Chapter 2

Transformations and Expectations

2.1 a. $f_x(x) = 42x^5(1-x)$, $0 < x < 1$; $y = x^3 = g(x)$, monotone, and $\mathcal{Y} = (0,1)$. Use Theorem 2.1.5.

$$\begin{aligned} f_Y(y) &= f_x(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = f_x(y^{1/3})\frac{d}{dy}(y^{1/3}) = 42y^{5/3}(1-y^{1/3})(\frac{1}{3}y^{-2/3}) \\ &= 14y(1-y^{1/3}) = 14y - 14y^{4/3}, \quad 0 < y < 1. \end{aligned}$$

To check the integral,

$$\int_0^1 (14y - 14y^{4/3})dy = 7y^2 - 14\frac{y^{7/3}}{7/3}\Big|_0^1 = 7y^2 - 6y^{7/3}\Big|_0^1 = 1 - 0 = 1.$$

b. $f_x(x) = 7e^{-7x}$, $0 < x < \infty$, $y = 4x+3$, monotone, and $\mathcal{Y} = (3,\infty)$. Use Theorem 2.1.5.

$$f_Y(y) = f_x(\frac{y-3}{4})\left|\frac{d}{dy}(\frac{y-3}{4})\right| = 7e^{-(7/4)(y-3)}\left|\frac{1}{4}\right| = \frac{7}{4}e^{-(7/4)(y-3)},\ 3 < y < \infty.$$

To check the integral,

$$\int_3^\infty \frac{7}{4}e^{-(7/4)(y-3)}dy = -e^{-(7/4)(y-3)}\Big|_3^\infty = 0 - (-1) = 1.$$

c. $F_Y(y) = P(0 \le X \le \sqrt{y}) = F_X(\sqrt{y})$. Then $f_Y(y) = \frac{1}{2\sqrt{y}}f_X(\sqrt{y})$. Therefore

$$f_Y(y) = \frac{1}{2\sqrt{y}}30(\sqrt{y})^2(1-\sqrt{y})^2 = 15y^{\frac{1}{2}}(1-\sqrt{y})^2, \quad 0 < y < 1.$$

To check the integral,

$$\int_0^1 15y^{\frac{1}{2}}(1-\sqrt{y})^2dy = \int_0^1 (15y^{\frac{1}{2}} - 30y + 15y^{\frac{3}{2}})dy = 15(\frac{2}{3}) - 30(\frac{1}{2}) + 15(\frac{2}{5}) = 1.$$

2.2 In all three cases, Theorem 2.1.5 is applicable and yields the following answers.

a. $f_Y(y) = \frac{1}{2}y^{-1/2}$, $0 < y < 1$.

b. $f_Y(y) = \frac{(n+m+1)!}{n!m!}e^{-y(n+1)}(1-e^{-y})^m$, $0 < y < \infty$.

c. $f_Y(y) = \frac{1}{\sigma^2}\frac{\log y}{y}e^{-(1/2)((\log y)/\sigma)^2}$, $0 < y < \infty$.

2.3 $P(Y=y) = P(\frac{X}{X+1} = y) = P(X = \frac{y}{1-y}) = \frac{1}{3}(\frac{2}{3})^{y/(1-y)}$, where $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{x}{x+1}, \ldots$.

2.4 a. $f(x)$ is a pdf since it is positive and

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 \frac{1}{2}\lambda e^{\lambda x}dx + \int_0^\infty \frac{1}{2}\lambda e^{-\lambda x}dx = \frac{1}{2} + \frac{1}{2} = 1.$$

b. Let X be a random variable with density $f(x)$.

$$P(X<t)=\begin{cases}\int_{-\infty}^{t}\frac{1}{2}\lambda e^{\lambda x}dx & \text{if } t<0\\ \int_{-\infty}^{0}\frac{1}{2}\lambda e^{\lambda x}dx+\int_{0}^{t}\frac{1}{2}\lambda e^{-\lambda x}dx & \text{if } t\geq 0\end{cases}$$

where, $\int_{-\infty}^{t}\frac{1}{2}\lambda e^{\lambda x}dx=\frac{1}{2}e^{\lambda x}\big|_{-\infty}^{t}=\frac{1}{2}e^{\lambda t}$ and $\int_{0}^{t}\frac{1}{2}\lambda e^{-\lambda x}dx=-\frac{1}{2}e^{-\lambda x}\big|_{0}^{t}=-\frac{1}{2}e^{-\lambda t}+\frac{1}{2}$. Therefore,

$$P(X<t)=\begin{cases}\frac{1}{2}e^{\lambda t} & \text{if } t<0\\ 1-\frac{1}{2}e^{-\lambda t}dx & \text{if } t\geq 0\end{cases}$$

c. $P(|X|<t)=0$ for $t<0$, and for $t\geq 0$,

$$\begin{aligned}P(|X|<t) &= P(-t<X<t) = \int_{-t}^{0}\frac{1}{2}\lambda e^{\lambda x}dx+\int_{0}^{t}\frac{1}{2}\lambda e^{-\lambda x}dx\\ &= \frac{1}{2}\left[1-e^{-\lambda t}\right]+\frac{1}{2}\left[-e^{-\lambda t}+1\right] = 1-e^{-\lambda t}.\end{aligned}$$

