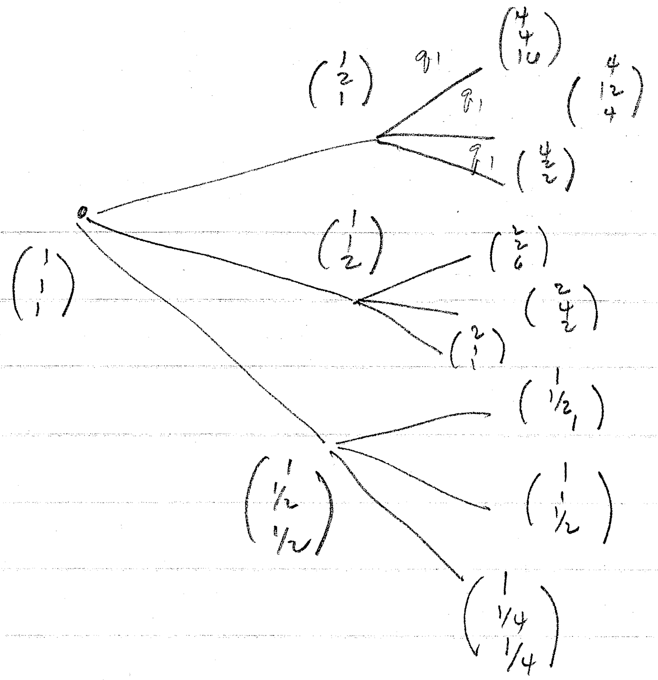
$$\lambda^{(w)} = \frac{6}{5}\nu\left(w\,\frac{1}{S_2^0}\right)$$
$$= \frac{6}{5}\,\frac{1}{S_2^0(w)}\,\nu(w)$$

$$\lambda(\text{top}) = \frac{6}{5}\,\frac{1}{2}\,\frac{1}{6} = \frac{1}{10}$$
$$\lambda(\text{2nd}) = \frac{6}{5}\,\frac{1}{2}\,\frac{1}{6} = \frac{1}{10}$$
$$\lambda(\text{3rd}) = \frac{6}{5}\cdot\frac{1}{1}\cdot\frac{2}{9} = \frac{4}{15}$$
$$\lambda(\text{bot}) = \frac{6}{5}\cdot 1\cdot\frac{4}{9} = \frac{8}{15}$$

Forward Price; $E_\lambda[S_2^1] = \frac{1}{10}\cdot 6 + \frac{1}{10}\cdot 2 + \frac{4}{15}\cdot 1 + \frac{8}{15}\cdot\frac{1}{4}$
$$= \frac{6}{5}$$

HW:

$$\begin{pmatrix} S^0 \\ S^1 \\ S^2 \end{pmatrix}$$



Ex 3:

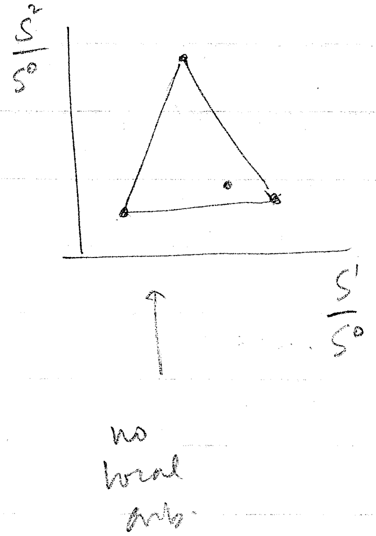
(a) Show this mkt admits no arb.

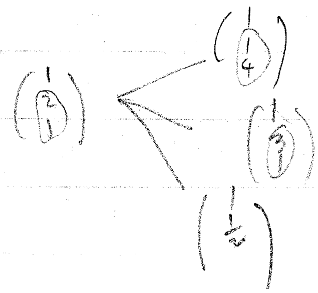
use S^0 as numeraire

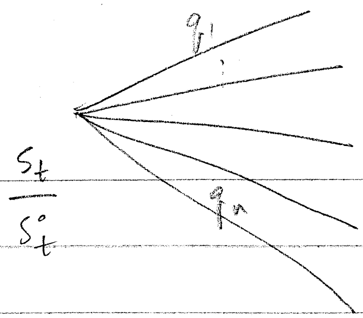
(b) find ν, λ [illegible] 2-fwd measures.

(c) find the 2-fwd price of S^1 at $t=0$.

(d) futures prices of S^1 at $t=0$.







$$\frac{S_{t+1}}{S^0_{t+1}} = \left(1, \frac{S^1_{t+1}}{S^0_{t+1}}, \dots, \frac{S^k_{t+1}}{S^0_{t+1}}\right)$$

$$\sum_{j=1}^{n} q_j \left(1, \frac{S^1_{t+1}(j)}{S^0_{t+1}(j)}, \dots, \frac{S^k_{t+1}(j)}{S^0_{t+1}(j)}\right) = \left(1, \frac{S^1_t}{S^0_t}, \dots, \frac{S^k_t}{S^0_t}\right)$$

$k+1$ eqns

in n unknowns $q_1, \dots, q_n$

$\underbrace{k+1}_{\#\text{ brands}} < \underbrace{n}_{\#\text{ arms}}$

If $k+1 < n \Rightarrow \infty$ many solutions

$\Rightarrow$ inc. market by thm 3

$k+1 = n \Rightarrow$ expect complete market if no arb.

$k+1 > n \Rightarrow$ expect arbitrage or redundancy.

Back to Baxter & Rennie Chap. 3.

HW p. 49 #1,2

current HW due Wed.

Exam Monday.

linear algebra pictures

3. (a) efficient frontier is a series portfolios which are fully invested, each portfolio has minimum risk among all portfolio with the same expected return.

(b) min: $h_p^T V h_p$ subject to $h_p^T f = f_p$ and $h_p^T e = 1$

Let $u = h_p^T V h_p - \lambda(h_p^T e - 1) - \mu(h_p^T f - f_p)$

$\nabla u = 0$, $\frac{\partial}{\partial \lambda} u = 0$, $\frac{\partial}{\partial \mu} u = 0$

$\Rightarrow 2Vh_p = \lambda e + \mu f \Rightarrow h_p = \frac{\lambda}{2} V^{-1} e + \frac{\mu}{2} V^{-1} f$

$h_p = (\frac{\lambda}{2} \frac{1}{\sigma_C^2}) h_C + (\frac{\mu}{2} \frac{f_Q}{\sigma_Q^2}) h_Q$ —> plug into constraint

$$\Rightarrow \begin{cases} \frac{\lambda}{2} \frac{1}{\sigma_C^2} + \frac{\mu}{2} \frac{f_Q}{\sigma_Q^2} = 1 \\ \frac{\lambda}{2} \frac{1}{\sigma_C^2} f_C + \frac{\mu}{2} \frac{f_Q^2}{\sigma_Q^2} = f_p \end{cases}$$

$\Rightarrow \lambda = \frac{f_Q - f_p}{f_Q - f_C} (2\sigma_C^2)$, $\mu = \frac{f_C - f_p}{f_C - f_Q} \times \frac{\sigma_Q^2}{f_Q} \times 2$

$\therefore h_p = (\frac{f_Q - f_p}{f_Q - f_C}) h_C + (\frac{f_C - f_p}{f_Q - f_C}) h_Q$

4. $U_R(p) = \alpha_p - \lambda \omega_p^2$, subject to $h_p^T \beta = 0$

$\nabla(U_R(p)) = \alpha - 2\lambda V_R h_p$

$= \alpha - 2\lambda(V - \beta \sigma_B^2 \beta^T V) h_p = 0$

Bring $h_p^T \beta = 0 \Rightarrow 2\lambda V h_p = \alpha \Rightarrow h_p = \frac{1}{2\lambda} V^{-1}\alpha = \frac{h_A}{2\lambda \sigma_A^2}$

$(\beta^T h_p)^T = (h_p^T \beta_p)$

$$a = \frac{V h_a}{\sigma_A^2}$$

$$\alpha = \frac{V h_A}{\sigma_A^2}$$

$$V^{-1}\alpha = \frac{h_A}{\sigma_A^2}$$

$\Rightarrow \alpha - 2\lambda V h = \mu \beta$

1. Short Answer:

(a) Q is the whole market, all the market in the world.

