Estimating Value at Risk with the Kalman Filter *

Cristina Sommacampagna †

Abstract

In this paper we develop a new approach to Value-at-Risk estimating the betas of the assets in the portfolio with the Kalman filter. This technique is applied to a portfolio of assets of an insurance company and is compared with the performances of two *traditional* methodologies: the approach based on the variance-covariance matrix of returns and the approach based on OLS Sharpe betas. The back testing analysis shows that the proposed technique is able to capture the dynamics of financial markets and is flexible enough to match the hedging purposes of a financial institution.

Keywords: Value-at-Risk; Kalman filter; Sharpe beta.

In questo articolo si sviluppa un nuovo approccio per il calcolo del Value-at-Risk che utilizza il filtro di Kalman per stimare il beta dei titoli di un portafoglio. Tale tecnica viene applicata al portafoglio azionario di una società assicurativa e confrontata con i metodi *tradizionali* basati sulla matrice di varianza-covarianza dei rendimenti e il beta di Sharpe stimato con i minimi quadrati ordinari. L'analisi di back testing evidenzia che la metodologia proposta è in grado di cogliere la dinamica del mercato finanziario e di adattarsi con flessibilità alle esigenze di copertura di un'istituzione finanziaria.

Parole Chiave: Value-at-Risk; filtro di Kalman; beta di Sharpe.

[JEL codes: C51, C52, G10]

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†Visiting scholar at Haas Business School, University of California Berkeley, USA. E-mails: *csommacampagna@berkeley.edu* and *cristina.sommacampagna@economia.univr.it*. On the web: *http://dse.univr.it/sommacampagna/*.

1 Introduction

One of the main objective of risk management consists in identifying and evaluating the measure of risk arising from both the operational and financial activities of the company. In order to satisfy these needs, many information and measurement instruments have been developed; one of them is Value-at-Risk (VaR). As far as financial risk is concerned, the VaR expresses in percentage terms the maximum loss which is likely to be exceeded on the portfolio, given a certain probability and time horizon. Many approaches to VaR estimation have been developed: non parametric and parametric, Jorion [9]. In the first class we find methods that imply the full generation of returns distribution: Historical Simulation and Monte Carlo Simulation. The Historical Simulation refers to the past empirical distribution of returns in order to simulate their future realizations. Considering a single asset: the first simulation consists in assuming that tomorrow asset return will equal the return we registered in the first day of the considered historical series; the second simulation consists in assuming that tomorrow asset return will equal the return we registered in the second day of the same historical series; and so on, Jorion [9] and Alexander [2]. In this way we build up a distribution for future returns; VaR is determined as a quantile of this distribution. The main negative aspect of this approach is that it can not account for currently likely events that never happened in the past. This is not a good assumption especially when we are dealing with market structural breaks. When we use Monte Carlo Simulation we assign probability distributions to some risky factors, which are usually the prices of the assets of the portfolio, and generate N hypothetical values for the factors. From the risky factors simulated values we derive returns and consequently a return distribution. Then VaR is determined as quantile of the distribution, Jorion [9] and Alexander [2]. In this approach, the choice of the model for the risky factor plays a key role.

Other approaches are more recent: the Filtered Historical Simulation method, Barone-Adesi [1], which combines Historical Simulation with the estimation of volatility using a GARCH model and with a bootstrap approach on the standardized residuals to generate new scenarios; the Extreme Value Theory (EVT) based approach, Embrechts [6], Kllezi [11], McNeil [12] and Neftci [13]. This second technique is based on the estimation of the Generalized Pareto Distribution to describe the behavior of the extreme values since it is proved, empirically, that distribution of returns is affected by fat tails. The parametric methods make the measure of risk (which is strictly related to the volatility of the portfolio through the VaR calculation) depending on the estimation of a single parameter. With regard to portfolios of equities, one parametric approach requires the estimation of the variance-covariance matrix of asset returns; in this case, the main problem is represented by the extremely ill-conditioned var-cov matrix. A second approach is the Diagonal Model of Sharpe, [14], which uses the ordinary least square (OLS) technique to estimate the parameter beta, i.e. the regression coefficient relating the single asset returns to the returns on the market index, Morgan [10]. However, as we will show below, the empirical evidence shows that the relation linking asset returns among them or the single asset return to the corresponding index return are actually time varying. Given this evidence, in this work we apply an alternative technique to estimate the parameter beta: the Kalman filter. The application of this methodology concerns the equity portfolio of an insurance company[1]. The portfolio composition is observed weekly during the period 31/12/99 - 12/04/01. This period experiences high volatility and the sharp increase in market values. The portfolio Value-at-Risk is daily calculated using i) the variance-covariance approach, ii) the parametric Sharpe beta approach with ordinary least square technique and iii) the Kalman filter estimation technique. Carrying out a back testing analysis we evaluate the performances of the three different estimation methods for Value-at-Risk. The VaR estimated with the Kalman filter appears to be more effective than the other two methods in capturing market volatility changes. The structure of the paper is the following. The second and the third sections review the literature of both classical variance-covariance and parametric Sharpe beta approaches to the estimation of Value-at-Risk. We test the instability of the parameter beta and provide a justification to the use of the Kalman filter for its estimation. In section 4 we describe the Kalman filter methodology and review applications of this technique to the estimation of the parameter beta. In section 5 we show the results of VaR calculation and back testing analysis. Finally, section 6 contains some concluding remarks. In the appendix we show the procedure for aggregating the VaRs.