2.5 To apply Theorem 2.1.8. Let $A_0=\{0\}$, $A_1=(0,\frac{\pi}{2})$, $A_3=(\pi,\frac{3\pi}{2})$ and $A_4=(\frac{3\pi}{2},2\pi)$. Then $g_i(x)=\sin^2(x)$ on A_i for $i=1,2,3,4$. Therefore $g_1^{-1}(y)=\sin^{-1}(\sqrt{y})$, $g_2^{-1}(y)=\pi-\sin^{-1}(\sqrt{y})$, $g_3^{-1}(y)=\sin^{-1}(\sqrt{y})+\pi$ and $g_4^{-1}(y)=2\pi-\sin^{-1}(\sqrt{y})$. Thus

$$\begin{aligned}f_Y(y) &= \frac{1}{2\pi}\left|\frac{1}{\sqrt{1-y}}\frac{1}{2\sqrt{y}}\right|+\frac{1}{2\pi}\left|-\frac{1}{\sqrt{1-y}}\frac{1}{2\sqrt{y}}\right|+\frac{1}{2\pi}\left|\frac{1}{\sqrt{1-y}}\frac{1}{2\sqrt{y}}\right|+\frac{1}{2\pi}\left|-\frac{1}{\sqrt{1-y}}\frac{1}{2\sqrt{y}}\right|\\ &= \frac{1}{\pi\sqrt{y(1-y)}},\quad 0\leq y\leq 1\end{aligned}$$

To use the cdf given in (2.1.6) we have that $x_1=\sin^{-1}(\sqrt{y})$ and $x_2=\pi-\sin^{-1}(\sqrt{y})$. Then by differentiating (2.1.6) we obtain that

$$\begin{aligned}f_Y(y) &= 2f_X(\sin^{-1}(\sqrt{y})\frac{d}{dy}(\sin^{-1}(\sqrt{y})-2f_X(\pi-\sin^{-1}(\sqrt{y})\frac{d}{dy}(\pi-\sin^{-1}(\sqrt{y})\\ &= 2(\frac{1}{2\pi}\frac{1}{\sqrt{1-y}}\frac{1}{2\sqrt{y}})-2(\frac{1}{2\pi}\frac{-1}{\sqrt{1-y}}\frac{1}{2\sqrt{y}})\\ &= \frac{1}{\pi\sqrt{y(1-y)}}\end{aligned}$$

2.6 Theorem 2.1.8 can be used for all three parts.

a. Let $A_0=\{0\}$, $A_1=(-\infty,0)$ and $A_2=(0,\infty)$. Then $g_1(x)=|x|^3=-x^3$ on A_1 and $g_2(x)=|x|^3=x^3$ on A_2. Use Theorem 2.1.8 to obtain

$$f_Y(y)=\frac{1}{3}e^{-y^{1/3}}y^{-2/3},\quad 0<y<\infty$$

.

b. Let $A_0=\{0\}$, $A_1=(-1,0)$ and $A_2=(0,1)$. Then $g_1(x)=1-x^2$ on A_1 and $g_2(x)=1-x^2$ on A_2. Use Theorem 2.1.8 to obtain

$$f_Y(y)=\frac{3}{8}(1-y)^{-1/2}+\frac{3}{8}(1-y)^{1/2},\quad 0<y<1$$

.

c. Let $A_0 = \{0\}$, $A_1 = (-1, 0)$ and $A_2 = (0, 1)$. Then $g_1(x) = 1 - x^2$ on A_1 and $g_2(x) = 1 - x$ on A_2. Use Theorem 2.1.8 to obtain

$$f_Y(y) = \frac{3}{16}(1 - \sqrt{1-y})^2 \frac{1}{\sqrt{1-y}} + \frac{3}{8}(2-y)^2, \quad 0 < y < 1$$

.

2.7 Theorem 2.1.8 does not directly apply.

a. Theorem 2.1.8 does not directly apply. Instead write

$$\begin{aligned} P(Y \le y) &= P(X^2 \le y) \\ &= \begin{cases} P(-\sqrt{y} \le X \le \sqrt{y}) & \text{if } |x| \le 1 \\ P(1 \le X \le \sqrt{y}) & \text{if } x \ge 1 \end{cases} \\ &= \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx & \text{if } |x| \le 1 \\ \int_1^{\sqrt{y}} f_X(x)dx & \text{if } x \ge 1 \end{cases}. \end{aligned}$$

Differentiation gives

$$f_y(y) = \begin{cases} \frac{2}{9}\frac{1}{\sqrt{y}} & \text{if } y \le 1 \\ \frac{1}{9} + \frac{1}{9}\frac{1}{\sqrt{y}} & \text{if } y \ge 1 \end{cases}.$$

b. If the sets $B_1, B_2, \ldots, B_K$ are a partition of the range of Y, we can write

$$f_Y(y) = \sum_k f_Y(y) I(y \in B_k)$$

and do the transformation on each of the B_k. So this says that we can apply Theorem 2.1.8 on each of the B_k and add up the pieces. For $A_1 = (-1, 1)$ and $A_2 = (1, 2)$ the calculations are identical to those in part (a). (Note that on A_1 we are essentially using Example 2.1.7).

