Project 2: Importance Sampling Using Tilted Distributions

Jaime Frade

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Contents

List of Figures

Chapter 1

Introduction

1.1 Importance Sampling Estimator

1.1.1 Problem Statement

The objective of this assignment is to estimate the right tail of an exponential random variable using ideas from importance sampling and tilted distributions.

The problem to solve for this project is as follows: Let $\mathbf{X}$ be an exponential random varibale with intensity λ (mean $\frac{1}{\lambda}$). For an constant $a > 0$, the goal is to estimate the probability:

$$\theta = Pr\{X > a\} = \lambda \int_{a}^{\infty} e^{-\lambda\, x} dx \tag{1.1}$$

Since the event $X > a$ occurs with a very small probability, i.e. 0 is very small, the use of classical Monte Carlo approach will not be efficient. the usual approach would be completely inadequate since approximating μ to any reasonable degree of accuracy would require n to be very large. Using a much smaller value of n, would lead result in an estimate, $\hat{\theta} = 0$. Instead, it is better to use importance sampling with a tilted density serving as the density to sample from. In this paper, a method to reduce the variance of the esimator will be presented and numerical calcuations of accuarcy and computation time will be compared with classical approaches.

Chapter 2

Methodology

2.1 Importance Sampling: The Importance Sampling Estimator

To estimate $\theta = E[h(X)]$, where X has a probability density function of f(x). Let $g(\ldots)$ be another pdf with the property that $g(x) \neq 0$ whenever $f(x) \neq 0$. That is g has the same support as f. Then

$$\begin{aligned} \theta &= E[h(x)] = \int h(x)f(x)dx \\ &= \int h(x)\frac{f(x)}{g(x)}g(x)dx \end{aligned} \tag{2.1}$$

Since g is a pdf, then the expectation with respect to $g(\ldots)$ is

$$\theta = E\left[\frac{h(X)f(X)}{g(X)}\right] \tag{2.2}$$

Using Monte Carlo simulation method generates n samples of X from the density, $f(\ldots)$, and set $\hat{\theta}_n = \sum h(X)/n$. An alternative method is to generate n vales of X from the density, $g(\dot{)}$, and set

$$\hat{\theta}_{n,is} = \sum_{j=1}^{n} \frac{h(X_j)f(X_j)}{ng(X_j)} \tag{2.3}$$

then $\hat{\theta}_{n,is}$ is the importance sampling estimator of θ.

2.2 Determine the form of the tilted density

For a scalar $t > 0$, define the tilted density as:

$$f_t(x) = \frac{e^{tx}f(x)}{M(t)}, \quad \text{where} \quad f(x) = \lambda\, e^{\lambda\, x} \tag{2.4}$$

Then

$$
\begin{aligned}
M'(x) &= \int_0^\infty e^{tx} f(x) dx \\
&= \lambda \int_0^\infty e^{tx} e^{\lambda/,x} dx \\
&= e^{-x(\lambda-t)} dx \\
&= -\left(\frac{\lambda}{\lambda - t}\right)(0-1) \\
&= \left(\frac{\lambda}{\lambda - t}\right) && (2.5)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_t(x) &= \frac{e^{tx} f(x)}{\left(\frac{\lambda}{\lambda-t}\right)} \\
&= (\lambda - t) e^{-(\lambda-t)x} && (2.6)
\end{aligned}
$$

$f_t(x)$, the tilted density, has a exponential distribution with mean $= \frac{1}{(\lambda - t)}$, or using the recipical form, the intensity of $(\lambda - t)$.

2.3 From the tilted density, how to obtain samples

Will obtain samples by using the inverse transform method. Generate unform random variables from [0,1] and X_i's from $\frac{1}{(\lambda-t)} \log(U)$.

2.4 State the optimal amount of tilt t to estimate θ, for a given a.

From the mean of $f_t(x)$ can obtain the t^*, the optimal amount of tilt. The mean of $f_t(x)$ is $\frac{1}{(\lambda-t)}$, which is equal to a. Therefore,

$$
t^* = \lambda - \left(\frac{1}{a}\right) \tag{2.7}
$$

2.5 State the expression for $\tilde{\theta}_n$ Monte Carlo estimator that uses these samples.

By sampling X_i's be from the optimally tilted density, say $f_t^*(x)$, the X_i's will have an exponential distribution with mean $\frac{1}{(\lambda - t^*)}$.