(b) $IR = IC\sqrt{BR}$, since BR double $\Rightarrow$ IR become $\sqrt{2}$ times

(c) $\beta_{P,Q} f_Q = f_P$ is the expected excess return under CAPM

(d) $\frac{\Delta p_i}{p_i} \approx I\, \sigma_i \sqrt{\frac{V}{V_d}}$

(e) the Beta = 1 portfolio with min risk is Benchmark portfolio

2. $r_1 = \beta_1 r_B + \theta_1$

$r_2 = \beta_2 r_B + \theta_2$

Assumption: $\frac{Cov(\theta_1, \theta_2)}{w_1 w_2} = 1$

$$\rho_{12} = \frac{Cov(r_1, r_2)}{\sigma_1 \sigma_2} = \frac{Cov(\beta_1 r_B + \theta_1, \beta_2 r_B + \theta_2)}{\sigma_1 \sigma_2}$$

$$= \frac{\beta_1 \beta_2 \sigma_B^2 + Cov(\theta_1, \theta_2)}{\sigma_1 \sigma_2}$$

$w_1 = \sqrt{\sigma_1^2 - \beta_1^2 \sigma_B^2}$, $w_2 = \sqrt{\sigma_2^2 - \beta_2^2 \sigma_B^2}$

since ~~Cov(...)~~ $= Cov(\theta_1, \theta_2) = w_1 w_2$

$$\therefore \rho_{12} = \frac{\beta_1 \beta_2 \sigma_B^2 + \sqrt{(\sigma_1^2 - \beta_1 \sigma_B^2)(\sigma_2^2 - \beta_2 \sigma_B^2)}}{\sigma_1 \sigma_2}$$

MAP 6621 HW-3 Ya-Chun Chia.

p.133 #4

1. $\lambda_R = 0.12$, $IR = 0.6$

$VA[w_p] = w_p \cdot IR - \lambda_R w_p^2$

$\max VA[w_p] = \frac{IR^2}{4\lambda_R}$, at $w^* = \frac{IR}{2\lambda_R}$

$\therefore$ (1) $w^* = \frac{IR}{2\lambda_R} = \frac{0.6}{2(0.12)} = \frac{5}{2}$

(2) $VA = \frac{IR^2}{4\lambda_R} = \frac{w^* IR}{2} = \frac{0.36}{0.48} = \frac{3}{4}$.

p.133 #5

$IR_{ym} = 0.3$ $\therefore$ $w_{ym} = \frac{0.3}{2(0.12)} = \frac{5}{4}$

$V^*_{ym} = \frac{IR_{ym}^2}{4\lambda_R} = \frac{0.09}{0.48} = \frac{3}{16}$

Using the setting of previous problem

then $VA = w_p IR - \lambda_R w_p^2$

$= (2.5)(0.3) - (0.12)(2.5)^2 = 0$

then we'll lose $\frac{3}{16}$ value added.

p.145 #1

We know $\left(\frac{f_Q}{\sigma_Q}\right) = SR_Q$, $\left(\frac{f_B}{\sigma_B}\right) = SR_B$, and $f = \beta \cdot f_B + \alpha$

$\left(\frac{f_Q}{\sigma_Q}\right)^2 = \frac{f_Q f_Q}{\sigma^2_Q} = \frac{f_Q f_C}{\sigma^2_C}$ (by 2A.35 $\frac{f_Q}{\sigma^2_Q} = \frac{f_C}{\sigma^2_C}$)

$= \frac{f_Q e_q}{\sigma^2_q}$ (by 2A.29. $\frac{f_C}{\sigma^2_C} = \frac{e_q}{\sigma^2_q}$)

$= \frac{h_Q^T f e_q}{\sigma^2_q} = \frac{h_q^T f}{\sigma^2_q}$ (by $h_Q = \frac{h_q}{e_q}$) $= \frac{f_q}{\sigma^2_q} = \frac{1}{\sigma^2_q} = f^T V^{-1} f$

$= (\beta \cdot f_B + \alpha)^T V^{-1} (\beta \cdot f_B + \alpha)$

$= f_B^2 \beta^T V^{-1} \beta + f_B \beta^T V^{-1} \alpha + \alpha^T V^{-1} \beta f_B + \alpha^T V^{-1} \alpha$

$= f_B^2 / \sigma_B^2 + f_B (\beta^T V^{-1} \alpha + \alpha^T V^{-1} \beta) + \frac{1}{\sigma^2_A}$

(by 5A.7. $\sigma_A = w_A = \frac{1}{IR}$ and $\beta_A = \beta^T h_A = \frac{\beta^T V^{-1} \alpha}{\alpha^T V^{-1} \alpha} = 0$.

$\therefore \beta^T V^{-1} \alpha = 0$ $\therefore (\beta^T V^{-1} \alpha)^T = \alpha^T V^{-1} \beta = 0$)

$= \left(\frac{f_B}{\sigma_B}\right)^2 + IR^2$

3. R.f. EFF EFF β a series port which are fully invested each port has min risk among all portfolio w/ the same expected return

$\min h_p^T V h_p$ s.t. $h_p^T f = f_p$ and $h_p^T e = 1$

let $u = h_p^T V h_p - \lambda(h_p^T e - 1) - \mu(h_p^T f - f_p)$

$\nabla u = 0$, $\frac{\partial}{\partial \lambda} u = 0$, $\frac{\partial}{\partial \mu} u = 0 \Rightarrow 2 V h_p = \lambda e + \mu f \Rightarrow h_p = \frac{\lambda}{2} V^{-1} e + \frac{\mu}{2} V^{-1} f$

$h_p = \left(\frac{\lambda}{2} \frac{1}{\sigma_C^2}\right) h_C + \left(\frac{\mu}{2} \frac{f_Q}{\sigma_Q^2}\right) h_Q$

b $\begin{cases} \frac{\lambda}{2} \frac{1}{\sigma_C^2} + \frac{\mu}{2} \frac{f_Q}{\sigma^2 Q} = 1 \\ \frac{\lambda}{2} \frac{1}{\sigma_C^2} f_C + \frac{\mu}{2} \frac{f_Q^2}{\sigma^2 Q} = f_p \end{cases}$

4. $U_R(P) = \alpha_p - \lambda \omega_p^2$ s.t. $h_p^T \beta = 0$

$\nabla(U_R(p)) = \alpha - 2\lambda V_R h_p$

$= \alpha - 2\lambda(V - \beta \sigma_B^2 \beta^T) h_p = 0$

Bring $h_p^T \beta = 0 \Rightarrow 2\lambda V h_p = \alpha \Rightarrow h_p = \frac{1}{2\lambda} V^{-1} \alpha = \frac{h_A}{2\lambda \sigma_A^2}$

Technical Appendix 5

CHARACTERISTIC PORTFOLIO OF ALPHA

$\alpha = \{\alpha_1, \ldots, \alpha_N\}$ forecasts of residual returns

characteristic portfolio: $\boxed{h_A = \dfrac{V^{-1}\alpha}{\alpha^T V^{-1}\alpha}}$

portfolio A has alpha 1: $\boxed{h_A^T \alpha = 1}$

and has minimum risk

variance of portfolio A: $\boxed{\sigma_A^2 = h_A^T V h_A = \dfrac{1}{\alpha^T V^{-1}\alpha}}$

alpha in terms of portfolio A: $\boxed{\alpha = \dfrac{V h_A}{\sigma_A^2}}$

INFORMATION RATIOS

$Q \leftrightarrow A \leftrightarrow IR$

$\forall$ portfolio P, $\omega_P > 0$, $\boxed{IR_P = \dfrac{\alpha_P}{\omega_P}}$ $= 0$ if $\omega_P = 0$.