2 Classical Approach to Value-at-Risk Estimation

This method focuses on the estimation of the portfolio standard deviation. The model is based on two main assumptions: the distribution of returns is normal and portfolio returns are linear with respect to the portfolio risk factors. These two combined hypothesis imply a normal distribution for the portfolio returns. Since the portfolio returns have normal distribution and their mean is zero, the σ_p parameter, which is the standard deviation of returns of portfolio, describes the distribution itself and consequently determines the VaR value. The normal distribution assumption allows us to say, for example, that a negative return of the portfolio equal to $-1.65\sigma_p$ will be exceeded with a 5% probability. We can introduce a diagram to better explain this point (see figure 1). Considering the past returns of a generic portfolio, we can plot their frequency in a diagram and show their empirical distribution; we sum up the probability of the better realizations (so starting from the left of the diagram) until we get a cumulative probability equal to $(1-\alpha)$ where α is the confidence level we choose for the calculation of our VaR. The value that corresponds to that confidence level is the $\alpha\%$ Value-at-Risk. Therefore, we are able to determine the VaR of the portfolio. Given the second assumption of a linear relation between changes in

[1]Data have been provided by RAS S.p.A.

portfolio value and changes in the risk factors, the portfolio return will also be itself normally distributed and therefore its variance would represent the whole risk. In the simplest case, the portfolio risk factors coincide with the assets in the portfolio. In matrix terms, the variance of the portfolio returns is equal to $\sigma_p^2 = x'\Sigma x$ and the Value-at-Risk, in percentage terms, is equal to

$$VaR = z\sqrt{x'\Sigma x}\sqrt{\Delta t}, \tag{1}$$

where x is the vector of weights, Δt is the time horizon and Σ is the variance-covariance matrix of asset returns. The main problem related with the classical approach to the VaR estimation is that the var-cov matrix is extremely ill-conditioned. The Sharpe Diagonal Model, which is presented in the following section, provides a simpler structure for the estimation of the variance of portfolio.

3 The OLS-based Approach

The basic assumption underlying the OLS-based approach is that asset returns depend on a single common factor which is represented by the market. The model (called Sharpe Diagonal Model) is based on linear regression of asset returns against market index returns:

$$R_i = \alpha_i + \beta_i R_m + \varepsilon_i;$$

where R_i is the single asset return, R_m is the return of the market index in which the asset is traded, α_i is an additive constant, β_i is the beta coefficient which indicates the sensitivity of R_i to changes in R_m, ε_i is the error component (a random variable, with mean zero and variance equal to σ_ε^2, which assumes independently and identically distributed values). The model assumes that the risk of the portfolio can be divided into two components: the first component can be reduced by the mean of diversification; the second component depends on market risk. The parameter β_i can be interpreted as a measure of the riskiness of the single asset in the portfolio. The coefficient can also be interpreted as representing the relationship between the riskiness of the single asset and the market index volatility; as it is given by the covariance between the respective returns over the variance of the index returns. The estimation of the betas is obtained by means of the ordinary least square (OLS) method, which consists in minimizing the squared differences between observed and estimated values. It is possible to demonstrate that, when the number of assets in the portfolio increases, the variance of the portfolio converges to $\sigma_p^2 \rightarrow (x'\beta\beta'x)\sigma_m^2$; the variance depends on the only risky factor σ_m. The VaR of the portfolio can consequently be calculated as follows:

$$VaR = z\sqrt{x'\beta\beta' x\sigma_m^2}\sqrt{\Delta t}, \tag{2}$$

z is the quantile of the normal distribution that corresponds to the confidence level we choose (1,65 in the example of the previous section); x is the portfolio weight vector; β is the vector of estimated parameters; σ_m^2 is the estimated market index variance[2].

3.1 Evidence on the Instability of the Beta Parameter

The OLS method is static since it implicitly gives the same weight to each observation and the estimated value may significantly depend on the length of the series we use. For example, using the last 250 observations, as it is usually done in practice, we implicitly assume that there are significant changes in the parameter value every year. Moreover, the betas represent the correlation between the single asset and the market index. Structural breaks occur in the financial markets due to many possible factors. Political events, for example, can stress the economic equilibrium of a country and determine some consequences on financial equilibrium. Abrupt changes in the economic cycle, unexpected changes in the macroeconomic factors that drive the economy, and many other microeconomic and social factors, can have some influence on the financial markets equilibrium. Given this, since the beta parameter that we are examining represents the correlation between the single asset (that is related to a real economic activity) and the market index, it is likely that this relation will not stay constant through time, Wells [15]. To test the stability of the estimated parameter we can use a recursive least square analysis. Plotting the values of the parameters calculated in this way, we can observe significant changes in the values of the betas. As an example, we show in figure 2 the path of the beta values recursively estimated from April 1996 to February 2000 for the asset Siemens which is included in the Dax market index; clearly a structural break has occurred around the end of May 1997 in the asset or in the market behavior. The value of beta for the asset Siemens has jumped from a value of approximately 0.73 up to 0.87, within few months, around the date of 30 May 1997. To detect when the break has occurred, we use the CUSUM test (Brown, Durbin and Evans [5]) that is based on the comparison between the cumulated sum of residuals generated by a recursive regression process and a confidence bound corresponding to the 5% of probability. If values of the cumulated sum cross over the confidence interval, we conclude that the parameter is not stable. Figure 3 shows the results of the CUSUM test applied to the Siemens stock: the presence of a break is clear.

The Chow's breakpoint test can also be used to test the statistical significance of the date we want to interpret as a break. The test consists in comparing the sum of squared residuals coming from the regression of a single equation on the entire sample with the sum of squared residuals coming from the regression of the same equation on two sub samples. A significant difference between the two values shows there has been a structural break in the relation between the variables we consider. The null hypothesis is the absence of structural breaks.

[2] The relationship between the variance-covariance model and the Sharpe beta model is represented by the following equation: $\beta = \frac{\Sigma X}{X'\Sigma X}$. X is the weight vector, Σ is the variance-covariance matrix.

Table 1 reports the results of the test in terms of p-value and clearly shows that the null hypothesis has to be rejected.