2.8 For each function we check the conditions of Theorem 1.5.3.

a. (i) $\lim_{x \to 0} F(x) = 1 - e^{-0} = 0$, $\lim_{x \to -\infty} F(x) = 1 - e^{-\infty} = 1$.
(ii) $1 - e^{-x}$ is increasing in x.
(iii) $1 - e^{-x}$ is continuous.
(iv) $F_x^{-1}(y) = -\log(1 - y)$.

b. (i) $\lim_{x \to -\infty} F(x) = e^{-\infty}/2 = 0$, $\lim_{x \to \infty} F(x) = 1 - (e^{1-\infty}/2) = 1$.
(ii) $e^{-x/2}$ is increasing, $1/2$ is nondecreasing, $1 - (e^{1-x}/2)$ is increasing.
(iii) For continuity we only need check $x = 0$ and $x = 1$, and $\lim_{x \to 0} F(x) = 1/2$, $\lim_{x \to 1} F(x) = 1/2$, so F is continuous.
(iv)

$$F_X^{-1}(y) = \begin{cases} \log(2y) & 0 \le y < \frac{1}{2} \le y < 1, \\ 1 - \log(2(1-y)) & \frac{1}{2} \le y < 1 \end{cases}$$

c. (i) $\lim_{x \to -\infty} F(x) = e^{-\infty}/4 = 0$, $\lim_{x \to \infty} F(x) = 1 - e^{-\infty}/4 = 1$.
(ii) $e^{-x}/4$ and $1 - e^{-x}/4$ are both increasing in x.
(iii) $\lim_{x \downarrow 0} F(x) = 1 - e^{-0}/4 = \frac{3}{4} = F(0)$, so F is right-continuous.
(iv) $F_X^{-1}(y) = \begin{cases} \log(4y) & 0 \le y < \frac{1}{4} \\ -\log(4(1-y)) & \frac{1}{4} \le y < 1 \end{cases}$

2.9 From the probability integral transformation, Theorem 2.1.10, we know that if $u(x) = F_x(x)$, then $F_x(X) \sim \text{uniform}(0,1)$. Therefore, for the given pdf, calculate

$$u(x) = F_x(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ (x-1)^2/4 & \text{if } 1 < x < 3 \\ 1 & \text{if } 3 \leq x \end{cases}$$

.

2.10 a. We prove part b), which is equivalent to part a).

b. Let $A_y = \{x : F_x(x) \leq y\}$. Since F_x is nondecreasing, A_y is a half infinite interval, either open, say $(-\infty, x_y)$, or closed, say $(-\infty, x_y]$. If A_y is closed, then

$$F_Y(y) = P(Y \leq y) = P(F_x(X) \leq y) = P(X \in A_y) = F_x(x_y) \leq y.$$

The last inequality is true because $x_y \in A_y$, and $F_x(x) \leq y$ for every $x \in A_y$. If A_y is open, then

$$F_Y(y) = P(Y \leq y) = P(F_x(X) \leq y) = P(X \in A_y),$$

as before. But now we have

$$P(X \in A_y) = P(X \in (-\infty, x_y)) = \lim_{x \uparrow y} P(X \in (-\infty, x]),$$

Use the Axiom of Continuity, Exercise 1.12, and this equals $\lim_{x\uparrow y} F_X(x) \leq y$. The last inequality is true since $F_x(x) \leq y$ for every $x \in A_y$, that is, for every $x < x_y$. Thus, $F_Y(y) \leq y$ for every y. To get strict inequality for some y, let y be a value that is "jumped over" by F_x. That is, let y be such that, for some x_y,

$$\lim_{x \uparrow y} F_X(x) < y < F_X(x_y).$$

For such a y, $A_y = (-\infty, x_y)$, and $F_Y(y) = \lim_{x\uparrow y} F_X(x) < y$.

