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THE KALMAN FILTER APPROACH FOR TIME-VARYING β ESTIMATION

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Beta parameter is used in finance in the form of market model to estimate systematic risk. Such βs are assumed to be time invariant. Literature shows that now there is a considerable evidence that β risk is not constant over time. The aim of this article is the estimation of time-varying Italian industry parameter βs using the Kalman filter technique. This approach is applied to returns of the Italian market over the period 1991–2001.

Keywords: Time-varying β; Market index; Kalman filter

1. INTRODUCTION

The study about the market effect on the returns of single assets is one of the most investigated arguments in finance. The *Capital Asset Pricing Model* (CAPM) suggests that the market effect is due to the relationship between the asset returns and the *market portfolio* returns. The CAPM suggests that the asset sensibility to the variations of the market portfolio returns produces the single asset expected returns. Parameter β measures the asset sensibility to the variations on the market returns [1].

The market model is used to describe the relation between market returns [2]. Fama's article is notable because it is the first one to use the symbol β for the systematic risk. Indeed, in Finance, these two terms have become synonymous.

The proponents of the market model assumed that returns were Gaussian. Another topic of debate during the 1960s was whether this assumption could be justified [3–7]. The alternative to the Gaussian model was the family of stable distribution (of which the Gaussian is a special case). The main difference between the models considered was that the Gaussian has a finite second moment while the stable distributions studied did not. In an article in the *Journal of Business* in 1969, Jensen [8] discussed the implications of this debate for the capital asset pricing model. He showed that the systematic risk is more or less the same no matter which distribution returns actually follow.

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In the classical financial analysis parameter β is assumed to be time invariant, but there is considerable general evidence that the β stability assumption is invalid in several financial markets in US markets [9], in Malaysia [10] and in Australia [11]. In this work we will suppose that parameter β is time-variant and we will study the Italian financial market describing the relation between the assets return and the market index return by means of the market model. We will assume that parameter β follows a *Random Walk Model*. The variables involved are random, because we are not able to model completely all the asset return components, thus we will obtain a stochastic model. Consequently, we will need estimate parameter β. We will suppose that the random variables are Gaussian, but because of the above remark regarding the knowledge of the asset return components, we will assume that the covariance of the random variables is unknown. Before starting with the parameter β estimation we will need estimate such parameters, by means of the maximum likelihood function. Since all random variables present are supposed to be Gaussian, the estimation algorithm used is the Kalman Filter, which give the optimal estimation in the Gaussian case.

This article is organised as follows. In Section 2 the standard market model regression able to define an unconditional beta for any asset is presented whereas in Section 3 Kalman methodology by which conditional time dependent βs may be estimated is analysed. Section 3 is devoted to present time-varying βs generated for Italian data and finally Section 4 presents some conclusions based on the evidence obtained in this study.

2. THE MODEL

The relation between the asset return and the market index return can be estimated by the standard market model regression and can be expressed as follows:

$$R_{it} = \alpha_{it} + \beta_{it} R_{Mt} + \varepsilon_{it} \quad t = 1, \ldots, T \tag{1}$$

where:

- R_{it} is the return for the asset i
- R_{Mt} is the return for the market index
- α_t is a random variable that describe the component of the return for the asset i which is independent to the market return
- ε_{it} is the random disturbance vector such that:
  - $E(\varepsilon_{it}) = 0;\ \forall i, \forall t$
  - $E(\varepsilon_{it}\varepsilon_{jt}^T) = 0;\ \forall i, \forall j, \forall t, i \neq j$
  - $E(\varepsilon_{it}\varepsilon_{i\tau}^T) = 0;\ \forall i, \forall t, \forall \tau$
  - $E(\varepsilon_{it}R_{Mt}^T) = 0;\ \forall i, \forall t, t \neq \tau$

Equation (1) shows that the return for the asset i R_{it}, during the period t, depends on the return for the market index R_{Mt} on the same time. Moreover, the relation between these two variables is linear.

Coefficient β is the most important parameter present in the previous equation. It shows how asset returns vary with the market returns and is used to measure the asset systematic risk, or market risk.

3. KALMAN FILTER AND β ESTIMATION

The Kalman Filter is used in several fields to give the optimal estimate of random Gaussian variables. However, while emphasizing the relevance of the filter in modelling economic systems, we need to assume known and assigned *a priori* covariance matrices for the various noise processes in the model.