4 A New Approach based on the Kalman Filter

In this section, we develop a new procedure for the VaR calculation which is based on the estimate of the betas using the Kalman filter. The Kalman filter is a recursive algorithm which allows one to upgrade model estimates using new information, Hamilton [7] and Harvey [8]. Wells ([15]) applied this estimation technique to stock markets and estimated beta parameters of Stockholm exchange equities.
The Kalman filter is based on the representation of the dynamic system with a state space regression. Since we accept the hypothesis that the beta is not constant, we model its dynamics assuming an autoregressive process. In brief the Kalman filter algorithm can be summarized as follows. Let y_t be the vector $(n \times 1)$ of observed variables at time t whose dynamics depend on the possible observations of the state vector ξ_t of dimension $(r \times 1)$. The state-space representation of the dynamics of y_t is given by the following system of equations:

$$y_t = A'x_t + H'\xi_t + w_t \tag{3}$$

$$\xi_t = F\xi_{t-1} + v_t; \tag{4}$$

F, A' and H' are matrixes of parameters of dimensions $(r\times r)$, $(n\times k)$ and $(n\times r)$, respectively, and x_t is a vector of predetermined exogenous variables. Equation 4 is defined as state equation and the equation 3 as observation equation. The ξ_t vector is the state vector. v_t $(r \times 1)$ and w_t $(n \times 1)$ are white noises, vectors of random variables with zero mean and variance covariance equal to

$$E(v_t v_t') = \begin{cases} Q & \text{for } t = \tau \\ 0 & \text{for } t \neq \tau \end{cases} \tag{5}$$

$$E(w_t w_t') = \begin{cases} R & \text{for } t = \tau \\ 0 & \text{for } t \neq \tau \end{cases} \tag{6}$$

Q and R are matrixes of dimensions $(r \times r)$ and $(n \times n)$, respectively. We assume that errors are not correlated. The objective of state space formulation is to define the state vector ξ_t in a way that guarantees the minimization of the number of elements and the comprehension of all the available information at time t. To estimate the model we use the Maximum Likelihood technique. Casual starting values are assigned to the ξ_t vector and to the F, A' and H' matrixes of parameters and the estimation procedure maximizes the likelihood function.

In this work we apply the Kalman filter technique to the estimation of the Sharpe beta of a linear regression model. In the specific case, the state space model has this form:

$$R_t = \alpha_i + (\overline{\beta} + \beta_t) + \varepsilon_t$$

$$\beta_t = \theta\beta_{t-1} + w_t.$$

Specifying the state space in this way we can use the Kalman filter to estimate the beta parameter as if it was an unobservable variable. We assume the beta parameter follows a first order autoregressive process; a mean value for the beta parameter is also estimated as stated in the first equation; this means that we assume the parameter to be mean reverting towards $\overline{\beta}$, see Hamilton [7] and Harvey [8].

5 An Empirical Application

5.1 Data

The application is developed on a portfolio of equities of an insurance company[3]; the portfolio composition is observed during the period 31/12/1999 - 12/04/2001. The portfolio composition and the percentage weight of the assets are weekly refreshed. The average number of assets in the portfolio, during the observation period is 100. To estimate the Value-at-Risk of the portfolio, we use, for each asset, daily series of one year length (250 observations). For each methodology we calculate a VaR with different confidence levels and time horizons. The financial markets involved are those of: Finland, France, Germany, Italy, Japan, Netherlands, Spain, Sweden, Switzerland, U.K., U.S.A.. We create different groups of equities on the basis of the markets in which they are traded.

For each market we consider one or more indexes. For example, the IBM American shares are contained in the Dow Jones index and the American Intel shares are part of the NASDAQ index, so we relate them to the two different indexes respectively. For every group of equities we calculate the related VaR ($VaR_{index,i}$). Since the portfolio is denominated in more than one currency, in order to take into consideration the related risk effect, we also consider the series of exchange rates against Euro for: American Dollar, Japanese Yen, Swedish Crown, Swiss Franc, U.K. Sterling; so we also calculate a VaR for each currency ($VaR_{currency,i}$).

The aggregation of VaRs is described in details in the next section of the paper (see equations 7 and 8 and following part of the text).

We create a procedure which is able to recursively update the database of daily series of assets, markets and exchange rates. The procedure estimates the VaR every day for the three different methodologies.

5.2 Procedure

VaR is estimated for the period 31/12/1999 - 12/4/2001. Hence we obtain 325 values for each estimation approach, with confidence levels 95%, 97.5% and 99% and time horizons of one, five and ten days. The estimation approaches we apply have already been described: classical method based on the variance-covariance

[3]Data have been provided by RAS S.p.A.

matrix; ordinary least square method estimating the Sharpe beta with a linear regression; the Kalman filter method which, with a recursive process, estimates a time varying beta parameter. First of all, from the series of quotations we generate the series of daily returns:

$$r_t = \ln(P_t/P_{t-1}).$$

Second, we record the percentage weights of every asset through time for every day in which we want to calculate the VaR. As regards the classical approach, we estimate a variance-covariance matrix Σ for each group of equities, the group being created as described in the previous section; we apply equation 1 to each market and estimate the VaR for different confidence levels and time horizons. The VaR for currencies is calculated following this equation: $VaR_{currency} = z\sigma x\sqrt{\Delta t}$, where σ is the standard deviation of the series of percentage variations of the exchange rate, and x is the percentage of portfolio invested in that currency. Using the OLS methodology, we estimate the beta vector for each market and applying equation 2 we obtain the VaR for each group of assets. The OLS beta is estimated with a moving window of 250 data. The Kalman beta is also estimated with a moving window of 250 data and the same is for the var-cov matrix. The procedure of VaR estimation is repeated for 325 days, for all the three different methodologies, no comparative advantage is given to any of the three.

The estimation of the betas with the Kalman filter methodology is based on the assumption that they follow an autoregressive process with constant mean as we specified at the end of section 4.

We first calculate the VaR of the portfolio disregarding the diversification effect between market indexes and currencies: this means that we add up all the VaR components that we have calculated ($VaR_{index,i}$ and $VaR_{currency,i}$) as reflecting a conservative assumption of zero correlation between market indexes and between currencies. Then we derive a diversified VaR estimating the correlation matrixes both among market indexes $\Theta_{indexes}$ and currencies $\Theta_{currencies}$. The choice is justified considering Basel Committee on Banking Supervision [3] and [4].