$\boxed{IR = Max\{IR_P | P\}}$

Proposition 1

Portfolio A has zero beta: $\boxed{\beta_A = \beta^T \cdot h_A = 0}$

Portfolio A has max IR $\boxed{IR = IR_A = \sqrt{\alpha^T V^{-1}\alpha} \geq IR_P}$

Portfolio A has total and residual risk equal to $1 \div IR$: $\boxed{\omega_A = \sigma_A = 1/IR}$.

Any portfolio P that can be written as an exposure-l.c. of portfolios B and A has max IR

$\boxed{h_P = \beta_P \cdot h_B + \alpha_P \cdot h_A \Rightarrow IR_P = IR}$ e.g. Q

Portfolio Q is a mixture of benchmark B and A

$\boxed{h_Q = \beta_Q h_B + \alpha_Q h_A}$, $\beta_Q = \dfrac{f_B \cdot \sigma_Q^2}{f_Q \cdot \sigma_B^2}$, $\alpha_Q = \dfrac{\sigma_Q^2}{f_Q \cdot \omega_A^2}$

. Total Holdings in risky assets for Portfolio A $\boxed{e_A = \dfrac{\alpha_C \cdot \omega_A^2}{\sigma_C^2}}$

. θ_P = residual return of portfolio P: $\boxed{IR_P = IR_Q \cdot Corr(\theta_P, \theta_Q)}$

. IR is related to SR: $\boxed{IR = \dfrac{\alpha_Q}{\omega_Q} = SR\left(\dfrac{\omega_Q}{\sigma_Q}\right)}$

. $\boxed{\alpha = IR \cdot \dfrac{V \cdot h_A}{\omega_A}}$

10. $\boxed{SR_\beta^2 = SR^2 - IR^2}$

Identities $V_R = V - \beta\beta^T\sigma_B^2$

$U_R(P) = \alpha_P - \lambda\omega_P^2 -$

$= h_P^T\alpha - \lambda h_P^T V_R h_P$

$= h_A^T\alpha - \lambda h_A^T V h_A$

$= h_{Q}^T\alpha - \lambda h_R^T V h_R$

OPTIMAL POLICY and OPTIMAL VALUE ADDED -- Portfolio

$\boxed{\underset{h_P}{Max}\ VA = \alpha - \lambda_R \omega_P^2 = h_P^T\alpha - \lambda_R h_P^T V R h_P}$

Proposition 2

$\boxed{h_P^* = \beta_P h_B + \dfrac{IR}{2\lambda_R \omega_A} h_A}$ $\boxed{VA^* = \dfrac{IR^2}{4\lambda_R}}$ $\boxed{\omega_P^* = \dfrac{IR}{2\lambda_R}}$

The $\beta = 1$ ACTIVE FRONTIER

$\begin{cases} \text{efficient} \\ \beta = 1 \\ \text{min risk at fixed return} \end{cases}$

$$\Rightarrow \begin{cases} \sigma_P^2 = \sigma_B^2 = \omega_P^2 \\ f_P = f_B + \alpha_P \\ \omega_P^2 = \dfrac{\alpha_P^2}{IR^2} \end{cases} \Rightarrow \boxed{\sigma_P^2 = \sigma_B^2 + \frac{1}{IR^2}\cdot(f_P - f_B)^2}$$

The ACTIVE POSITION Y: No ACTIVE CASH, No ACTIVE β

$e_Y = 0$

. Residual holdings of portfolio C: $\boxed{h_{CR} = h_C - \beta h_B}$

. Portfolio Y: $\boxed{h_Y = \dfrac{h_A}{\omega_A} - \dfrac{IR_C}{IR} \cdot \dfrac{h_{CR}}{\omega_C}}$

Proposition 3:

1. Portfolio Y has a zero beta: $\boxed{\beta_Y = 0}$
2. Portfolio Y has total and residual variance $\boxed{\sigma_Y^2 = \omega_Y^2 = 1 - \left(\dfrac{IR_C}{IR}\right)}$
3. Portfolio Y has an alpha $\boxed{\alpha_Y = IR\left[1 - \left(\dfrac{IR_C}{IR}\right)^2\right]}$
4. Portfolio has zero cash position: $\boxed{e_Y = 0}$ (i.e. long risky = short risky)
5. $\boxed{IR_Y = IR \cdot \sqrt{1 - \left(\dfrac{IR_C}{IR}\right)^2} = IR \cdot \sqrt{1 - Corr^2\{\theta_Q, \theta_C\}}}$

The OPTIMAL NO ACTIVE BETA & NO ACTIVE CASH PORTFOLIO

$\boxed{MAX\ h_P^T\alpha - \lambda_R h_P^T V R h_P \quad s.t. \quad \underbrace{h_P^T\beta = 1}_{\text{no active beta}} \text{ and } \underbrace{h_P^T e = e_B}_{\text{No active cash}}}$

Proposition 4. $\boxed{h_P^* = h_B + \left(\dfrac{IR}{2\lambda_R}\right) h_Y}$

Proposition 5. Time T years, residual returns have same mean, stand. dev, THEN $\boxed{\dfrac{E(\theta)}{Std(\theta)} = \sqrt{K}\left(\dfrac{\alpha}{\omega}\right) = \sqrt{\dfrac{T}{\Delta t}}\left(\dfrac{\alpha}{\omega}\right)}$

30/40

EXAM I

MAP6437
Prof. Kercheval
Oct. 6, 2003

Name: Chuan Dai

Directions: Do all problems. No books or notes, except one sheet of notes in your own hand is allowed.

Begin each problem on a new sheet of paper. Arrange the solutions in numerical order and staple them together with this sheet on top.

There are four problems, each worth 10 points. For problems 2, 3 and 4, only clear and full explanations will receive full credit.

8

1. Short answer. (Two points each.)

(a) The CAPM implies Q = ?

(b) Doubling the breadth of a strategy will increase IR by a factor of ?

(c) For any portfolio P, $\beta_{P:Q} f_Q = ?$

(d) In our simple model of market impact, fractional price change is proportional to what function of trade size?

(e) What is the minimum risk beta=1 portfolio?

5

2. Assume stock 1 and stock 2 have perfectly corrrelated residual returns.

Derive an expression for the correlation of the excess returns of the two stocks as a function of β_1, β_2, σ_1, σ_2, and σ_B, where B denotes the benchmark with respect to which the stock betas are defined.

10

3. (a) Define what is meant by the "efficient frontier".

(b) Prove that every portfolio on the efficient frontier is a linear combination of $\mathbf{h}_C$ and $\mathbf{h}_Q$.

7

4. Assume $\mathbf{f}$ and V are given. Use the method of Lagrange Multipliers to find the portfolio P maximizing

$$U_R(P) = \alpha_P - \lambda\omega_P^2$$

subject to the constraint $\beta_P = 0$.

Chuan Dai

(1) (a) $Q = M$

(b) $IR_2 = IC\sqrt{2BR_1} = \sqrt{2}\, IC\sqrt{BR_1} = \sqrt{2}\, IR_1$

(c) $\beta_{P:Q} f_Q = f_P$

(d) $\frac{\Delta P_i}{P_i} = \tau \sigma_i \sqrt{\frac{V}{V_d}}$

$\frac{0}{2}$ (e) $\cancel{h_A}$ h_B. $\beta_A = 0$.

(2) $\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1 \sigma_2}$

5 $\sigma_{1,2} = cov(r_1, r_2) = cov(\beta_1 r_B + \varepsilon_1, \beta_2 r_B + \varepsilon_2) = \beta_1\beta_2 cov(r_B, r_B) + cov(\varepsilon_1, \varepsilon_2) = \beta_1\beta_2\sigma_B^2 + 1$ $\omega_1\omega$

so $\beta_{1,2} = \frac{\sigma_{1,2}}{\sigma_1\sigma_2} = \frac{\beta_1\beta_2\sigma_B^2 + 1}{\sigma_1\sigma_2}$ X

(3) (a) E.F. is defined by $\min h^T V h$ subject to $h^T f = f_P$ and $h^T e = 1$

that means all portfolio on E.F. are fully invested.