During the first 1970s there were the first applications of the Kalman filter to the problem to give an estimation for the systematic risk [13,14]. The model proposed for β was the *Random Walk Model* [15]. Their main problem was estimating the unknown variances in the observation and the system equations. Rather than estimate a constant variance for the observation equation, they used the OLS residuals as an estimate of the error sequence. Then the variance of the observation equation was calculated using different time lengths; that is, the variance at period t was based on the residuals from the first to the tth period. The estimate for the variance of the system noise was more problematical. They look this variance to be proportional to that of the observation equation. Thus the factor of proportionality was the ratio between the OLS estimate of the variance of the estimated β and the OLS residuals variance for the entire sample. Assuming that this ratio was constant, they could also generate a sequence of variance terms for the system equation.

Rosenberg [16] presented a maximum likelihood method for estimating the unknown parameters in the filter. His main interest in this article was, however, estimation and β appeared only in passing. Rosenberg considered the variances unknown but constant parameters and was mostly concerned with finding estimates that ‘‘converged’’ to the ‘‘true’’ variances: estimates of them were obtained through more or less complicated statistical procedures. Garbade [17] seems to have presented the first widely published application of Kalman filtering to the problem of estimating an econometric model other than the CAPM with varying coefficients. Using simulation studies he compares the relative strength of the *cusum* and the *cusumsq* tests against a likelihood ratio test based on the maximization of the likelihood associated with the varying coefficient model. Garbade and Rentzler [18] used the Kalman filter to estimate the variance of the system equation (1) when a random walk model is assumed. They present the likelihood function to be maximized, and the likelihood ratio test that could be used to test the hypothesis of β stationarity.

3.1. Random Walk Model

There have been rather many models for systematic risk described in literature. All of them can be represented by a simple two equation model. There are numerous studies assuming that asset prices follow the *Random Walk* model (RW).

There is a sizable body of literature [13,14] that asserts that β follows a random walk. Some article have developed parametric statistical tests designed to test if β follows this process [19,20]. The RW model can be expressed as follows

$$R_{it} = \alpha_t + \beta_{it} R_{Mt} + \varepsilon_{it} \tag{2}$$

$$\alpha_t = \alpha_{t-1} + u_{it} \tag{3}$$

$$\beta_t = \beta_{it-1} + \eta_{it} \tag{4}$$

As said before, in this article we assume that random variables ε_{it}, u_{it} and η_{it} are Gaussians:

$$\varepsilon_{it} \sim N(0, \Omega) \tag{5}$$

$$u_{it} \sim N(0, Q_1) \tag{6}$$

$$\eta_{it} \sim N(0, Q_2) \tag{7}$$

Initial conditions are

$$\beta_0 \sim N(\beta_0, P_0) \tag{8}$$

We remark that no knowledge is assumed on value of the variances Q_1, Q_2 and Ω. Before proceeding is helpful to represent the RW model in the state space.

3.2. System Equations

Observation equation:

$$y(t) = C(t)x(t) + \varepsilon_t = C(t)x(t) + G\psi(t) \tag{9}$$

Previous equation represents the market model with time-varying coefficients.

Matrix $C(t)$ has dimensions $T \times 2$ so that each row will represent the observations at certain point in time, has the following structure

$$C(t) = \begin{bmatrix} 1 & | & R_{Mt} \end{bmatrix} \tag{10}$$

and is assumed to be known.

The *state vector* $x(t)$ has dimensions 2×1 and represents the α and β coefficients at time t:

$$x(t) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{11}$$

ε_t is the part of the return $y(t)$ to the asset which is not modelled. The variance of ε_t is unknown, assumed finite and modelled by matrix G

$$G = \sqrt{\Omega}. \tag{12}$$

state equation, whose most general form is

$$x(t) = \Phi x(t-1) + \xi_t = Ax(t-1) + F\chi(t). \tag{13}$$

The covariance matrix of ζ (matrix F) is assumed diagonal, finite and its elements are unknown.

In the model adopted in the present work (RW), matrix Φ is the identity matrix

$$\Phi = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{14}$$

while vector ζ_t models the random part of the state vector:

$$\xi_t = \begin{bmatrix} u_t \\ \eta_t \end{bmatrix}. \tag{15}$$

The covariance matrix of the state noise has the following structure

$$F = \begin{bmatrix} \sqrt{Q_1} & 0 \\ 0 & \sqrt{Q_2} \end{bmatrix}. \tag{16}$$

There are three parameters that must be estimated: Ω and the diagonal elements of F which are the variances of the stochastic terms. We represent these unknown parameters as a vector $\vartheta = (\Omega, Q_1, Q_2)$.