Then we apply the following equations according to Morgan [10]:

$$VaR_{equity} = \sqrt{VaR'\Theta_{indexes}VaR}; \tag{7}$$

for each market index and methodology we discussed; the currency VaR is given by:

$$VaR_{currency} = \sqrt{VaR'\Theta_{currencies}VaR}. \tag{8}$$

Finally, the total VaR for the trading days between 31/12/1999 and 12/04/2001 is calculated as $VaR_{total} = VaR_{equity} + VaR_{currency}$, according to Basel Committee on Banking Supervision [4] and [3].

The whole procedure is discussed, more in detail, in appendix A.

5.3 Back Testing Analysis

The back testing analysis consists in testing the reliability of a Value-at-Risk model by evaluating the difference between the values estimated by the VaR system and the ex-post mark-to-market portfolio value. For example, in the case of a VaR with one day horizon and 95% confidence interval, the back testing analysis verifies whether daily losses above the VaR estimate occur only in the 5% of the cases. The same analysis is applied to the VaR calculated with the OLS and the variance-covariance matrix approaches in order to allow us a comparative analysis of the Kalman methodology. The Basle Committee has published a document (Basel Committee on Banking Supervision [4]) about the use of back testing which aimed at encouraging the use of this technique. The Basle Committee recommends a daily estimation of the Value-at-Risk. When the time horizon is five or ten trading days, it is not possible to disregard the fact that the portfolio composition may change in the mean time.

Usually the portfolio return is daily recorded and an historical series is created; this historical series can not be used for a back testing analysis. For this reason, for our back testing analysis, we assume that the composition of the portfolio remains unchanged from the moment in which we estimate Value-at-Risk for the five or ten days after the evaluation. This allows us to appropriately compare the VaR estimation with the return of the unchanged portfolio.

The back testing analysis can be interpreted as a static statistical test on the validity of the Value-at-Risk estimation methodology. Recognizing that this test has a limited power in distinguishing well specified models from not well specified ones, the Basle document provides a table for the interpretation of the back testing results: the number of errors made by the VaR method is associated with a cumulated probability to encounter that number of errors. The probability is divided into different "zones" on the base of which the VaR model can be: accepted as correct ("green zone", up to a cumulated probability of 95%), refused as not correct ("red zone", cumulated probability higher than 99,99%), be in an intermediate zone where it is not possible to conclude the model is correct or not ("yellow zone", cumulated probability between 95% and 99,99%) (table 2). The zones are determined as to balance the two kind of statistical errors: first type error - the VaR model is classified as incorrect while it is not; second type error - the VaR model is classified as correct when it is badly specified.

5.3.1 Results

The results of the back testing analysis are reported in tables 3 and 4; the number of errors we have registered for every type of VaR is expressed as percentage over the total number of VaR estimations. The result we should obtain is a 1%, 2.5% and 5% of errors for the 99%, 97.5% and 95% VaR respectively.

We define as VarCov the variance-covariance method, OLS the method that estimates static betas, Kalman the method that estimates betas using the Kalman filter. In the case of the diversified VaR, the estimates obtained applying the Kalman method appear to be, in terms of back testing, considerably more accurate than the OLS and VarCov estimates: the percentage error is lower if we consider the diversified VaR are a one day horizon (99%) and higher if we consider the diversified VaR on a ten day horizon (95%), but in both cases it is nearer to the target value. The diversified one day horizon, 99% confidence interval Kalman VaR is the only one included in the "green zone" of table 2, with a percentage error of 1,54%; it can consequently be considered a well specified model; OLS and VarCov, instead, are in the "yellow zone", which means that they are not completely well specified.

If we consider the non-diversified VaR in the case of one day-horizon, 95% and 97.5%, five day-horizon, 95% and 97.5%, ten day-horizon 97.5%, the Kalman and OLS estimation techniques give approximately the same results and seem to be better than the VarCov method; in all the other cases, the three methods exhibit almost the same percentages of error and it is impossible to draw conclusions. We observe that, when the time horizon lengthen, the percentage of errors of the three methods tends to decrease considerably; this fact is probably due to the fact that the relation between the daily volatility and the volatility for a longer time period is smaller than the coefficient $\sqrt{\Delta t}$ which is commonly used. In general, we observe that VarCov presents percentages of error more distant from the target values of 1%, 2.5% and 5%. The empirical analysis above has been carried out for a particular period characterized by high volatility and decreasing quotations following a boom of the market. Figures 4 and 5 provide evidence of the changes in the portfolio composition: it has been adjusted as to assume a more defensive position on the market; the beta of the portfolio has gradually and clearly decreased. We observe that the paths of Kalman and OLS portfolio's betas are similar, but the first plot is smoother than the second one. This is due to the fact that the Kalman beta is more sensitive to the market variations and faster in reacting to them.

The series generated by the Kalman VaR show higher volatility. OLS and Kalman VaRs are characterized by a smoother trend if compared with the VarCov series (figures 6-9). The difference in mean between VarCov and Kalman in the case of non-diversified, one day-horizon, 99% confidence interval VaR is equal to 0.38% and the maximum gap is 0.62%. The mean difference, if we consider VarCov and OLS, is equal to 0.42% and the maximum gap is 0.72%.

The three methods are different also in terms of volatility of estimations. The standard deviation of Kalman beta series is 0.0664, the standard deviation for the OLS beta of portfolio is 0.0468. The higher volatility of Kalman betas, for equation 2, has consequences on the VaR estimations, as it is clear from figures 6-9. With a 99% confidence interval and a one day unwind period, the non-diversified Kalman VaR has a standard deviation of 0.17, the non-diversified OLS VaR of 0.14. The estimate with the Kalman filter methodology, as we expect, is characterized by a higher volatility affecting the VaR calculation; this is particularly clear in the case of diversified VaR with one day-horizon, 99% confidence level.

6 Conclusions

In this paper we have developed a methodology to calculate VaR based on the Kalman filter approach for the estimation of portfolio betas. The empirical analysis provides evidence that this approach is more sensitive to market volatility changes than alternative methods using OLS betas or variance-covariance matrices. This is particularly true in periods characterized by high market volatility, as the one considered in our empirical work. Therefore, the technique advanced in the paper seems to provide significant improvements with respect to more "traditional" approaches and might be employed to better control daily changes in the risk of the portfolio. An interesting extension of the approach would consist in the dynamic estimation of the correlation matrices through the use of stochastic volatility models. This is an interesting direction for future research.