2.11 a. Using integration by parts with $u = x$ and $dv = xe^{\frac{-x^2}{2}}dx$ then

$$\mathrm{E}X^2 = \int_{-\infty}^{\infty} x^2 \frac{1}{2\pi} e^{\frac{-x^2}{2}} dx = \frac{1}{2\pi}\left[-xe^{\frac{-x^2}{2}}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx \right] = \frac{1}{2\pi}(2\pi) = 1.$$

Using example 2.1.7 let $Y = X^2$. Then

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi}} e^{\frac{-y}{2}} + \frac{1}{\sqrt{2\pi}} e^{\frac{-y}{2}} \right] = \frac{1}{\sqrt{2\pi y}} e^{\frac{-y}{2}}.$$

Therefore,

$$\mathrm{E}Y = \int_0^{\infty} \frac{y}{\sqrt{2\pi y}} e^{\frac{-y}{2}} dy = \frac{1}{\sqrt{2\pi}}\left[-2y^{\frac{1}{2}} e^{\frac{-y}{2}}\Big|_0^{\infty} + \int_0^{\infty} y^{\frac{-1}{2}} e^{\frac{-y}{2}} dy \right] = \frac{1}{\sqrt{2\pi}}(\sqrt{2\pi}) = 1.$$

This was obtained using integration by parts with $u = 2y^{\frac{1}{2}}$ and $dv = \frac{1}{2}e^{\frac{-y}{2}}$ and the fact the $f_Y(y)$ integrates to 1.

b. $Y = |X|$ where $-\infty < x < \infty$. Therefore $0 < y < \infty$. Then

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(|X| \leq y) = P(-y \leq X \leq y) \\ &= P(x \leq y) - P(X \leq -y) = F_X(y) - F_X(-y). \end{aligned}$$

Therefore,

$$F_Y(y) = \frac{d}{dy}F_Y(y) = f_X(y) + f_X(-y) = \frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}} + \frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}} = \sqrt{\frac{2}{\pi}}e^{\frac{-y}{2}}.$$

Thus,

$$\mathrm{E}Y = \int_0^\infty y\sqrt{\frac{2}{\pi}}e^{\frac{-y}{2}}dy = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-u}du = \sqrt{\frac{2}{\pi}}\left[-e^{-u}\Big|_0^\infty\right] = \sqrt{\frac{2}{\pi}},$$

where $u = \frac{y^2}{2}$.

$$\mathrm{E}Y^2 = \int_0^\infty y^2\sqrt{\frac{2}{\pi}}e^{\frac{-y}{2}}dy = \sqrt{\frac{2}{\pi}}\left[-ye^{\frac{-y}{2}}\Big|_0^\infty + \int_0^\infty e^{\frac{-y}{2}}dy\right] = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}} = 1.$$

This was done using integration by part with $u = y$ and $dv = ye^{\frac{-y}{2}}dy$. Then $\mathrm{Var}(Y) = 1 - \frac{2}{\pi}$.

2.12 We have $\tan x = y/d$, therefore $\tan^{-1}(y/d) = x$ and $\frac{d}{dy}\tan^{-1}(y/d) = \frac{1}{1+(y/d)^2}\frac{1}{d}dy = dx$. Thus,

$$f_Y(y) = \frac{2}{\pi d}\,\frac{1}{1+(y/d)^2}, \quad 0 < y < \infty.$$

This is the Cauchy distribution restricted to $(0, \infty)$, and the mean is infinite.

2.13 $P(X = k) = (1-p)^k p + p^k(1-p)$, $k = 1, 2, \ldots$. Therefore,

$$\begin{aligned} \mathrm{E}X &= \sum_{k=1}^\infty k[(1-p)^k p + p^k(1-p)] = (1-p)p\left[\sum_{k=1}^\infty k(1-p)^{k-1} + \sum_{k=1}^\infty kp^{k-1}\right] \\ &= (1-p)p\left[\frac{1}{p^2} + \frac{1}{(1-p)^2}\right] = \frac{1-2p+2p^2}{p(1-p)}. \end{aligned}$$

2.14

$$\begin{aligned} \int_0^\infty (1 - F_X(x))dx &= \int_0^\infty P(X > x)dx \\ &= \int_0^\infty \int_x^\infty f_X(y)dydx \\ &= \int_0^\infty \int_0^y dx f_X(y)dy \\ &= \int_0^\infty y f_X(y)dy = \mathrm{E}X, \end{aligned}$$

where the last equality follows from changing the order of integration.

2.15 Assume without loss of generality that $X \le Y$. Then $X \vee Y = Y$ and $X \wedge Y = X$. Thus $X + Y = (X \wedge Y) + (X \vee Y)$. Taking expectations

$$\mathrm{E}[X + Y] = \mathrm{E}[(X \wedge Y) + (X \vee Y)] = \mathrm{E}(X \wedge Y) + \mathrm{E}(X \vee Y).$$

Therefore $\mathrm{E}(X \vee Y) = \mathrm{E}X + \mathrm{E}Y - \mathrm{E}(X \wedge Y)$.

2.16 From Exercise 2.14,

$$\mathrm{E}T = \int_0^\infty \left[ae^{-\lambda t} + (1-a)e^{-\mu t}\right]dt = \frac{-ae^{-\lambda t}}{\lambda} - \frac{(1-a)e^{-\mu t}}{\mu}\Big|_0^\infty = \frac{a}{\lambda} + \frac{1-a}{\mu}.$$

2.17 a. $\int_0^m 3x^2 dx = m^3 \stackrel{set}{=} \frac{1}{2} \quad \Rightarrow \quad m = \left(\frac{1}{2}\right)^{1/3} = .794.$

b. The function is symmetric about zero, therefore $m = 0$ as long as the integral is finite.