From there, will substitute t^*. This will make the X_i's have exponential distribution with mean a. Using the method described above, will generate the X_i's to have a density of $a \log(U)$.

Therefore the expression for ,

$$
\begin{aligned}
\tilde{\theta_n} &= \frac{\lambda}{n(\lambda - t^*)} \sum_{i=1}^{n} I_{[a,\infty)}(x_i) e^{-t^* x_i} \\
&= \frac{a\,\lambda}{n} \sum_{i=1}^{n} I_{[a,\infty)}(x_i) e^{-(\lambda - \frac{1}{a}) x_i} \qquad (2.8)
\end{aligned}
$$

2.6 Compare the $\tilde{\theta_n}$ with $\hat{\theta_n}$, the classical Monte Carlo estimator that uses samples from $f(x)$ directly.

From $\hat{\theta_n}$, the classical Monte Carlo estimator, the X_i's have an exponential distribution with mean $\frac{1}{\lambda}$

$$
\hat{\theta_n} = \frac{1}{n} \sum_{i=1}^{n} I_{[a,\infty)}(x_i) \qquad (2.9)
$$

Chapter 3

Matlab Code

```
clear all; clc;
a=8;
lambda = 1;n=10000;
t=lambda -(1/a);
mt= lambda/(lambda-t);
u=rand(1,n);
x1=-log(u);
x = -a*log(u);
%classical estimate
tic
for i=1:length(x1)
    flag=0;
    if x1(i)>=a
      flag=flag+1;
    end
    if flag==1
        gx1(i)=1;
    else
        gx1(i)= 0;
    end
    thetahat(i)=(1/i)*sum(gx1);
end
toc
figure(1)
plot(thetahat)
xlabel('n');
ylabel('thetahat')


%tilted sampling
tic
for i=1:length(x)
    flag=0;
    if x(i)>=a
        flag=flag+1;
```

```
    end
    if flag==1
        gx(i)=exp(-t*x(i));
    else
        gx(i)= 0;
    end
    thetatilda(i)=(mt/i)*sum(gx);
end
toc;
figure(2)
plot(thetatilda)
xlabel('n');
ylabel('thetatilda')

thetahat(i)
thetatilda(i)
```

Chapter 4

Results

4.1 Comparing Classical MC and Important-Sampling

4.1.1 Plots of Convergence

Implement the classical and the important-sampling estimators in Matlab. Compare the two estimators on the basis of: (a) computation time and (b) accuracy of estimation. In other words, take a sample size, say n = 10000, and study the computation time and the accuracy of the two estimators for that value of n. Choose a = 6 and $\lambda = 1$ for this study.

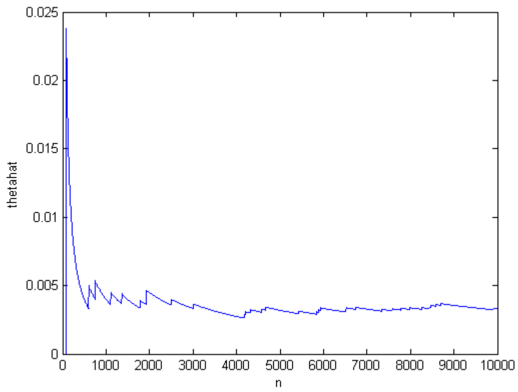

Figure 4.1: Convergence of Classical MC for a=6 and $\lambda = 1$

4.1.2 Computation Times

Classical MC	0.566971 seconds
Important-sampling of tilted	0.530406 seconds

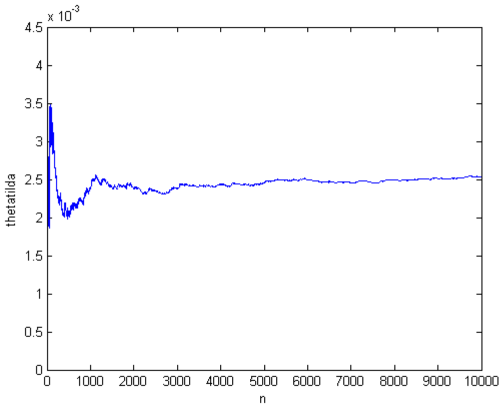

Figure 4.2: Convergence of Tilted Density for a=6 $\lambda = 1$

4.1.3 Accuarcy and Consistency

True Estimates	0.0025
Classical MC	0.0033
Important-sampling of tilted	0.0025

4.1.4 Conclusion

Analyzing the graphs, the classical MC approach took longer to converage. Also, in comparasion with computation time, the important-sampling titled was faster then the classical estimator. In accuarcy the important-sampling titled estimate was more accuate (closer to true value) and more consistent then the classical MC approach.