3.3. The Maximum Likelihood Function

Our problem is estimating the unknown the variances of the error terms. The data on which the estimations will be based consists of observations of the returns to individual securities and the market portfolio. By adding and subtracting $C(t)A\hat{x}(t-1)$, where $A\hat{x}(t-1)$ indicates the unbiased estimate of the state vector at time t conditional on information available at time t–1, Eq. (9) may be rewritten as

$$y(t) = C(t)A\hat{x}(t-1) + C(t)x(t) - C(t)A\hat{x}(t-1) + G\psi(t). \tag{17}$$

We rewrite the previous equation as

$$\nu_o(t) = y(t) - C(t)A\hat{x}(t-1) = C(t)(x(t) - A\hat{x}(t-1)) + G\psi(t). \tag{18}$$

$\nu_o(t)$ is the output innovation sequence, which is zero mean, Gaussian and white.

In fact, taking expectations conditional upon the information set at t–1, the last two terms vanish as both $\psi(t)$ and the expression in parentheses have an expected value of zero: the conditional expectation of $\nu_o(t)$ is thus 0.

We indicate with $\psi_{\nu o}(t)$ the covariance of $\nu_o(t)$, whose expression at time t is:

$$\Psi_{\nu_o}(t) = E\left\{\nu_o(t)\nu_o^T(t)\right\} = C(t)P_p(t;\vartheta)C(t)^T G(\vartheta)G(\vartheta)^T, \tag{19}$$

where $P_p(t;\vartheta)$ is the covariance of the estimate of the state at time t conditional on the information available at time t–1. The final term in the above equation is the variance of the error term.

It is important to note that $\nu_o(t)$ and $y(t)$ contain the same information.

The likelihood function to be maximized is thus

$$L_{OT} = \frac{1}{(2\pi)^{q(T+1)/2}\prod_{t=0}^{T}\left|\Psi_{\nu_o}(t;\vartheta)\right|^{1/2}} \cdot \exp\left\{\frac{1}{2}\sum_{t=0}^{T}\nu_o^T(t)\Psi_{\nu_o}^{-1}(t;\vartheta)\nu_o(t)\right\}, \tag{20}$$

The maximum likelihood estimation $\hat{\vartheta}$ is the vector such that L_{OT} reaches the maximum value. Otherwise maximize L_{OT} and minimize its logarithm, taking out all the multiplicative constants, is the same. We can therefore define the following cost index to be minimized in order to obtain the maximum likelihood estimation

$$J(\vartheta) = \frac{1}{2}\sum_{t=0}^{T} \log\left|\Psi_{\nu_o}(t;\vartheta)\right| + \frac{1}{2}\sum_{t=0}^{T} [y(t) - C(t)A\hat{x}(t-1)]^T \Psi_{\nu_o}^{-1}(t;\vartheta)\cdot[y(t) - C(t)A\hat{x}(t-1)]. \tag{21}$$

The above function has been minimized by means of the gradient method.

When the maximum likelihood estimation $\hat{\vartheta}$ of the parameter vector is calculated, the optimum estimation of the state vector is obtained by means of the Kalman filter, by using the system matrices evaluated for $\hat{\vartheta}$.

3.4. Filter Equations

Kalman filter is a recursive algorithm, which give the optimal estimation of the state vector at time t only by means of the output at same time t and the optimal estimate of the state at the previous time $t-1$. [12]

The filter need to be initialized; initial conditions for the state vector and for the prediction covariance matrix are:

$$\hat{x}(0|-1) = E\{x(0)\} = 0 \quad P_p(0) = E\{x(0)x(0)^T\} = \Psi_{x(0)}$$

Afterwards, it is possible to proceed with the estimation algorithm. At each time t, following steps are reiterated:

1. Prediction covariance matrix:

$$P_p(t) = AP(t-1)A^T + F(\hat{\vartheta})F^T(\hat{\vartheta}) \tag{22}$$

2. Filter gain:

$$K(t) = P_p(t)C^T(t)\left(C(t)P_p(t)C^T(t) + G(\hat{\vartheta})G^T(\hat{\vartheta})\right)^{-1} \tag{23}$$

3. Filter covariance:

$$P(t) = [I - K(t)C(t)]P_p(t) \tag{24}$$

4. Optimal prediction of the state vector:

$$\hat{x}(t|t-1) = A\hat{x}(t-1) \tag{25}$$

5. Optimal estimate of the state vector:

$$\hat{x}(t) = \hat{x}(t|t-1) + K(t)(y(t) - C(t)\hat{x}(t|t-1)) \tag{26}$$

where $K(t)$ is the filter gain, $P(t)$ and $P(t|t-1)$ are the filter and prediction covariances, respectively. If the matrix $C(t)P_p(t)C^T(t)$ is singular, we can use the Moore–Penrose pseudoinverse.