(b) $\begin{cases} \nabla(h^T V h) = u_1 \nabla(h^T f) + u_2 \nabla(h^T e) \\ h^T f = f_P \\ h^T e = 1 \end{cases} \Rightarrow \begin{aligned} 2Vh &= u_1 f + u_2 e \\ h &= \frac{u_1}{2} V^{-1} f + \frac{u_2}{2} V^{-1} e \end{aligned}$

since $h_C = \frac{V^{-1}e}{e^T V^{-1} e}$ $h_Q = \frac{1}{e_Q}\frac{V^{-1}f}{f^T V^{-1} f}$ $\Rightarrow h = c_1 h_Q + c_2 h_C$

$\begin{cases} (c_1 h_Q + c_2 h_C)^T f = f_P \\ (c_1 h_Q + c_2 h_C)^T e = 1 \end{cases} \Rightarrow \begin{cases} c_1 f_Q + c_2 f_C = f_P \\ c_1 + c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{f_Q - f_P}{f_Q - f_C} \\ c_2 = \frac{f_P - f_C}{f_Q - f_C} \end{cases}$

$\Rightarrow h_P = \left(\frac{f_Q - f_P}{f_Q - f_C}\right) h_C + \left(\frac{f_P - f_C}{f_Q - f_C}\right) h_Q$

Chuan Dai

(4) $U_R(P) = \alpha_P - \lambda \omega_P^2 = h_P^T \alpha - \lambda h_P^T V_R h_P = h_A^T \alpha - \lambda h_A^T V h_A$

subject to $h_P^T \beta = 0$ where $V_R = V - \beta\beta^T \sigma_B^2$

$$\begin{cases} f = \alpha - 2\lambda V_R h_P = u\beta \\ h_P^T \beta = 0 \end{cases} \Rightarrow \begin{cases} h_P = \frac{V_R^{-1}\alpha}{2\lambda} - \frac{u}{2\lambda} V_R^{-1}\beta \\ h_P^T \beta = 0 \end{cases}$$

$$\left(\frac{V_R^{-1}\alpha - u V_R^{-1}\beta}{2\lambda}\right)^T \beta = 0 \Rightarrow \alpha^T V_R^{-1}\beta - u\beta^T V_R^{-1}\beta = 0$$

$$\Rightarrow u = \frac{\alpha^T V_R^{-1}\beta}{\beta^T V_R^{-1}\beta}$$

$$\Rightarrow h_P = \frac{1}{2\lambda}\left[V_R^{-1}\alpha - \frac{\alpha^T V_R^{-1}\beta}{\beta^T V_R^{-1}\beta} \cdot V_R^{-1}\beta\right]$$ where $V_R = V - \beta\beta^T\sigma_B^2$

---

this was assuming $\beta = 1$!

$\max U_P(P) = \max_{h_A} h_A^T \alpha - \lambda h_A^T V h_A$ sub to $h_P^T \beta = (h_A + h_B)^T \beta = h_A^T \beta + h_B^T \beta =$

$\Rightarrow h_A^T \beta = -1$ ✗

$$\begin{cases} \alpha - 2\lambda V h_A = u\beta \\ h_A^T \beta = 1 \end{cases} \Rightarrow$$

$h_A = \frac{V^{-1}\alpha}{2\lambda} - \frac{u}{2\lambda} V^{-1}\beta$ since $h_A = \frac{V^{-1}\alpha}{\alpha^T V^{-1}\alpha} = \sigma_A^2 V^{-1}\alpha$

$h_A = c_1 h_A + c_2 h_B$ $\quad h_B = \frac{V^{-1}\beta}{\beta^T V^{-1}\beta} = \sigma_B^2 V^{-1}\beta$

$$\Rightarrow \begin{cases} c_1 = \frac{1}{2\lambda\sigma_A^2} \\ c_2 = 1 - \frac{1}{2\lambda\sigma_A^2} \end{cases}$$

$(1 - c_1) h_A = c_2 h_B$ $\quad h_A = \left(\frac{c_2}{1-c_1}\right) h_B$

$h_A^T \beta = \frac{c_2}{1-c_1} h_B^T \beta = 1 \Rightarrow c_2 = c_1 - 1$

$h_A = h$

$h_P = h_A + h_B$

$= c_1 h_A + (c_1 - 1) h_B$

$= c_1(h_A - h_B) - h_B$

$= c_1(h_A - h_B)$

$= \frac{1}{2\lambda\sigma_A^2}(h_A - h_B)$

7

Financial Engineering EXAM II Practice (Fall 2002 exam)

1.(10 points) Indicate TRUE or FALSE for each item.

X(a) In a complete market, all instruments are tradable.-

(b) In the continuous Black-Scholes case, a strategy holding ϕ_t units of the stock and ψ_t units of the bond is self-financing if and only if $S_t d\phi_t + B_t d\psi_t = 0$.

(c) In a risk-neutral world, all stocks have the same expected returns.✓

(d) In our market models, you cannot short cash.✓

X(e) Having zero drift is equivalent to being a martingale.

2.(10 points) Consider a futures contract on a stock S in a discrete, viable market. Suppose the margin account for the contract is deterministic.

Prove that the futures price is equal to the forward price.

3. (20 points) Consider a discrete market model.

(a) Prove: if the market admits an arbitrage, then it admits a local arbitrage.

(b) Prove that if a market admits a strong arbitrage, then it admits an arbitrage, but not conversely.

4. (20 points) Consider a Black-Scholes stock model

$$
\begin{aligned}
S_t &= S_0 \exp(\sigma W_t + \mu t) \\
B_t &= \exp(rt)
\end{aligned}
$$

where W_t is a Brownian motion with respect to a measure P.

(a) What is the measure Q with respect to which the discounted stock is a martingale? (Indicate its Radon-Nikodym derivative with respect to P.)

(b) Express the Q-brownian motion $\tilde{W}_t$ in terms of W_t and express S_t in terms of $\tilde{W}_t$.

(c) Find the time zero value of a claim that pays 1 if $S_T > 2S_0$ and zero otherwise. Express your answer as an explicit integral, but do not evaluate it.

(d) Compute an explicit value for $E_Q[S_T]$.

3/4

Homework #1 Chuan Dai ①

① Show that $cov(r_m, \theta_p) = 0$

$cov(r_m, \theta_p)$

$= cov(r_m, r_p - \beta_p r_m)$

$= cov(r_m, r_p) - cov(r_m, \beta_p r_m)$

$= cov(r_m, r_p) - \beta_p cov(r_m, r_m)$

$= cov(r_m, r_p) - \beta_p var(r_m)$

$= cov(r_m, r_p) - \frac{cov(r_m, r_p)}{\cancel{var(r_m)}} \cdot \cancel{var(r_m)}$

$= cov(r_m, r_p) - cov(r_m, r_p)$

$= 0$

② Given factor model $\vec{r} = X\vec{b} + \vec{u}$ & $cov(u_i, u_j) = 0$ for $i = j$, proof $V = XFX^T + \Delta$ $V_{ij} = cov(r_i, r_j)$

$cov(r_i, r_j) = cov(\sum_{k=1}^{k} x_{ik} b_k + u_i, \sum_{k=1}^{k} x_{jk} b_k + u_j)$

if $i \neq j$ $cov(r_i, r_j) = cov(\sum_{k=1}^{k} x_{ik} b_k, \sum_{k=1}^{k} x_{jk} b_k)$