4.2 Comparation for different values of a and $\lambda = 1$

Plot the convergence of $\hat{\theta_n}$ and $\tilde{\theta_n}$ versus n for the following values of a = 2, 4, 6,and 8, with $\lambda = 1$. Plot them in the separate plots and clearly label your plots. Show your final estimates for these cases in a table.

4.2.1 a = 2 and $\lambda = 1$

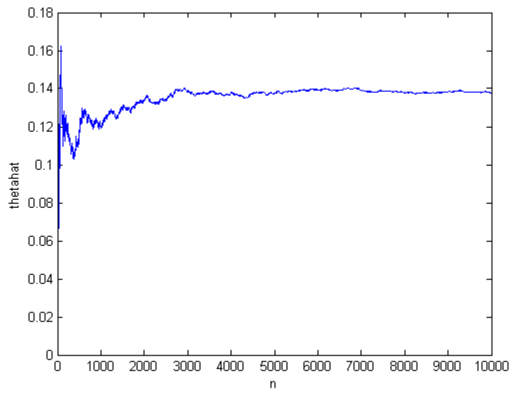

Figure 4.3: Convergence of Classical MC for a=2 and $\lambda = 1$

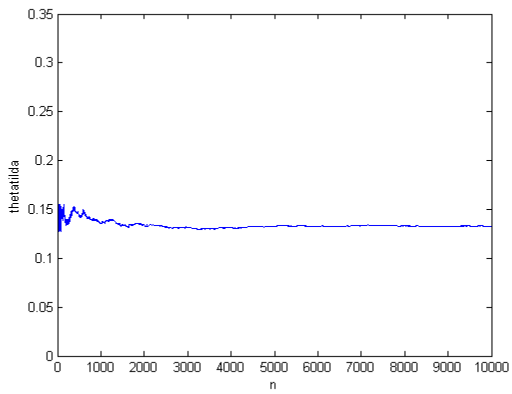

Figure 4.4: Convergence of Tilted Density for a=2 $\lambda = 1$

4.2.2 $a = 4$ and $\lambda = 1$

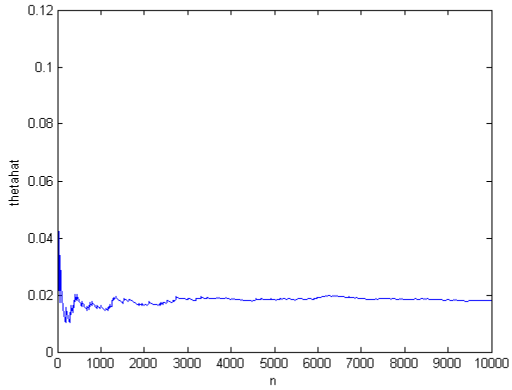

Figure 4.5: Convergence of Classical MC for a=4 and $\lambda = 1$

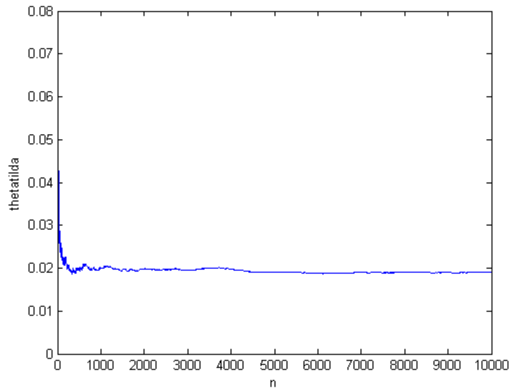

Figure 4.6: Convergence of Tilted Density for a=4 $\lambda = 1$

4.2.3 $a = 6$ and $\lambda = 1$

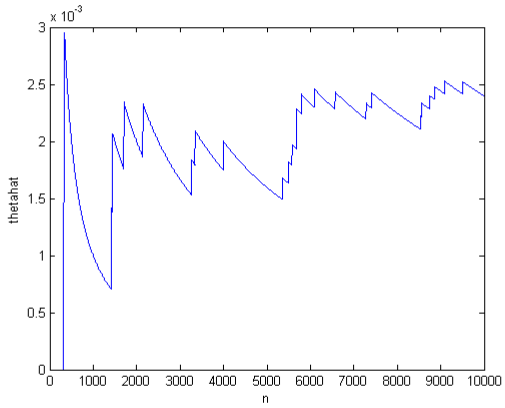

Figure 4.7: Convergence of Classical MC for a=6 and $\lambda = 1$

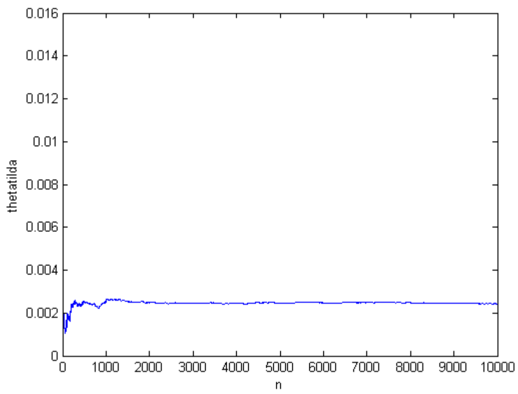

Figure 4.8: Convergence of Tilted Density for a=6 $\lambda = 1$

4.2.4 $a = 8$ and $\lambda = 1$

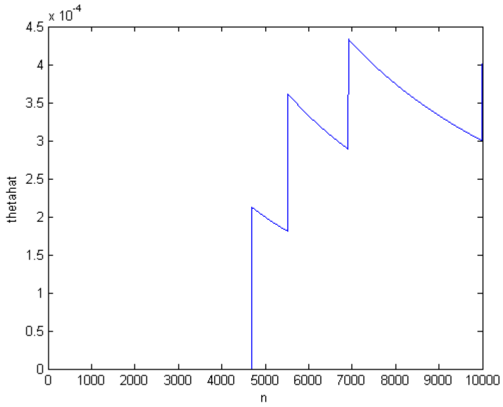