The optimal estimate of parameter β at each time t is easily determined by extracting the second component in the vector $\hat{x}(t)$.

3.5. Goodness of the Proposed Method

We assess the accuracy of the forecast using the MAE (Mean Absolute Forecasting Error indices) and MSE (Mean Square Forecasting Error) indices [21,22]:

a. *Mean Absolute Forecasting Error:* once we forecast $\hat{R}_{it}$ it is possible to measure estimation accuracy using a measure of forecast error which compares the forecast to actual by

$$\mathrm{MAE}_i = \sum_{t=1}^{T} \frac{\left|\hat{R}_{it} - R_{it}\right|}{T} \tag{27}$$

A potential problem with the use of MAE measure is it weighs all errors equally. An alternative approach is to give an heavier penalty on outliers then the MAE measure with the use of squared term by the following index:

b. *Mean Square Forecasting Error:*

$$\mathrm{MSE}_i = \sum_{t=1}^{T} \frac{\left(\hat{R}_{it} - R_{it}\right)^2}{T} \tag{28}$$

4. EMPIRICAL RESULTS

The concept of β is well known in the financial community and its values are estimated by various technical service organizations. Generally speaking, we expect aggressive companies or highly leveraged companies to have high βs, whereas companies whose performance is unrelated to the general market behaviour are expected to have low βs. In this article the data used are weekly price relative information for 20 Italian Stock Exchange industries provided by TraderLink s.r.l. Our full sample period extends from May 1991 to June 2001. The data were expressed in Italian lyres and percentage returns were created for the analysis. The standard market model was estimated for each of the Italian industry, firstly using the domestic market index as shown in the second column of Table I. Notice that estimates are significantly different from zero with the exception of Paper industry and different from unity with the exception of Transport, Chemicals, Mineral and Textiles industries. The highest β value is for Public Utilities category with 1.584 whereas the lowest β value is for Paper category with -0.019.

Literature suggests that this market model parameters are likely to be unstable over time and our tests on stability applied on these data confirm this general finding. For this reason it is appropriate to analyze time-varying β risk using the Kalman filter model able to generate conditional β risk estimates for the 20 Italian Stock Exchange (ISX) industries as shown in the last column of Table I as well as the range of β

TABLE I β estimates of Italian industry

ISX industry	*β estimate*	*Mean conditional β (high/low)*
Food	0.443	0.798 (2.428/0.623)
Insurance	0.881	1.288 (1.635/1.109)
Transport	1.025	0.872 (1.101/0.526)
Banks	1.204	0.915 (1.418/0.599)
Paper	−0.019	0.843 (1.591/0.361)
Chemicals	0.989	0.904 (1.274/0.645)
Building materials	0.898	1.055 (1.659/0.636)
Distribution	1.524	1.215 (1.601/0.937)
Publishing	1.525	0.778 (1.769/0.145)
Electronics	0.779	0.851 (1.090/0.732)
Diversified financials	1.758	0.616 (2.067/0.130)
Financial holdings	0.733	1.210 (1.491/1.100)
Real estate	1.149	0.845 (1.593/0.696)
Equipments	1.373	0.832 (1.486/0.501)
Miscellaneous industries	0.597	0.842 (2.768/−0.481)
Mineral	1.005	0.856 (1.331/0.408)
Public utilities	1.584	0.878 (1.187/0.693)
Financial services	0.263	0.815 (1.233/0.594)
Textiles	1.049	0.785 (2.432/0.533)
Tourism and leisure	1.398	0.933 (1.459/0.599)

TABLE II MAE and MSE forecast error results

ISX industry	*MAE*	*MSE*
Food	5.89E-05	1.08E-08
Insurance	8.94E-05	1.53E-08
Transport	1.82E-05	6.52E-10
Banks	5.45E-05	6.13E-09
Paper	2.90E-04	1.59E-07
Chemicals	1.35E-04	3.50E-08
Building materials	2.07E-04	8.67E-08
Distribution	1.42E-04	3.74E-08
Publishing	1.35E-04	4.11E-08
Electronics	9.62E-05	1.71E-08
Diversified financials	1.47E-04	6.45E-08
Financial holdings	7.87E-05	1.12E-08
Real estate	1.71E-04	6.93E-08
Equipments	1.43E-04	4.10E-08
Miscellaneous industries	4.72E-04	5.31E-07
Mineral	1.11E-04	2.39E-08
Public utilities	4.82E-05	4.13E-09
Financial services	6.54E-05	8.15E-09
Textiles	1.19E-04	3.36E-08
Tourism and leisure	1.97E-04	8.33E-08