A Aggregating Value at Risk

Define the portfolio rate of return as

$$R_p = w_1 R_1 + w_2 R_2 ... + w_N R_N = \begin{bmatrix} w_1 & w_2 & . & w_N \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ . \\ R_N \end{bmatrix} = w'R \tag{9}$$

where R_1 and w_1 represent, respectively, the rate of return and the weight on asset 1 and so on asset 2 until N, w' represents the transposed vector of weights and R is the vector containing individual asset returns. The portfolio variance is

$$V(R_P) = \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} w_i w_j \sigma_{ij} \tag{10}$$

where σ_i^2 is the variance of asset i and σ_{ij} is the covariance between asset i and j. The portfolio variance can be written as

$$\sigma_p^2 = \begin{bmatrix} w_1 & w_2 & . & w_N \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & ... & \sigma_{1N} \\ . & & & & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & ... & \sigma_{NN} \end{bmatrix} \begin{bmatrix} w_1 \\ . \\ w_N \end{bmatrix} = w'\Sigma w \tag{11}$$

defining Σ as the covariance matrix. Define the correlation ρ between asset i and j as

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}. \tag{12}$$

In the two assets portfolio case the portfolio variance is

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{12} \sigma_1 \sigma_2. \tag{13}$$

Defining the Value at Risk (VaR) as

$$VaR_p = \alpha \sigma_p \sqrt{t} W \tag{14}$$

where α can be set at 1.65 for a one-tail 95 percent confidence level, $\sqrt{t}$ is the square root of number of days defined as the unwinding period and W is the initial portfolio value. According to Jorion [9], the portfolio VaR is then

$$VaR_p = \alpha \sigma_p \sqrt{t} W = \alpha \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{12} \sigma_1 \sigma_2} \sqrt{t} W. \tag{15}$$

When the correlation ρ is zero, the Value at Risk reduces to

$$VaR_p = \sqrt{\alpha^2 w_1^2 \sigma_1^2 t W^2 + \alpha^2 w_2^2 \sigma_2^2 t W^2} = \sqrt{VaR_1^2 + VaR_2^2}; \quad (16)$$

when the correlation is exactly unity, we have

$$VaR_p = \sqrt{VaR_1^2 + VaR_2^2 + 2VaR_1 VaR_2} = VaR_1 + VaR_2. \quad (17)$$

The portfolio VaR can be expressed also in terms of correlation matrix, Θ, according to Morgan [10] and we obtain given $\rho_{11} = \rho_{22} = 1$ and $\rho_{21} = \rho_{12}$

$$\begin{aligned}
VaR_p &= \alpha\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2}\sqrt{t}W = \\
&= \sqrt{\alpha^2 w_1^2\sigma_1^2 tW^2 + \alpha^2 w_2^2\sigma_2^2 tW^2 + 2\rho_{12}\left(\alpha w_1\sigma_1\sqrt{t}W\right)\left(\alpha w_2\sigma_2\sqrt{t}W\right)} \\
&= \sqrt{VaR_1^2 + VaR_2^2 + 2\rho_{12}\left(VaR_1\right)\left(VaR_2\right)} \\
&= \sqrt{\rho_{11}VaR_1^2 + \rho_{22}VaR_2^2 + \rho_{12}VaR_1VaR_2 + \rho_{21}VaR_1VaR_2}.
\end{aligned} \quad (18)$$

Then, the portfolio VaR can be written in matrix notation

$$VaR_p = \sqrt{\begin{bmatrix} VaR_1 & VaR_2 \end{bmatrix}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}\begin{bmatrix} VaR_1 \\ VaR_2 \end{bmatrix}} = \sqrt{VaR'\Theta VaR} \quad (19)$$

where VaR' represents the transposed vector of $VaRs$.

Now, consider the previous portfolio where the two assets 1 and 2 are a USD asset and a Yen asset respectively and the Euro is the reference currency for portfolio value. Define with σ_{usd}^2, σ_{yen}^2 and $\rho_{usd,yen}$, respectively, the volatility of USD, the volatility of the Yen and the correlation between the two currencies. The portfolio VaR in terms of currency risk is

$$\begin{aligned}
VaR_{currency} &= \alpha\sigma_{currency}\sqrt{t}W \quad (20) \\
&= \alpha\sqrt{w_1^2\sigma_{usd}^2 + w_2^2\sigma_{yen}^2 + 2w_1w_2\rho_{usd,yen}\sigma_{usd}\sigma_{yen}}\sqrt{t}W.
\end{aligned}$$

Then we can also write

$$\begin{aligned}
VaR_{currency} &= \sqrt{\begin{bmatrix} VaR_{usd} & VaR_{yen} \end{bmatrix}\begin{bmatrix} \rho_{usd,usd} & \rho_{usd,yen} \\ \rho_{yen,usd} & \rho_{yen,yen} \end{bmatrix}\begin{bmatrix} VaR_{usd} \\ VaR_{yen} \end{bmatrix}} \\
&= \sqrt{VaR'\Xi VaR} \quad (21)
\end{aligned}$$

where VaR' represents the transposed vector of currency $VaRs$ and Ξ is the correlation matrix among currencies.

When the correlation between the market risk and currency risk is assumed equal to one, that is exactly what regulators recommend according to the Basel Committees' rules, Basel Committee on Banking Supervision [3] and [4], the total portfolio VaR is

$$VaR_{TOT} = \alpha \sigma_{TOT} \sqrt{t} W = VaR_P + VaR_{currency}, \tag{22}$$

where $\sigma_{TOT} = \sigma_p + \sigma_{currency}$.