Figure 4.9: Convergence of Classical MC for a=8 and $\lambda = 1$

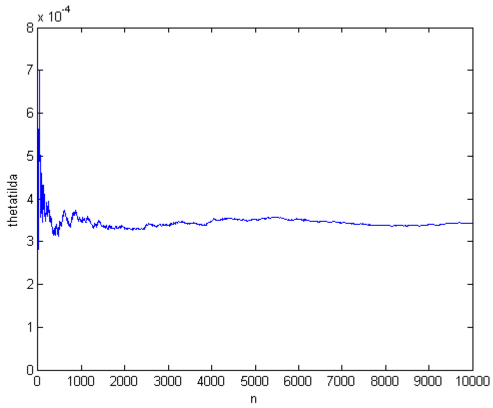

Figure 4.10: Convergence of Tilted Density for a=8 $\lambda = 1$

4.2.5 Table of Estimates

a	$\hat{\theta}_n$	$\tilde{\theta}_n$	True value	Comutational time $\hat{\theta}_n$	Comutational time $\tilde{\theta}_n$
2	0.1377	0.1328	0.1353	0.568419 seconds.	0.530554 seconds.
4	0.0181	0.0185	0.0183	0.478063 seconds.	0.467150 seconds.
6	0.0024	0.0025	0.0025	0.566394 seconds.	0.443624 seconds.
8	4.0000e-004	3.4270e-004	3.34e-004	0.553530 seconds.	0.499536 seconds.

Chapter 5

Conclusion

In this paper, it was found that the the tilted density esitmator for θ calculated using importance sampleing techniques was closer to the true value calculated from the intergral. In comparsion with the classical approach, the Monte Carlo approach, it was pointed in the tables and plots in the paper that the tilted density had shorter computational time , the accuracy was more presice , and converagnce was faster and more stable. For rare events or events with small probabiltiy, such as the the tail of the exponential density, the method implemented here proved to be a better estimator.