observations (in parentheses) where the large range of observations was evidenced in the Miscellaneous industries (2.768/−0.481) while the smallest was found for Electronics industry (1.090/0.732). Kalman β estimates need a start-up value and in the initial stages of estimation could generate very large and negative parameter values; to avoid this problem we exclude the first fifty weeks of observations. To evaluate the performance of β estimates we calculate the MAE and MSE metrics presented above (27), (28). The forecast $\hat{R}_{it}$ were compared to the actual R_{it} and the summary mean square error MSE and mean absolute error MAE measures are presented in Table II. Notice that the utilized technique produced in all 20 industries low level of forecast error demonstrating the effectiveness of the chosen estimation approach. In particular Paper and Transport industry show the lowest and highest level of forecast errors. Figures 1 and 2 present plots of the conditional betas generated in these two industries using the Kalman filter method.

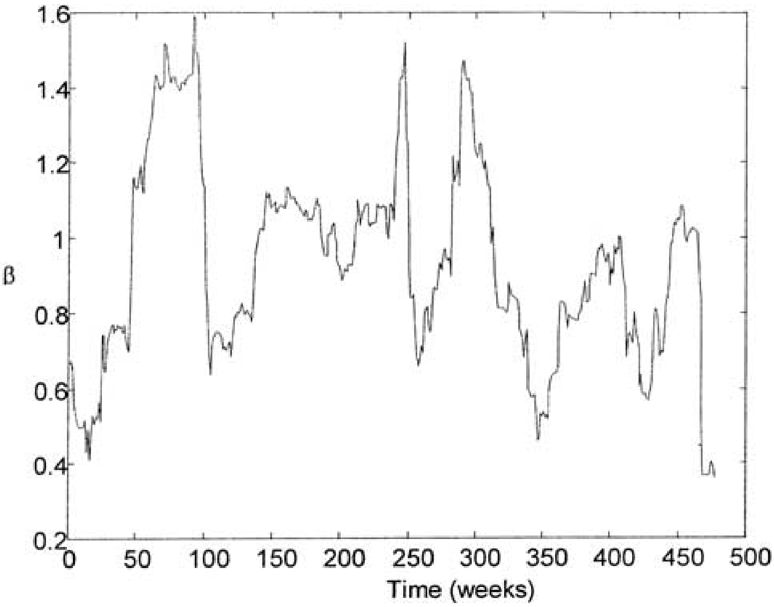

FIGURE 1 Kalman conditional β estimates for the Italian Paper Industry.

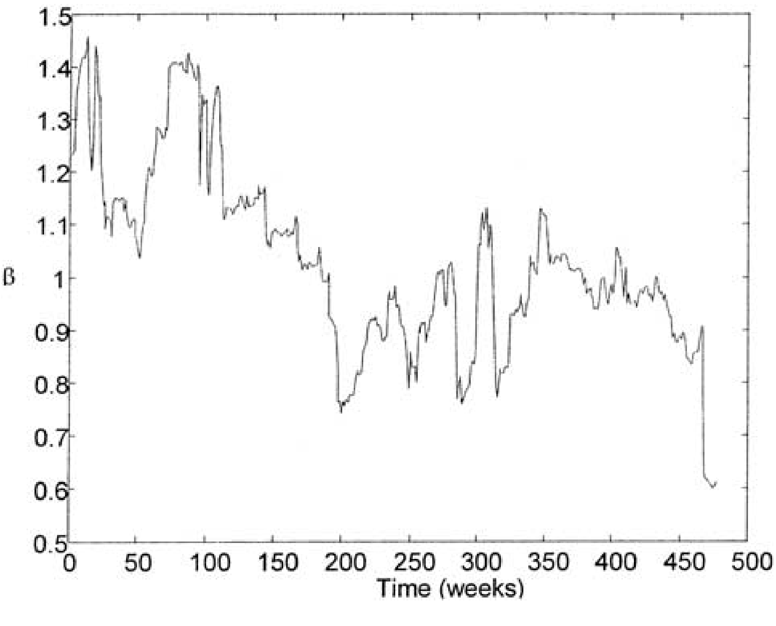

FIGURE 2 Kalman conditional β estimates for the Italian Transport Industry.

5. CONCLUSIONS

In this article we face the problem of estimation of systematic risk β. Unconditional point estimates are generally not stable and so exclude some important information. The presented results show that it is possible to estimate conditional time-dependent βs applying the Kalman filter technique to a sample of returns on Italian industry portfolios over the period 1991–2001.

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