Jaime Frade
STA5166

2. Show that

$$E[Y] = \int_0^\infty P\{Y > y\}dy - \int_0^\infty P\{Y < -y\}dy \tag{1}$$

$$\begin{aligned} E[Y] &= \int_{-\infty}^\infty x f_y(x)dx \\ &= \int_{-\infty}^0 x f_y(x)dx + \int_0^\infty x f_Y(x)dx \\ &= -\left[-\int_{-\infty}^0 x f_Y(x)dx\right] + \int_0^\infty x f_Y(x)dx \end{aligned} \tag{2}$$

Show the following for (2) to hold:

$$\int_0^\infty P\{Y > y\}dy = \int_0^\infty x f_Y(x)dx \tag{3}$$

$$\int_0^\infty P\{Y < -y\}dy = -\int_{-\infty}^0 x f_Y(x)dx \tag{4}$$

By definition,

$$P\{Y > y\} = \int_y^\infty f_Y(x)dx$$

Intergrating both sides,

$$\int_0^\infty P\{Y > y\}dy = \int_0^\infty \int_y^\infty f_Y(x)dx\, dy \tag{5}$$

Changing the order of intergration in (5),

$$\begin{aligned} \int_0^\infty P\{Y > y\}dy &= \int_0^\infty \int_0^x f_Y(x)dy/, dx \\ &= \int_0^\infty \left(\int_0^x dy\right) f_Y(x)dx \\ &= \int_0^\infty x f_Y(x)dx \end{aligned} \tag{6}$$

By definition,

$$P\{Y < -y\} = \int_{-\infty}^{-y} f_Y(x)dx$$

Intergrating both sides,

$$\int_0^\infty P\{Y < -y\}dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x)dx\, dy \tag{7}$$

Let $x = -x'$. Rewrite (7).

$$\int_0^\infty P\{Y < -y\}dy = \int_0^\infty \int_y^\infty f_Y(-x')dx'\, dy \tag{8}$$

Changing the order of intergration in (8),

$$\begin{aligned}
\int_0^\infty P\{Y < -y\}dy &= \int_0^\infty \int_0^{x'} f_Y(-x')dy\,dx' \\
&= \int_0^\infty \left(\int_0^{x'} dy\right) f_Y(-x')dx' \\
&= \int_0^\infty x' f_Y(-x')dx' \qquad (9)
\end{aligned}$$

Let $-x = x'$. Rewrite (9).

$$\begin{aligned}
\int_0^\infty P\{Y < -y\}dy &= \int_0^\infty x' f_Y(-x')dx' \\
&= \int_0^{-\infty} -x f_Y(x)(-dx) \\
&= -\int_{-\infty}^0 x f_Y(x)dx \qquad (10)
\end{aligned}$$

Therefore by (6) and (10), will obtain (2).

$$\begin{aligned}
E[Y] &= -\left[-\int_{-\infty}^0 x f_Y(x)dx\right] + \int_0^\infty x f_Y(x)dx \\
&= \int_0^\infty x f_Y(x)dx - \left[-\int_{-\infty}^0 x f_Y(x)dx\right] \\
&= \int_0^\infty P\{Y > y\}dy - \int_0^\infty P\{Y < -y\}dy \qquad (11)
\end{aligned}$$

5. Use the result that for a nonnegative random variable Y

$$E[Y] = \int_0^\infty P\{Y > t\}dt$$

to show for a nonnegative random variable X

$$E[X^n] = \int_0^\infty nx^{n-1}P\{X > x\}dx \tag{12}$$

By definition,

$$E[X^n] = \int_0^\infty P\{X^n > t\}dt \tag{13}$$

8. Given that $P\{0 \leq X \leq c\} = 1$, show

$$Var(X) \leq \frac{c^2}{4} \tag{14}$$

To illustrate (14), will use the following results.

$$\begin{aligned} E[X^2] &= E[X \cdot X] \\ &\leq E[c \cdot X] \\ &= cE[X] \end{aligned} \tag{15}$$

$$\begin{aligned} Var(X) &= E[X^2] - (E[x])^2 \\ &\leq cE[X] - (E[x])^2 \\ &= c^2 \frac{E[X]}{c} - c^2 \left(\frac{E[X]}{c}\right)^2 \\ &= c^2 \left(\frac{E[X]}{c} - \left(\frac{E[X]}{c}\right)^2\right) \\ &= c^2 \left(\frac{E[X]}{c}\left[1 - \frac{E[X]}{c}\right]\right) \end{aligned} \tag{16}$$

Let $\alpha = \dfrac{E[X]}{c}$ in (16) to obtain,

$$Var(X) \leq c^2[\alpha(1-\alpha)] \tag{17}$$

The maximum value for $\alpha = \dfrac{1}{2}$, solved by finding vertex of $f(x) = x - x^2$.

$$Var(X) \leq \frac{c^2}{4} \tag{18}$$

3. Given,

$$f(x) = \begin{cases} C(2x - x^3) & 0 < x < \frac{5}{2} \\ 0 & \text{else} \end{cases} \tag{19}$$

The above function (19) is not a p.d.f, because $f(x)$ must be nonnegative within domain.
Determine C.

By definition,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Therefore,

$$\begin{aligned} \int_0^{\frac{5}{2}} C(2x - x^3)dx &= 1 \\ C\left[x^2 - \frac{x^4}{4}\right]\Big|_0^{\frac{5}{2}} &= 1 \\ C\left(-\frac{225}{64}\right) &= 1 \\ C &= -\frac{64}{225} \end{aligned} \tag{20}$$

$$f(x) = \begin{cases} -\frac{64}{225}(2x - x^2) & 0 < x < \frac{5}{2} \\ 0 & \text{else} \end{cases} \tag{21}$$

Given,

$$f(x) = \begin{cases} C(2x - x^2) & 0 < x < \dfrac{5}{2} \\ 0 & \text{else} \end{cases} \tag{22}$$

The above function (22) is not a p.d.f, because $f(x)$ must be nonnegative within domain.
Determine C.

By definition,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Therefore,

$$\begin{aligned} \int_0^{\frac{5}{2}} C(2x - x^2)dx &= 1 \\ C\left[x^2 - \frac{x^3}{3}\right]\Big|_0^{\frac{5}{2}} &= 1 \\ C\left(\frac{25}{24}\right) &= 1 \\ C &= \frac{24}{25} \end{aligned} \tag{23}$$

$$f(x) = \begin{cases} \frac{24}{25}(2x - x^2) & 0 < x < \dfrac{5}{2} \\ 0 & \text{else} \end{cases} \tag{24}$$

4. Given,

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases} \tag{25}$$

a. Determine the following

$$\begin{aligned} P(x > 20) &= \int_{20}^{\infty} \frac{10}{x^2} dx \\ &= 1 - \int_{0}^{20} \frac{10}{x^2} dx \\ &= 1 - \int_{10}^{20} \frac{10}{x^2} dx \\ &= 1 - \left[-\frac{10}{x} \right] |_{10}^{20} \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned} \tag{26}$$

b. What is the c.d.f of $f(x)$?
by defintion,

$$\frac{d}{dx} F(x) = f(x) \tag{27}$$