$+ cov(\cancel{\sum_{k=1}^{k} x_{ik} b_k}, u_j) + cov(\cancel{\sum_{k=1}^{k} x_{jk} b_k}, u_i)$ $\to 0$, $\to 0$

$+ cov(\cancel{u_i}, u_j)$ $\to 0$

$= \sum_{k=1}^{k} \sum_{k=1}^{k} x_{ik} x_{jk} cov(b_k, b_k)$ X

if $i = j$ $cov(r_i, r_i) = var(r_i)$

$= var(\sum_{k=1}^{k} x_{ik} b_k) + var(u_i)$

$+ 2 cov(\cancel{\sum_{k=1}^{k} x_{ik} b_k}, u_i)$ $\to 0$

so

$$V = \underbrace{\begin{bmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \ddots & \vdots \\ x_{N1} & \cdots & x_{Nk} \end{bmatrix}}_{N \times K} \underbrace{\begin{bmatrix} cov(b_1, b_1) & \cdots & cov(b_1, b_k) \\ \vdots & \ddots & \vdots \\ cov(b_k, b_1) & \cdots & cov(b_k, b_k) \end{bmatrix}}_{K \times K} \underbrace{\begin{bmatrix} x_{11} & \cdots & x_{N1} \\ \vdots & \ddots & \vdots \\ x_{1k} & \cdots & x_{Nk} \end{bmatrix}}_{K \times N}$$

$$+ \underbrace{\begin{bmatrix} var(u_1) & & 0 \\ & \ddots & \\ 0 & & var(u_N) \end{bmatrix}}_{N \times N}$$

$$= X F X^T + \Delta$$

ex3 defend the choice of $U = f_p - \lambda \sigma_p^2$

proof: $U = h_p^T f - \lambda \left(\sqrt{h_p^T V h_p}\right)^n$

An utility fcn must be ① strict increasing
② quasi-concave ③ continuous

Suppose there only one asset

$u = hf - \lambda (h\sqrt{V})^n \Rightarrow u' = f - \lambda n \sqrt{V^n} h^{n-1} > 0$

$u'' = -\lambda n(n-1)\sqrt{V^n}\, h^{n-2} < 0$ (-∞<

in this case only $n = 2$
can gurantee $u'' < 0$) independent of h
so n must equal 2

← otherwise it may depend on h

④

④ show the Residual utility fcn

$V_R(P) = \alpha_P - \lambda w_P^2$

$= h_P^T \alpha - \lambda h_P^T V_R h_P$

$= h_A^T \alpha - \lambda h_A^T V h_A$

given ($V_R = V - \beta\beta^T \sigma_B^2$)

proof: assume $\beta_P = 1$ $\beta_A = 0$

we have $h_A = h - h_B = h_R$ and $\vec{f}_P = \beta_P \vec{f}_B + \vec{\alpha} \Rightarrow \vec{\alpha} = \vec{f}_P - \vec{f}_B$

$\alpha_P = f_P - \beta_P f_B = f_P - f_B = h_P^T(\vec{f} - \vec{f}_B) = h_P^T \vec{\alpha}$

$\left.\begin{array}{l} h^T = h_A^T + h_B^T \Rightarrow h_A^T \alpha + h_B^T \alpha = h_P^T \alpha \\ h_B^T \alpha = 0 \end{array}\right\} \Rightarrow h_P^T \alpha = h_A^T \alpha$

$\sigma_P^2 = \beta_P^2 \sigma_B^2 + w_P^2 \Rightarrow w_P^2 = \sigma_P^2 - \sigma_B^2$

$\lambda h_P^T V_R h_P = \lambda h_P^T (V - \beta\beta^T \sigma_B^2) h_P$

$= \lambda (h_P^T V h_P - h_P^T \beta\beta^T \sigma_B^2 h_P)$

$= \lambda (\sigma_P^2 - \sigma_B^2)$

$= \lambda w_P^2$

$\lambda h_A^T V h_A = \lambda (h_P^T - h_B^T) V (h_P - h_B)$

$= \lambda (h_P^T V h_P - h_B^T V h_B)$

$= \lambda (\sigma_P^2 - \sigma_B^2)$

$= \lambda w_P^2$

P19 ① Is there an expected returns vector f that will lead to portfolio Q 100% invested in GE, with no other position?

Yes, since Fact from portfolio theory
for any portfolio P, $f_P = \beta_{PQ} f_Q$
or $f = \beta_{Q} f_Q$,
if we know 100% invested GE is the optimal portfolio Q, there exist unique $\beta_Q f_Q$, so if we know

$f = \beta_Q f_Q$ Q-optimal risky portfolio

$$\Rightarrow f = \beta_{GE} f_{GE} = \frac{V h_{GE}}{h_{GE}^T V h_{GE}} f_{GE}$$

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P19 ②

(a)

$$f_B = \beta_{B:a} f_a = .965 \times 6\% = 5.79\%$$

$$f_c = \beta_{C:a} f_a = .831 \times 6\% = 4.986\%$$

$$f_a = 6\%$$

(b)

$$f_B = \beta_{B:B} f_B + \alpha_B \Rightarrow \alpha_B = 0$$

$$f_a = \beta_{a:B} f_B + \alpha_a \Rightarrow \alpha_a = f_a - \beta_{a:B} f_B = 6\% - 1.004 \times 5.79\% = 0.186$$

$$f_c = \beta_{C:B} f_B + \alpha_c \Rightarrow \alpha_c = f_c - \beta_{C:B} f_B = 4.986\% - .865 \times 5.79\% = -0.0$$

(c)

$$\sigma_B^2 = \beta_{B:B}^2 \sigma_B^2 + w_B^2 \Rightarrow w_B = 0$$

$$\sigma_a^2 = \beta_{a:B}^2 \sigma_B^2 + w_a^2 \Rightarrow w_a = \sqrt{\sigma_a^2 - \beta_{a:B}^2 \sigma_B^2}$$

$$= \sqrt{(15.82)^2 - (1.004)^2 (15.50)^2}$$

$$= 2.8454$$

$$\sigma_c^2 = \beta_{C:B}^2 \sigma_B^2 + w_c^2 \Rightarrow w_c = \sqrt{\sigma_c^2 - \beta_{C:B}^2 \sigma_B^2}$$

$$= \sqrt{(14.42)^2 - (.865)^2 (15.50)^2}$$

$$= 5.308$$

(d)

$$SR(B) = \frac{f_B}{\sigma_B} = 5.79/15.50 = 0.3735$$

$$SR(Q) = \frac{f_Q}{\sigma_Q} = 6/15.82 = 0.3793$$

$$SR(C) = \frac{f_C}{\sigma_C} = 4.986/14.42 = 0.3458$$

(e)

$$IR(B) = \frac{\alpha_B}{w_B} \Rightarrow IR(B) = 0$$

$$IR(Q) = \frac{\alpha_Q}{w_Q} = 0.1868/2.8454 = 0.0656$$

$$IR(C) = \frac{\alpha_C}{w_C} = -0.0224/5.308 = -0.0042$$

P19 (3)

$$IR_A = IR_B \Rightarrow IC_A\sqrt{BR_A} = IC_B\sqrt{BR_B}$$

$$\Rightarrow .02\sqrt{250 \cdot 4} = IC_B\sqrt{4 \cdot 4}$$

$$\Rightarrow IC_B = 0.1581$$

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P41 ①

ⓐ since we want to maximize $\alpha - \lambda w^2$ and residual return are uncorrelated

$$h_n^* = \frac{1}{2\lambda}\frac{\alpha_n}{w_n^2}$$

for $\alpha = 1\%$ for buys and $\alpha = -1\%$ for sell

so the $h_{AT\&T} = \frac{1}{2\lambda} \cdot \frac{1\%}{w_{AT\&T}^2}$ is the largest holding

since residual risk is 15.89% for ATT

ⓑ $\alpha = IC \times \text{volatility} \times \text{score}$

$= 0.09 \times w \times \text{Score}$

$$h_n^* = \frac{0.09 \times w_n S_n}{2\lambda \cdot w_n^2} = \frac{0.09}{2\lambda}\frac{S_n}{w_n}$$