A formula can be derived that add the VaR measures of the two positions in equity and currency

$$\begin{aligned} \sigma_{TOT}^2 &= \sigma_P^2 + \sigma_{CUR}^2 + 2\rho_{P,CUR}\sigma_P\sigma_{CUR} \\ &= (\sigma_P + \sigma_{CUR})^2 - 2\sigma_P\sigma_{CUR} + 2\rho_{P,CUR}\sigma_P\sigma_{CUR} \\ &= (\sigma_P + \sigma_{CUR})^2 - 2(1 - \rho_{P,CUR})\sigma_P\sigma_{CUR} \end{aligned} \tag{23}$$

and so

$$VaR_{TOT} = \alpha \sqrt{(\sigma_P + \sigma_{CUR})^2 - 2(1 - \rho_{P,CUR})\sigma_P\sigma_{CUR}} \sqrt{t} W \ . \tag{24}$$

The total VaR is the sum of the VaRs if we assume that the two risk factors are perfectly correlated $\rho_{P,CUR} = 1$ and

$$VaR_{TOT} = \alpha (\sigma_P + \sigma_{CUR}) \sqrt{t} W = VaR_P + VaR_{CUR}, \tag{25}$$

obtaining an undiversified total VaR in order to be prudent (see Alexander [2] and Jorion [9]).

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Chow Breakpoint Test: 5/30/1997			
F-statistic	14.987	Probability	0.000
Log likelihood ratio	29.786	Probability	0.000

Table 1: Chow Breakpoint test for Siemens asset, Mar 1996-Feb 2000.

Zone	Number of exception	Associated Probability
Green Zone	0 1 2 3 4 5	0 0.31 0.62 0.92 1.23 1.60
Yellow Zone	6 7 8 9 10 11	1.85 2.15 2.46 2.77 3.08 3.38
Red Zone	More than 11	>3.38

Table 2: Basel Committee ”zones”.

A.	diversified 1 day-horizon VaR		
	95%	97.5%	99%
Kalman	5.56	2.47	1.54
OLS	5.56	2.78	1.85
VarCov	4.94	2.47	1.85
B.	**diversified 5 day-horizon VaR**		
	95%	97.5%	99%
Kalman	7.19	3.13	0.94
OLS	6.56	3.13	0.94
VarCov	5.62	2.50	0.94
C.	**diversified 10 day-horizon VaR**		
	95%	97.5%	99%
Kalman	4.13	0.95	0.63
OLS	3.81	0.95	0.63
VarCov	3.49	0.95	0.63

Table 3: Back-testing Results: Diversified Value-at-Risk.

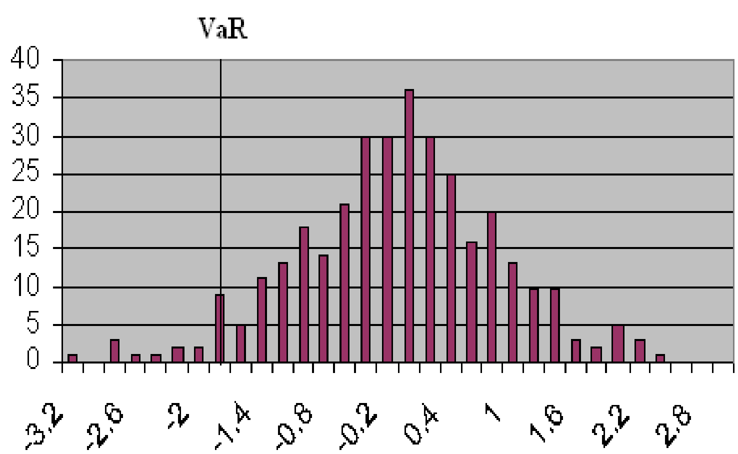

Figure 1: Value-at-Risk diagram.

A.	non diversified 1 day-horizon VaR		
	95%	97.5%	99%
Kalman	2.47	1.54	0.31
OLS	2.47	1.54	0.31
VarCov	1.85	0.93	0.31
B.	**non diversified 5 day-horizon VaR**		
	95%	97.5%	99%
Kalman	2.81	0.63	0.00
OLS	2.81	0.63	0.00
VarCov	1.25	0.00	0.00
C.	**non diversified 10 day-horizon VaR**		
	95%	97.5%	99%
Kalman	0.95	0.63	0.32
OLS	0.95	0.63	0.32
VarCov	0.95	0.32	0.32

Table 4: Back-testing Results: Non-diversified Value-at-Risk.

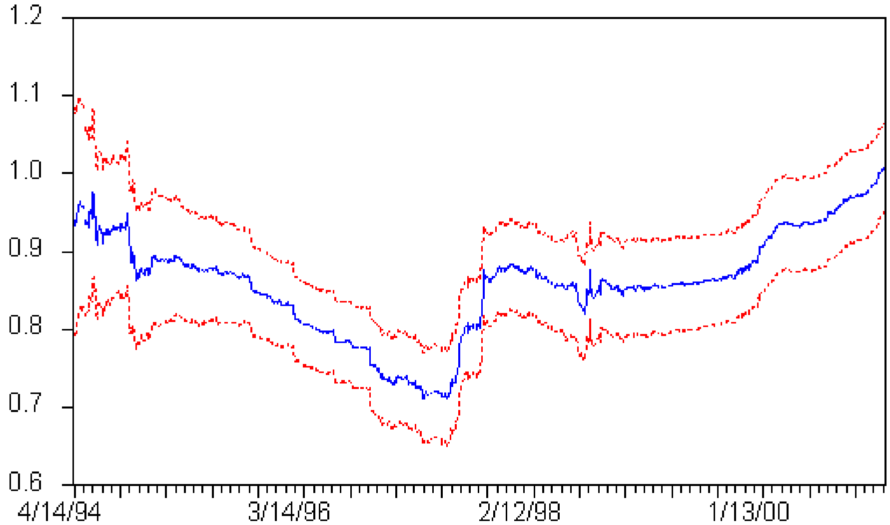

Figure 2: Recursive beta estimation for Siemens asset, Apr 1996-Feb 2000. The dashed lines indicate $\pm 2 Standard Error$.