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{1}{\pi}\tan^{-1}(x)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1.$$

This is the Cauchy pdf.

2.18 $\mathrm{E}|X-a| = \int_{-\infty}^{\infty} |x-a| f(x)dx = \int_{-\infty}^{a} -(x-a)f(x)dx + \int_a^{\infty}(x-a)f(x)dx$. Then,

$$\frac{d}{da}\mathrm{E}|X-a| = \int_{-\infty}^{a} f(x)dx - \int_a^{\infty} f(x)dx \stackrel{\text{set}}{=} 0.$$

The solution to this equation is $a =$ median. This is a minimum since $d^2/da^2\mathrm{E}|X-a| = 2f(a) > 0$.

2.19

$$\begin{aligned}
\frac{d}{da}\mathrm{E}(X-a)^2 &= \frac{d}{da}\int_{-\infty}^{\infty}(x-a)^2 f_X(x)dx = \int_{-\infty}^{\infty}\frac{d}{da}(x-a)^2 f_X(x)dx \\
&= \int_{-\infty}^{\infty} -2(x-a)f_X(x)dx = -2\left[\int_{-\infty}^{\infty} x f_X(x)dx - a\int_{-\infty}^{\infty} f_X(x)dx\right] \\
&= -2[\mathrm{E}X - a].
\end{aligned}$$

Therefore if $\frac{d}{da}\mathrm{E}(X-a)^2 = 0$ then $-2[\mathrm{E}X-a] = 0$ which implies that $\mathrm{E}X = a$. If $\mathrm{E}X = a$ then $\frac{d}{da}\mathrm{E}(X-a)^2 = -2[\mathrm{E}X-a] = -2[a-a] = 0$. $\mathrm{E}X = a$ is a minimum since $d^2/da^2\mathrm{E}(X-a)^2 = 2 > 0$. The assumptions that are needed are the ones listed in Theorem 2.4.3.

2.20 From Example 1.5.4, if $X =$ number of children until the first daughter, then

$$P(X=k) = (1-p)^{k-1}p,$$

where p = probability of a daughter. Thus X is a geometric random variable, and

$$\begin{aligned}
\mathrm{E}X &= \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p - \sum_{k=1}^{\infty}\frac{d}{dp}(1-p)^k = -p\frac{d}{dp}\left[\sum_{k=0}^{\infty}(1-p)^k - 1\right] \\
&= -p\frac{d}{dp}\left[\frac{1}{p}-1\right] = \frac{1}{p}.
\end{aligned}$$

Therefore, if p = $\frac{1}{2}$,the expected number of children is two.

2.21 Since $g(x)$ is monotone

$$\mathrm{E}g(X) = \int_{-\infty}^{\infty} g(x)f_X(x)dx = \int_{-\infty}^{\infty} y f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)dy = \int_{-\infty}^{\infty} y f_Y(y)dy = \mathrm{E}Y,$$

where the second equality follows from the change of variable $y = g(x)$, $x = g^{-1}(y)$ and $dx = \frac{d}{dy}g^{-1}(y)dy$.

2.22 a. Using integration by parts with $u = x$ and $dv = xe^{-x^2/\beta^2}$ we obtain that

$$\int_0^{\infty} x^2 e^{-x^2/\beta^2} dx^2 = \frac{\beta^2}{2}\int_0^{\infty} e^{-x^2/\beta^2} dx.$$

The integral can be evaluated using the argument on pages 104-105 (see 3.3.14) or by transforming to a gamma kernel (use $y = -\lambda^2/\beta^2$). Therefore, $\int_0^{\infty} e^{-x^2/\beta^2} dx = \sqrt{\pi}\beta/2$ and hence the function integrates to 1.

b. $\mathrm{E}X = 2\beta/\sqrt{\pi}$ $\qquad$ $\mathrm{E}X^2 = 3\beta^2/2$ $\qquad$ $\mathrm{Var}X = \beta^2\left[\frac{3}{2}-\frac{4}{\pi}\right]$.

2.23 a. Use Theorem 2.1.8 with $A_0 = \{0\}$, $A_1 = (-1, 0)$ and $A_2 = (0, 1)$. Then $g_1(x) = x^2$ on A_1 and $g_2(x) = x^2$ on A_2. Then

$$f_Y(y) = \frac{1}{2}y^{-1/2}, \quad 0 < y < 1.$$

b. $\mathrm{E}Y = \int_0^1 y f_Y(y)dy = \frac{1}{3}$ $\qquad$ $\mathrm{E}Y^2 = \int_0^1 y^2 f_Y(y)dy = \frac{1}{5}$ $\qquad$ $\mathrm{Var}Y = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}$.

