STA 5106: Project # 2

Fall 2008/Due Date: December 11

Importance Sampling Using Tilted Distributions

1. **Goal**: Our goal is to estimate the right tail of an exponential random variable using ideas from importance sampling and tilted distributions.

2. **Problem Statement**: Let X be an exponential random variable with intensity λ (mean $\frac{1}{\lambda}$). For an constant $a > 0$, our goal is to estimate the probability:

$$\theta = Pr\{X > a\} = \lambda \int_a^\infty e^{-\lambda x} dx \ .$$

   Since the event $X > a$ occurs with a very small probability, i.e. θ is very small, the use of classical Monte Carlo approach will not be efficient. Instead, it is better to use importance sampling with a tilted density serving as the density to sample from.

3. **Methodology**: Perform the following steps:

   (a) State the general idea behind importance sampling.

   (b) For a scalar $t > 0$, define the tilted density as:

$$f_t(x) = \frac{e^{tx} f(x)}{M(t)}, \quad \text{where} \quad f(x) = \lambda e^{-\lambda x} \ .$$

   Compute $M(t) = \lambda \int_0^\infty e^{tx} e^{-\lambda x} dx$. Determine what form does the tilted density takes. How can we sample from this tilted density?

   (c) What should be the optimal amount of tilt t to estimate θ for a given a? Call it t^*.

   (d) Let $X_i \sim f_{t^*}(x)$ be independent random samples from the optimally tilted density. What should be the expression for $\tilde{\theta}_n$ Monte Carlo estimator that uses these samples.

   Compare it with the expression for $\hat{\theta}_n$, the classical Monte Carlo estimator that uses samples from $f(x)$ directly.

4. **Experimental Results**:

   (a) Implement the classical and the important-sampling estimators in Matlab. Compare the two estimators on the basis of: (a) computation time and (b) accuracy of estimation. In other words, take a sample size, say $n = 10000$, and study the computation time and the accuracy of the two estimators for that value of n. Choose $a = 6$ and $\lambda = 1$ for this study.

(b) Plot the convergence of $\hat{\theta}_n$ and $\tilde{\theta}_n$ versus n for the following values of $a = 2, 4, 6,$ and 8, with $\lambda = 1$. Plot them in the separate plots and clearly label your plots. Show your final estimates for these cases in a table.

5. **Report**: Prepare a full report of your experiments including introduction, methodology, matlab programs, results, and conclusions. Points are allocated to the level of presentation in the report.