so the $h_{AT\&T} = \frac{0.09}{2\lambda} \times \frac{1}{w_{AT\&T}}$ is the largest holding

since residual risk is 15.89% for ATT

11/11 5/7 +lee

Fin Eng HW # 2 Chuan Dai ①

① verify $\nabla(h^T V h) = 2hV$

$$h^T V h = \underset{1\times N}{[h_1, h_2, \cdots, h_N]} \underset{N\times N}{\begin{bmatrix} V_{11} & \cdots & V_{1N} \\ \vdots & \ddots & \\ V_{N1} & \cdots & V_{NN} \end{bmatrix}} \underset{N\times 1}{\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{bmatrix}}$$

$$= \left[\sum_{i=1}^N h_i V_{i1}, \sum_{i=1}^N h_i V_{i2}, \cdots \sum_{i=1}^N h_i V_{iN}\right] \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{bmatrix}$$

$$= \sum_{j=1}^N h_j \sum_{i=1}^N h_i V_{ij}$$

$$= \sum_{j=1}^N \sum_{i=1}^N h_i V_{ij} h_j$$

$$f = f(\vec{x}) \Rightarrow \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots \frac{\partial f}{\partial x_N}\right)$$

$$\nabla(h^T V h) = \nabla\left(\sum_{j=1}^N \sum_{i=1}^N h_i V_{ij} h_j\right) = \begin{bmatrix} \dfrac{\partial \sum_{j=1}^N \sum_{i=1}^N h_i V_{ij} h_j}{\partial h_1} \\ \vdots \\ \dfrac{\partial \sum_{j=1}^N \sum_{i=1}^N h_i V_{ij} h_j}{\partial h_N} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^N h_i V_{i1} + \sum_{j=1}^N h_j V_{1j} \\ \vdots \\ \sum_{i=1}^N h_i V_{iN} + \sum_{j=1}^N h_j V_{Nj} \end{bmatrix} \xrightarrow[\text{is symmetric } V_{ij} = V_{ji}]{\text{since cov metrix } V} = \begin{bmatrix} 2\sum_{j=1}^N V_{1j} h_j \\ \vdots \\ 2\sum_{j=1}^N V_{Nj} h_j \end{bmatrix}$$

$$\begin{aligned}\mathrm{cov}(r_A, r_B) &= \mathrm{cov}(\beta_A r_M + \alpha_A,\ \beta_B r_M + \alpha_B) \\ &= \beta_A \beta_B [\mathrm{cov}(\cancel{r_M}, \alpha_B)^{\to 0} + \mathrm{cov}(\cancel{r_M}, \alpha_A)^{\to 0} + \mathrm{cov}(\cancel{\alpha_A}, \alpha_B)^{\to 0} + \mathrm{cov}(r_M, \\ &= \beta_A \beta_B \,\mathrm{var}\, r_M \\ &= 1.15 \times 0.95 \times 20^2 \\ &= 437\end{aligned}$$

$$\rho_{A,B} = \frac{\mathrm{cov}(r_A, r_B)}{\sigma_A \cdot \sigma_B} = \frac{437}{35 \times 33} = 0.38$$

P38 1 ~ 4

① $\left.\begin{array}{l}\beta_c = h_c^T \beta \\ h_c = \sigma_c^2 V^{-1} e \\ \beta = \dfrac{V h_M}{\sigma_M^2}\end{array}\right\} \Rightarrow \beta_c = (\sigma_c^2 V^{-1} e)^T \dfrac{V h_M}{\sigma_M^2}$

$$= \frac{\sigma_c^2 e^T h_M}{\sigma_M^2}$$

since market portfolio is fully invested

$e^T h_M = 1$

$$\therefore \beta_c = \frac{\sigma_c^2}{\sigma_M^2}$$

(3) prove Q is the only fully invested portfolio with max SR

$$h_Q = \frac{h_q}{e_q}$$

$$h_Q^T e = \left(\frac{h_q}{e_q}\right)^T e = \frac{h_q^T e}{e_q} = \frac{e_q}{e_q} = 1$$

$\therefore$ it's fully invested

$$SR_Q = \frac{f_Q}{\sigma_Q} = \frac{h_Q^T f}{\sqrt{h_Q^T V h_Q}} = \frac{\frac{h_q^T}{e_q} f}{\frac{1}{e_q}\sqrt{h_q^T V h_q}} = \frac{f_q}{\sigma_q}$$

$$= \sqrt{f^T V^{-1} f}$$

$\therefore$ it has max SR

4 $\max_{f} f_p - \lambda \sigma_p^2$ subject it's fully invested

$$\Rightarrow \max_{h_p} h_p^T f - \lambda h_p^T V h_p \quad \text{subject to } h_p^T e = 1$$

$$\Rightarrow \text{solve for } \begin{cases} \nabla (h_p^T f - \lambda h_p^T V h_p) = \nabla (u h_p^T e) \\ h_p^T e = 1 \end{cases}$$

$$\Rightarrow \begin{cases} f - 2\lambda V h_p = u e \\ h_p^T e = 1 \end{cases}$$

$$\Rightarrow \begin{cases} h_p = \frac{V^{-1} f}{2\lambda} - \frac{u V^{-1} e}{2\lambda} \\ h_p^T e = 1 \end{cases}$$

$$\Rightarrow h_p^T e = \left(\frac{V^{-1} f - u V^{-1} e}{2\lambda}\right)^T e = 1$$

$$V^T = V \Rightarrow (V^{-1})^T = V^{-1}$$

$$\Rightarrow f^T V^{-1} e - u e^T V^{-1} e = 2\lambda$$

$$\Rightarrow \frac{u}{\sigma_c^2} = f^T V^{-1} e - 2\lambda$$

$$\Rightarrow u = \sigma_c^2 (f^T V^{-1} e - 2\lambda)$$

$$= f^T \sigma_c^2 V^{-1} e - 2\lambda \sigma_c^2$$

$$= f^T h_c - 2\lambda \sigma_c^2$$

$$= f_c - 2\lambda \sigma_c^2$$

$$h_p^* = \frac{V^{-1} f - (f_c - 2\lambda \sigma_c^2) V^{-1} e}{2\lambda}$$

$$f^* = f_p^* = h_p^{*T} f = \frac{1}{2\lambda} [V^{-1} f - (f_c - 2\lambda \sigma_c^2) V^{-1} e]^T f$$

$$= \frac{1}{2\lambda} [f^T V^{-1} f - (f_c - 2\lambda \sigma_c^2) e^T V^{-1} f]$$

$$= \frac{1}{2\lambda} [f^T V^{-1} f - f_c e^T V^{-1} f + 2\lambda \sigma_c^2 e^T V^{-1} f]$$

see page

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note P54 1~4

① $U_R(z) = U_R(0) + \Delta U_R(2z - z^2)$

since $U_R(0) = U_R(0) + \Delta U_R(2\cdot 0 - 0^2) = 0$

$\Rightarrow U_R(0) = 0$

$\Rightarrow U_R(0) = \Delta U_R(2z - z^2)$

since $z = \frac{200\%}{200\%}$

$U(1) = \Delta U_R(2\cdot 1 - 1) = 200 \Rightarrow \Delta U_R = 200$

$z = \frac{100\%}{200\%} = \frac{1}{2}$

$\Rightarrow U(\frac{1}{2}) = \Delta U_R(2\times\frac{1}{2} - (\frac{1}{2})^2) = \frac{3}{4}\Delta U_R = 150$