2.24 a. $\mathrm{E}X = \int_0^1 x a x^{a-1}dx = \int_0^1 a x^a dx = \left.\frac{a x^{a+1}}{a+1}\right|_0^1 = \frac{a}{a+1}$.

$\mathrm{E}X^2 = \int_0^1 x^2 a x^{a-1}dx = \int_0^1 a x^{a+1}dx = \left.\frac{a x^{a+2}}{a+2}\right|_0^1 = \frac{a}{a+2}$.

$\mathrm{Var}X = \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2 = \frac{a}{(a+2)(a+1)^2}$.

b. $\mathrm{E}X = \sum_{x=1}^n \frac{x}{n} = \frac{1}{n}\sum_{x=1}^n x = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$.
$\mathrm{E}X^2 = \sum_{i=1}^n \frac{x^2}{n} = \frac{1}{n}\sum_{i=1}^n x^2 = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$.
$\mathrm{Var}X = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4} = \frac{n^2+1}{12}$.

c. $\mathrm{E}X = \int_0^2 x\frac{3}{2}(x-1)^2dx = \frac{3}{2}\int_0^2(x^3 - 2x^2 + x)dx = 1$.
$\mathrm{E}X^2 = \int_0^2 x^2\frac{3}{2}(x-1)^2dx = \frac{3}{2}\int_0^2(x^4 - 2x^3 + x^2)dx = \frac{8}{5}$.
$\mathrm{Var}X = \frac{8}{5} - 1^2 = \frac{3}{5}$.

2.25 a. $Y = -X$ and $g^{-1}(y) = -y$. Thus $f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| = f_X(-y)|-1| = f_X(y)$ for every y.

b. To show that $M_X(t)$ is symmetric about 0 we must show that $M_X(0+\epsilon) = M_X(0-\epsilon)$ for all $\epsilon > 0$.

$$\begin{aligned} M_X(0+\epsilon) &= \int_{-\infty}^{\infty} e^{(0+\epsilon)x} f_X(x)dx = \int_{-\infty}^{0} e^{\epsilon x} f_X(x)dx + \int_{0}^{\infty} e^{\epsilon x} f_X(x)dx \\ &= \int_{0}^{\infty} e^{\epsilon(-x)} f_X(-x)dx + \int_{-\infty}^{0} e^{\epsilon(-x)} f_X(-x)dx = \int_{-\infty}^{\infty} e^{-\epsilon x} f_X(x)dx \\ &= \int_{-\infty}^{\infty} e^{(0-\epsilon)x} f_X(x)dx = M_X(0-\epsilon). \end{aligned}$$

2.26 a. There are many examples; here are three. The standard normal pdf (Example 2.1.9) is symmetric about $a = 0$ because $(0-\epsilon)^2 = (0+\epsilon)^2$. The Cauchy pdf (Example 2.2.4) is symmetric about $a = 0$ because $(0-\epsilon)^2 = (0+\epsilon)^2$. The uniform$(0, 1)$ pdf (Example 2.1.4) is symmetric about $a = 1/2$ because

$$f((1/2)+\epsilon) = f((1/2)-\epsilon) = \begin{cases} 1 & \text{if } 0 < \epsilon < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le \epsilon < \infty \end{cases}.$$

b.

$$\begin{aligned} \int_a^{\infty} f(x)dx &= \int_0^{\infty} f(a+\epsilon)d\epsilon && \text{(change variable, } \epsilon = x - a\text{)} \\ &= \int_0^{\infty} f(a-\epsilon)d\epsilon && (f(a+\epsilon) = f(a-\epsilon) \text{ for all } \epsilon > 0) \\ &= \int_{-\infty}^{a} f(x)dx. && \text{(change variable, } x = a - \epsilon\text{)} \end{aligned}$$

Since

$$\int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 1,$$

it must be that

$$\int_{-\infty}^{a} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx = 1/2.$$

Therefore, a is a median.

c.

$$\begin{aligned} \mathrm{E}X - a &= \mathrm{E}(X-a) = \int_{-\infty}^{\infty}(x-a)f(x)dx \\ &= \int_{-\infty}^{a}(x-a)f(x)dx + \int_{a}^{\infty}(x-a)f(x)dx \\ &= \int_{0}^{\infty}(-\epsilon)f(a-\epsilon)d\epsilon + \int_{0}^{\infty}\epsilon f(a+\epsilon)d\epsilon \end{aligned}$$

With a change of variable, $\epsilon = a - x$ in the first integral, and $\epsilon = x - a$ in the second integral we obtain that

$$\begin{aligned} \mathrm{E}X - a &= \mathrm{E}(X-a) \\ &= -\int_{0}^{\infty}\epsilon f(a-\epsilon)d\epsilon + \int_{0}^{\infty}\epsilon f(a-\epsilon)d\epsilon \qquad (f(a+\epsilon) = f(a-\epsilon) \text{ for all } \epsilon > 0) \\ &= 0. \qquad \text{(two integrals are same)} \end{aligned}$$

Therefore, $\mathrm{E}X = a$.

d. If $a > \epsilon > 0$,

$$f(a-\epsilon) = e^{-(a-\epsilon)} > e^{-(a+\epsilon)} = f(a+\epsilon).$$

Therefore, $f(x)$ is not symmetric about $a > 0$. If $-\epsilon < a \le 0$,

$$f(a-\epsilon) = 0 < e^{-(a+\epsilon)} = f(a+\epsilon).$$

Therefore, $f(x)$ is not symmetric about $a \le 0$, either.

e. The median of $X = \log 2 < 1 = \mathrm{E}X$.