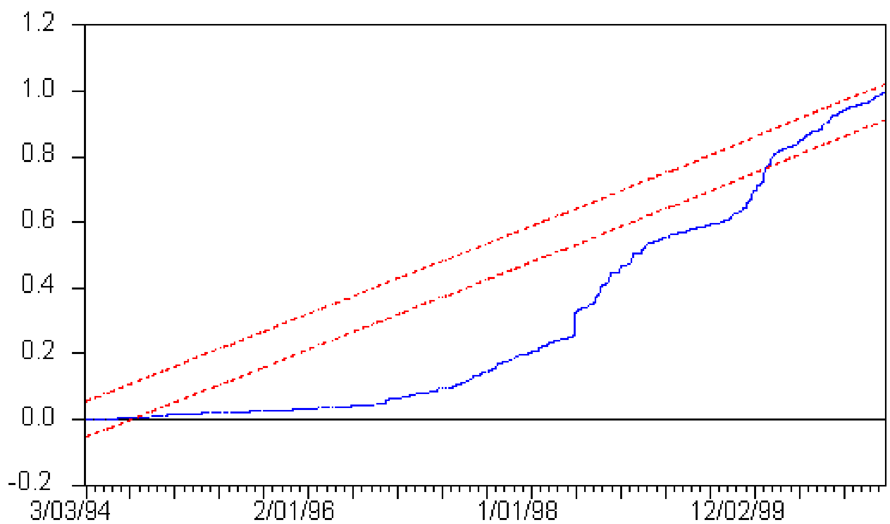

Figure 3: Cumulative Sum of Square Test for Siemens asset, Apr 1996-Feb 2000. The dashed lines indicate the 5% confidence interval.

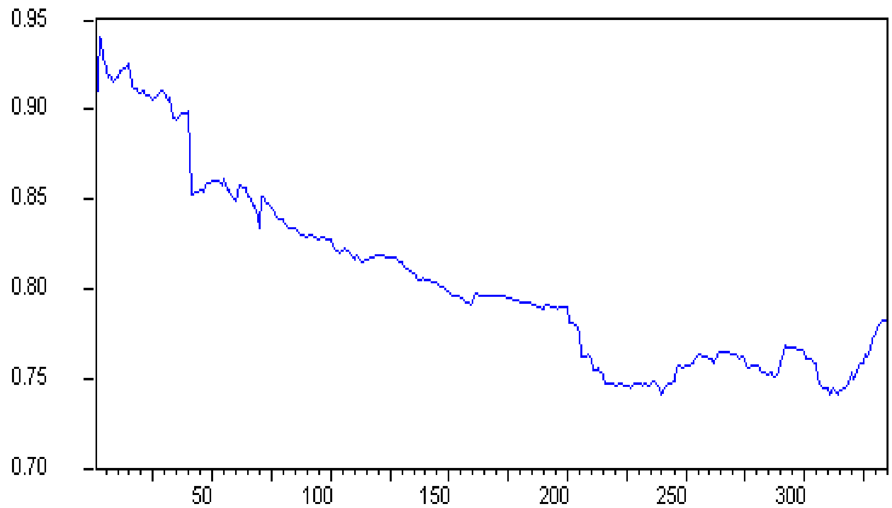

Figure 4: Beta of Portfolio with OLS technique.

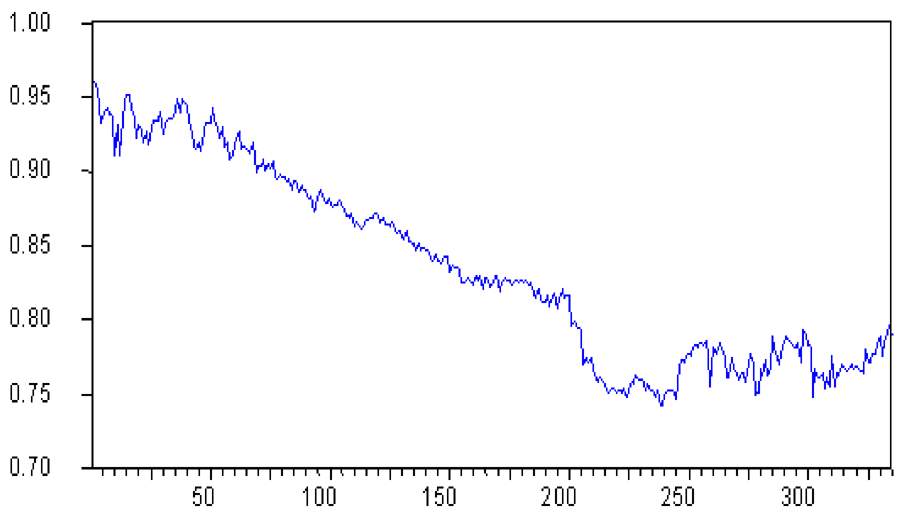

Figure 5: Beta of Portfolio with Kalman technique.

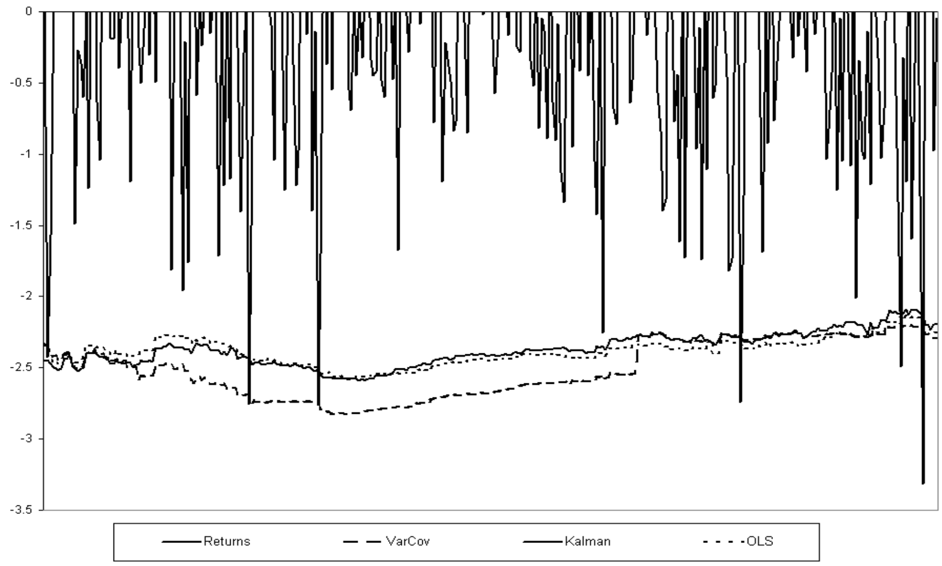

Figure 6: Diversified one day-horizon VaR with 99% confidence level (percentage values).