∴ the added value is 150 basic point

② $U(z) = \Delta U_R(2z - z^2) = 100$

$\Rightarrow 2z - z^2 = \frac{1}{2}$

$\Rightarrow 2z^2 - 4z + 1 = 0$

$\Rightarrow z = \frac{4 \pm \sqrt{16 - 4\cdot 2}}{4} = 1 \pm \frac{\sqrt{2}}{2}$

turnover reach 100 basis point

$= 200\% \cdot (1 - \frac{\sqrt{2}}{2}) = 58.58\%$

or $200\%(1 + \frac{\sqrt{2}}{2}) = 341.42\%$

regarding transaction cost we should choose turnover rate 58.58%

$$= \sigma_c^2 e^T V^{-1} f + \frac{1}{2\lambda}\left[f^T V^{-1} f - f_c e^T V^{-1} f \right]$$

$$= h_c^T f + \frac{1}{2\lambda}\left[\frac{1}{\sigma_q^2} - f_c \frac{e^T \frac{V^{-1} f}{\sigma_q^2}}{\sigma_q^2} \right] \qquad f_c \sigma_q^2 = e_q \sigma_c^2$$

$$= f_c + \frac{1}{2\lambda}\left[\frac{f_c}{e_q \sigma_c^2} - f_c \frac{f_q}{\sigma_q^2} \right]$$

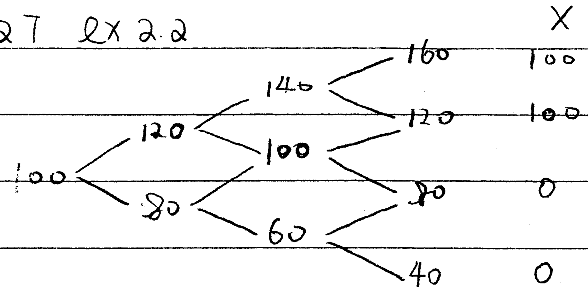
$$= f_c + \frac{1}{2\lambda} \frac{f_c(f_q - f_c)}{\sigma_c^2}$$

$$= f_c + \frac{1}{2\lambda k}$$

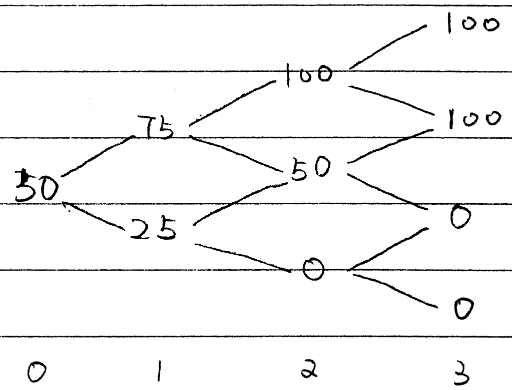
$|\frac{1}{2}| = 2\frac{1}{2}/3$

HW #3 Chuan Dai ①

① P27 ex 2.2



$$q = \frac{S_{now} - S_{down}}{S_{up} - S_{down}} = \frac{1}{2}$$ for all the node



for time 0

$$\phi = \frac{f_{up} - f_{down}}{S_{up} - S_{down}} = \frac{75-25}{120-80} = \frac{5}{4}$$ buy 1.25 unit, borrow 75

time 1 up to 120: $\phi = \frac{100-50}{140-100} = \frac{5}{4}$ doesn't change

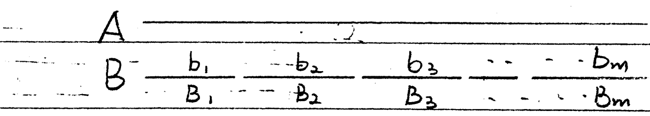
time 2 up to 140: $\phi = \frac{0}{160-120} = 0$ sell all the stock get 100

time 3 down

2 prove tower property

If $\mathcal{B}$ is a refinement of $\mathcal{a}$ then

$$E[E(X|\mathcal{B})|\mathcal{a}] = E[E(X|\mathcal{a})|\mathcal{B}] = E[X|\mathcal{a}]$$



$A \in \mathcal{a}$

Let $Y = E(Y|\mathcal{B})$

$$E[E(X|\mathcal{B}|\mathcal{a}](w) \overset{w \in A}{=\!=} \sum_{\eta \in A} Y(\eta)P(\eta) \Big/ P(A)$$

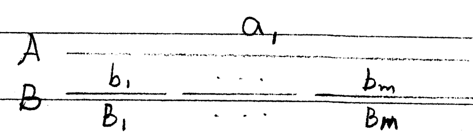
$$= \frac{b_1 P(B_1) + b_2 P(B_2) + \cdots b_m P(B_m)}{P(A)}$$

since $b_i = \dfrac{\sum_{\eta \in B_i} X(\eta)P(\eta)}{P(B_i)}$ plug into above

$$\Rightarrow \frac{\sum_{\eta \in B_1} X(\eta)P(\eta) + \sum_{\eta \in B_2} X(\eta)P(\eta) + \cdots \sum_{\eta \in B_m} X(\eta)P(\eta)}{P(A)}$$

$$= \frac{\sum_{\eta \in A} X(\eta)P(\eta)}{P(A)}$$

O.T.O.H



$A \in \mathcal{a}$

Let $W = E(X|\mathcal{a})$

$$E[E(X|\mathcal{a})|\mathcal{B}](w) \overset{w \in B_i}{=\!=} \sum_{\eta \in B_i} W(\eta)P(\eta) \Big/ P(B_i)$$

$$= \frac{a_1 \sum_{\eta \in B_i} P(\eta)}{P(B_i)}$$

$$= a_1$$

$3\frac{1}{2}/4$

①

HW 4

Chuan Dai

1. Prove $E_\nu[X|F_t] = E_\lambda[X\frac{d\nu}{d\lambda}|F_t] / E_\lambda[\frac{d\nu}{d\lambda}|F_t]$

proof:

$$E_\lambda[X\frac{d\nu}{d\lambda}|F_t] = \int_A X\frac{d\nu}{d\lambda}d\lambda / \int_A d\lambda$$

$A = ?$

$$= \int_A X d\nu / \int_A d\lambda$$

$$E_\lambda[\frac{d\nu}{d\lambda}|F_t] = \int_A \frac{d\nu}{d\lambda}d\lambda / \int_A d\lambda$$

$$= \int_A d\nu / \int_A d\lambda$$

$$E_\lambda[X\frac{d\nu}{d\lambda}|F_t] / E_\lambda[\frac{d\nu}{d\lambda}|F_t]$$

$$= \frac{\int_A X d\nu / \int_A d\lambda}{\int_A d\nu / \int_A d\lambda}$$

$$= \int_A X d\nu / \int_A d\nu$$

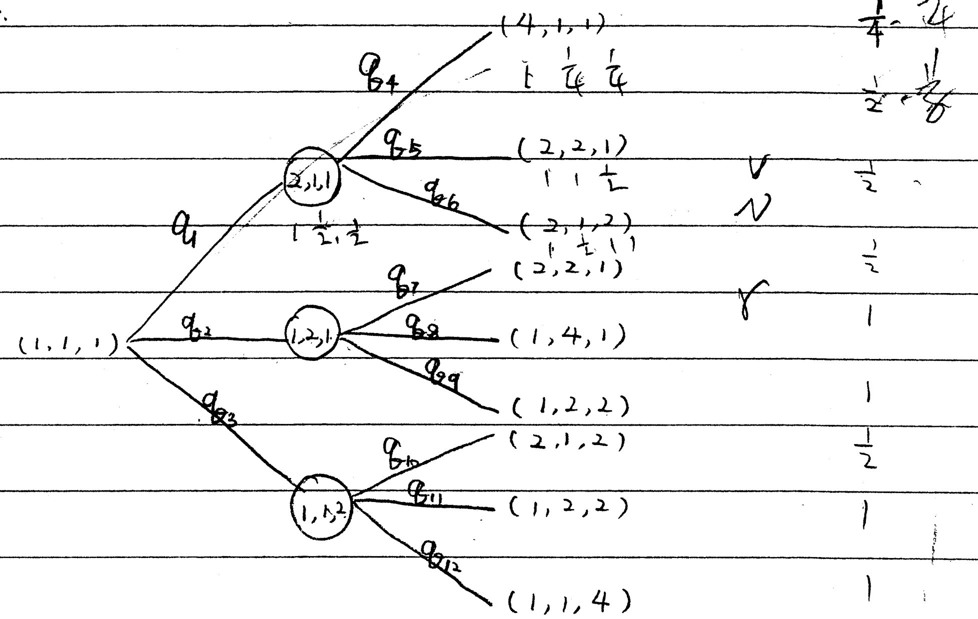
$$= E_\nu[X|F_t]$$

(A is the subset correspond to F_t)

an atom of $\mathcal{F}_t$.