2.27 a. The standard normal pdf.

b. The uniform on the interval $(0, 1)$.

c. For the case when the mode is unique. Let a be the point of symmetry and b be the mode. Let assume that a is not the mode and without loss of generality that $a = b+\epsilon > b$ for $\epsilon > 0$. Since b is the mode then $f(b) > f(b+\epsilon) \ge f(b+2\epsilon)$ which implies that $f(a-\epsilon) > f(a) \ge f(a+\epsilon)$ which contradict the fact the $f(x)$ is symmetric. Thus a is the mode.
For the case when the mode is not unique, there must exist an interval (x_1, x_2) such that $f(x)$ has the same value in the whole interval, i.e, $f(x)$ is flat in this interval and for all $b \in (x_1, x_2)$, b is a mode. Let assume that $a \notin (x_1, x_2)$, thus a is not a mode. Let also assume without loss of generality that $a = (b+\epsilon) > b$. Since b is a mode and $a = (b+\epsilon) \notin (x_1, x_2)$ then $f(b) > f(b+\epsilon) \ge f(b+2\epsilon)$ which contradict the fact the $f(x)$ is symmetric. Thus $a \in (x_1, x_2)$ and is a mode.

d. $f(x)$ is decreasing for $x \ge 0$, with $f(0) > f(x) > f(y)$ for all $0 < x < y$. Thus $f(x)$ is unimodal and 0 is the mode.

2.28 a.

$$\begin{aligned}\mu_3 &= \int_{-\infty}^{\infty} (x-a)^3 f(x)dx = \int_{-\infty}^{a} (x-a)^3 f(x)dx + \int_{a}^{\infty} (x-a)^3 f(x)dx \\ &= \int_{-\infty}^{0} y^3 f(y+a)dy + \int_{0}^{\infty} y^3 f(y+a)dy \qquad \text{(change variable } y = x-a) \\ &= \int_{0}^{\infty} -y^3 f(-y+a)dy + \int_{0}^{\infty} y^3 f(y+a)dy \\ &= 0. \qquad (f(-y+a) = f(y+a))\end{aligned}$$

b. For $f(x) = e^{-x}$, $\mu_1 = \mu_2 = 1$, therefore $\alpha_3 = \mu_3$.

$$\begin{aligned}\mu_3 &= \int_{0}^{\infty} (x-1)^3 e^{-x}dx = \int_{0}^{\infty} (x^3 - 3x^2 + 3x - 1)e^{-x}dx \\ &= \Gamma(4) - 3\Gamma(3) + 3\Gamma(2) - \Gamma(1) = 3! - 3\times 2! + 3\times 1 - 1 = 3.\end{aligned}$$

c. Each distribution has $\mu_1 = 0$, therefore we must calculate $\mu_2 = \mathrm{E}X^2$ and $\mu_4 = \mathrm{E}X^4$.

(i) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $\quad \mu_2 = 1$, $\quad \mu_4 = 3$, $\quad \alpha_4 = 3$.

(ii) $f(x) = \frac{1}{2}$, $-1 < x < 1$, $\quad \mu_2 = \frac{1}{3}$, $\quad \mu_4 = \frac{1}{5}$, $\quad \alpha_4 = \frac{9}{5}$.

(iii) $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$, $\quad \mu_2 = 2$, $\quad \mu_4 = 24$, $\quad \alpha_4 = 6$.

As a graph will show, (iii) is most peaked, (i) is next, and (ii) is least peaked.

2.29 a. For the binomial

$$\begin{aligned}\mathrm{E}X(X-1) &= \sum_{x=2}^{n} x(x-1)\binom{n}{x}p^x(1-p)^{n-x} \\ &= n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x} p^{x-2}(1-p)^{n-x} \\ &= n(n-1)p^2 \sum_{y=0}^{n-2} \binom{n-2}{y} p^{y}(1-p)^{n-2-y} = n(n-1)p^2,\end{aligned}$$

where we use the identity $x(x-1)\binom{n}{x} = n(n-1)\binom{n-2}{x}$, substitute $y = x-2$ and recognize that the new sum is equal to 1. Similarly, for the Poisson

$$\mathrm{E}X(X-1) = \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} = \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} = \lambda^2,$$

where we substitute $y = x-2$.