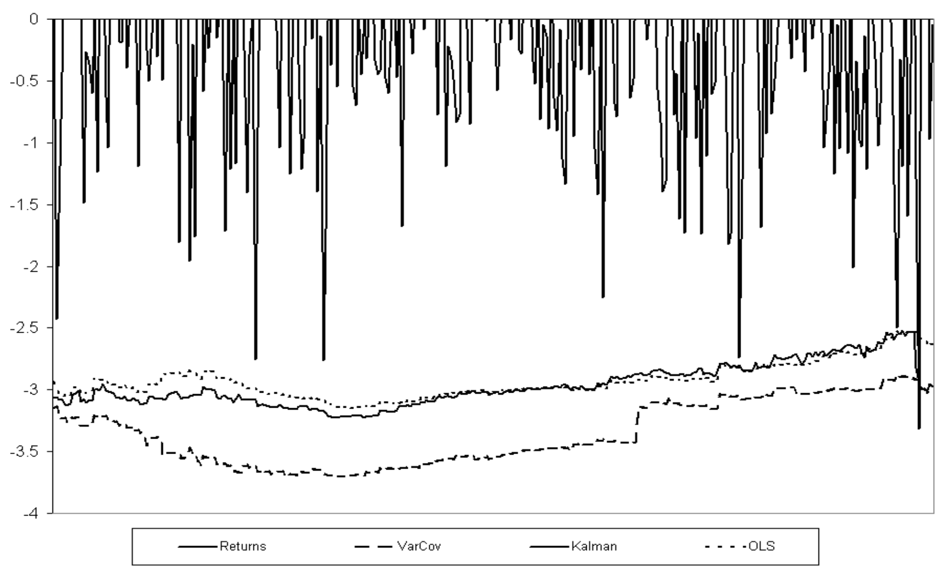

Figure 7: Non diversified one day-horizon VaR with 99% confidence level (percentage values).

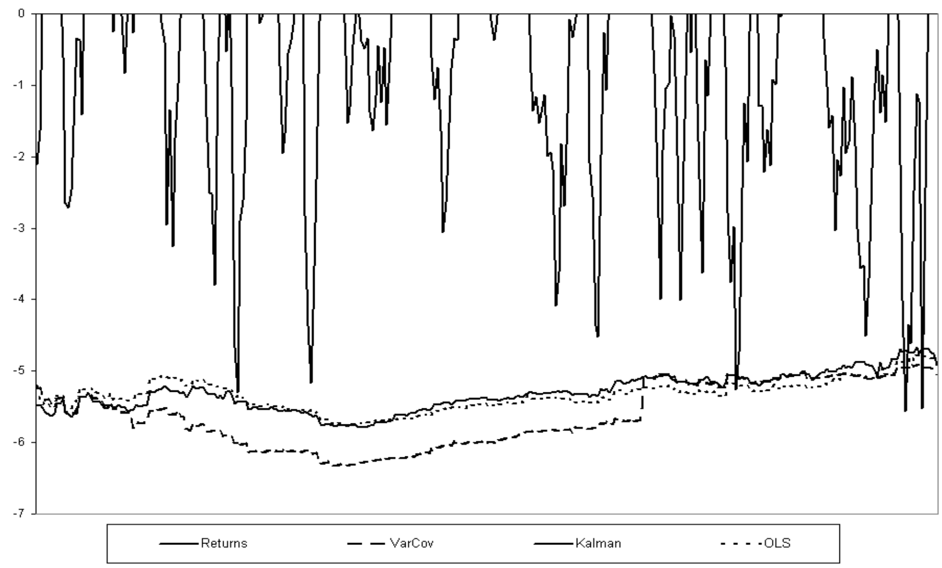

Figure 8: Diversified five day-horizon VaR with 99% confidence level (percentage values).

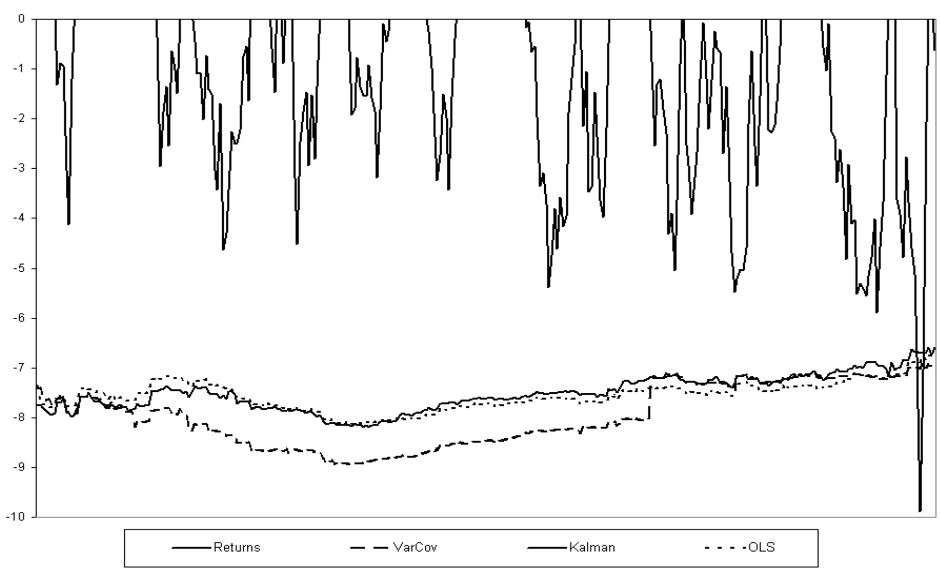

Figure 9: Diversified ten day-horizon VaR with 99% confidence level (percentage values).