③

3.



ⓐ prove there is no arbs

$$\begin{cases} q_1 + q_2 + q_3 = 1 \\ \frac{1}{2}q_1 + 2q_2 + q_3 = 1 \\ \frac{1}{2}q_1 + q_2 + 2q_3 = 1 \end{cases} \Rightarrow \begin{cases} q_1 = \frac{1}{2} \\ q_2 = \frac{1}{4} \\ q_3 = \frac{1}{4} \end{cases}$$

$$\begin{cases} q_4 + q_5 + q_6 = 1 \\ \frac{1}{4}q_4 + q_5 + \frac{1}{2}q_6 = \frac{1}{2} \\ \frac{1}{4}q_4 + \frac{1}{2}q_5 + q_6 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} q_4 = \frac{1}{2} \\ q_5 = \frac{1}{4} \\ q_6 = \frac{1}{4} \end{cases}$$

$$\begin{cases} q_7 + q_8 + q_9 = 1 \\ q_7 + 4q_8 + 2q_9 = 2 \\ \frac{1}{2}q_7 + q_8 + 2q_9 = 1 \end{cases} \Rightarrow \begin{cases} q_7 = \frac{1}{2} \\ q_8 = \frac{1}{4} \\ q_9 = \frac{1}{4} \end{cases}$$

4. # 1 P49

$Z \sim N(0,1)$ $X_t = \sqrt{t}\, Z$, is X_t a B.M

proof: ① $X_0 = \sqrt{0}\, Z = 0$

② $X_t = \sqrt{t}\, Z \sim N(0,t)$

③ $X_{t+s} - X_s = \sqrt{t+s}\, Z - \sqrt{s}\, Z$

$= (\sqrt{t+s} - \sqrt{s})\, Z \sim N(0, (\sqrt{t+s}-\sqrt{s})^2)$

depends on s

$\Rightarrow$ it's not a B.M

5. # 2 P49

W_t and $\widetilde{W}_t$ are two independent B.M

ρ is a constant $-1 < \rho < 1$

is $X_t = \rho W_t + \sqrt{1-\rho^2}\, \widetilde{W}_t$ a B.M

proof: ① $X_0 = \rho W_0 + \sqrt{1-\rho^2}\, \widetilde{W}_0 = 0$

② $X_t = \rho W_t + \sqrt{1-\rho^2}\, \widetilde{W}_t$

$$\left.\begin{array}{l} \rho W_t \sim N(0, \rho^2 t) \\ \sqrt{1-\rho^2}\, \widetilde{W}_t \sim N(0, (1-\rho^2) t) \end{array}\right\} \Rightarrow$$

$X_t \sim N(0,t)$

③ $X_{t+s} - X_s = \rho W_{t+s} + \sqrt{1-\rho^2}\, \widetilde{W}_{t+s} - \rho W_s - \sqrt{1-\rho^2}\, \widetilde{W}_s$

$= \rho(W_{t+s} - W_s) + \sqrt{1-\rho^2}\,(\widetilde{W}_{t+s} - \widetilde{W}_s)$

since W_t, $\widetilde{W}_t$ is B.M $\Rightarrow$

$$\begin{cases} W_{t+s} - W_t \sim N(0,t) \\ \widetilde{W}_{t+s} - \widetilde{W}_t \sim N(0,t) \end{cases}$$

7. Show $d(X_t \cdot Y_t) = X_t \, dY_t + Y_t \, dX_t + \sigma_t \rho_t \, dt$

Given $dX_t = \sigma_t \, dW_t + \mu_t \, dt$

$dY_t = \rho_t \, dW_t + \nu_t \, dt$

proof: Let $Z_t = \frac{1}{2}[(X_t+Y_t)^2 - (X_t^2 + Y_t^2)] = X_t Y_t$

$$\Rightarrow d(X_t \cdot Y_t) = \frac{1}{2} dZ_t = \frac{1}{2} d[(X_t+Y_t)^2 - (X_t^2+Y_t^2)]$$

$$d(X_t+Y_t) = (\sigma_t+\rho_t)dW_t + (\mu_t+\nu_t)dt$$

$$d[(X_t+Y_t)^2] = 2(X_t+Y_t)d(X_t+Y_t) + \frac{1}{2}2[d(X_t+Y_t)]^2$$

$$dX_t^2 = 2X_t \, dX_t + \frac{1}{2}2[dX_t]^2$$

$$dY_t^2 = 2Y_t \, dY_t + \frac{1}{2}2[dY_t]^2$$

$$\Rightarrow d(X_t Y_t) = \frac{1}{2}[2(X_t+Y_t)d(X_t+Y_t) + [d(X_t+Y_t)]^2 - 2X_t dX_t - (dX_t)^2 - 2Y_t dY_t - (dY_t)^2]$$

?

$$= \frac{1}{\cancel{2}}[\cancel{2}X_t dY_t + \cancel{2}Y_t dX_t + \cancel{2}dX_t dY_t]$$

$$= X_t dY_t + Y_t dX_t + (\sigma_t dW_t + \mu_t dt)(\rho_t dW_t + \nu_t$$

$$= X_t dY_t + Y_t dX_t + (\sigma_t \rho_t (dW_t)^2 + HOT)$$

$$= X_t dY_t + Y_t dX_t + \sigma_t \rho_t dt$$

Chuan Dai $\frac{2}{2} + 1EC$ (1)

5.1 for B~S model

$$S_t = S_0 \exp(\sigma W_t + \mu t).$$

$$B_t = \exp(rt)$$

$$Z_t = B_t^{-1} S_t = S_0 \exp[\sigma W_t + (\mu - r) dt]$$

by Itô

$$dS_t = S_t [\sigma dW_t + \mu dt + \tfrac{1}{2}\sigma^2 (dW_t)^2]$$

$$= S_t [\sigma dW_t + (\mu + \tfrac{1}{2}\sigma^2) dt]$$

$$dZ_t = Z_t [\sigma dW_t + (\mu - r) dt + \tfrac{1}{2}\sigma^2 (dW_t)^2]$$

$$= Z_t [\sigma dW_t + (\mu - r + \tfrac{1}{2}\sigma^2) dt]$$

5.2 $$d_t f(t,T) = \sigma d\widetilde{W}_t + \sigma^2 (T-t) dt$$

$$\Rightarrow f(t,T) = f(0,T) + \sigma \widetilde{W}_t + \int_0^t \sigma^2 (T-t) dt$$

$$= f(0,T) + \sigma \widetilde{W}_t + \sigma^2 t (T - \tfrac{1}{2} t)$$

$$r_t = f(t,t)$$

$$= f(0,t) + \sigma \widetilde{W}_t + \sigma^2 t (t - \tfrac{1}{2} t)$$

$$= f(0,t) + \sigma \widetilde{W}_t + \tfrac{1}{2}\sigma^2 t^2$$

$$dB_t = r_t B_t dt$$

$$B_t = \exp(\int_0^t r_s ds)$$

$$= \exp[\int_0^t (f(0,s) + \sigma \widetilde{W}_s + \tfrac{1}{2}\sigma^2 s^2) ds]$$

$$= \exp[\int_0^t f(0,u) du + \sigma \int_0^t \widetilde{W}_s ds + \tfrac{1}{6}\sigma^2 t^3]$$