b. $\mathrm{Var}(X) = \mathrm{E}[X(X-1)] + \mathrm{E}X - (\mathrm{E}X)^2$. For the binomial

$$\mathrm{Var}(X) = n(n-1)p^2 + np - (np)^2 = np(1-p).$$

For the Poisson

$$\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

c.

$$\mathrm{E}Y = \sum_{y=0}^{n} y\frac{a}{y+a}\binom{n}{y}\frac{\binom{a+b-1}{a}}{\binom{n+a+b-1}{y+a}} = \sum_{y=1}^{n} n\frac{a}{(y-1)+(a+1)}\binom{n-1}{y-1}\frac{\binom{a+b-1}{a}}{\binom{(n-1)+(a+1)+b-1}{(y-1)+(a+1)}}$$

$$
\begin{aligned}
&= \sum_{y=1}^{n} n \frac{a}{(y-1)+(a+1)} \binom{n-1}{y-1} \frac{\binom{a+b-1}{a}}{\binom{(n-1)+(a+1)+b-1}{(y-1)+(a+1)}} \\
&= \frac{\frac{na}{a+1}\binom{a+b-1}{a}}{\binom{a+1+b-1}{a+1}} \sum_{y=1}^{n} \frac{a+1}{(y-1)+(a+1)} \binom{n-1}{y-1} \frac{\binom{a+1+b-1}{a+1}}{\binom{(n-1)+(a+1)+b-1}{(y-1)+(a+1)}} \\
&= \frac{na}{a+b} \sum_{j=0}^{n-1} \frac{a+1}{j+(a+1)} \binom{n-1}{j} \frac{\binom{a+1+b-1}{a+1}}{\binom{(n-1)+(a+1)+b-1}{(j+(a+1))}} = \frac{na}{a+b},
\end{aligned}
$$

since the last summation is 1, being the sum over all possible values of a beta-binomial$(n-1, a+1, b)$. $\mathrm{E}[Y(Y-1)] = \frac{n(n-1)a(a+1)}{(a+b)(a+b+1)}$ is calculated similar to EY, but using the identity $y(y-1)\binom{n}{y} = n(n-1)\binom{n-2}{y-2}$ and adding 2 instead of 1 to the parameter a. The sum over all possible values of a beta-binomial$(n-2, a+2, b)$ will appear in the calculation. Therefore

$$
\mathrm{Var}(Y) = \mathrm{E}[Y(Y-1)] + \mathrm{E}Y - (\mathrm{E}Y)^2 = \frac{nab(n+a+b)}{(a+b)^2(a+b+1)}.
$$

2.30 a. $\mathrm{E}(e^{tX}) = \int_0^c e^{tx}\frac{1}{c}dx = \frac{1}{ct}e^{tx}\Big|_0^c = \frac{1}{ct}e^{tc} - \frac{1}{ct}1 = \frac{1}{ct}(e^{tc}-1)$.

b. $\mathrm{E}(e^{tX}) = \int_0^c \frac{2x}{c^2}e^{tx}dx = \frac{2}{c^2t^2}(cte^{tc} - e^{tc} + 1)$. (integration-by-parts)

c.

$$
\begin{aligned}
\mathrm{E}(e^{tx}) &= \int_{-\infty}^{\alpha} \frac{1}{2\beta} e^{(x-\alpha)/\beta} e^{tx} dx + \int_{\alpha}^{\infty} \frac{1}{2\beta} e^{-(x-\alpha)/\beta} e^{tx} dx \\
&= \frac{e^{-\alpha/\beta}}{2\beta} \frac{1}{(\frac{1}{\beta}+t)} e^{x(\frac{1}{\beta}+t)} \Bigg|_{-\infty}^{\alpha} + -\frac{e^{\alpha/\beta}}{2\beta}\frac{1}{(\frac{1}{\beta}-t)} e^{-x(\frac{1}{\beta}-t)} \Bigg|_{\alpha}^{\infty} \\
&= \frac{4e^{\alpha t}}{4-\beta^2t^2}, \quad -2/\beta < t < 2/\beta.
\end{aligned}
$$

d. $\mathrm{E}\left(e^{tX}\right) = \sum_{x=0}^{\infty} e^{tx}\binom{r+x-1}{x}p^r(1-p)^x = p^r \sum_{x=0}^{\infty}\binom{r+x-1}{x}\left((1-p)e^t\right)^x$. Now use the fact that $\sum_{x=0}^{\infty}\binom{r+x-1}{x}\left((1-p)e^t\right)^x\left(1-(1-p)e^t\right)^r = 1$ for $(1-p)e^t < 1$, since this is just the sum of this pmf, to get $\mathrm{E}(e^{tX}) = \left(\frac{p}{1-(1-p)e^t}\right)^r$, $t < -\log(1-p)$.

2.31 Since the mgf is defined as $M_X(t) = \mathrm{E}e^{tX}$, we necessarily have $M_X(0) = \mathrm{E}e^0 = 1$. But $t/(1-t)$ is 0 at $t=0$, therefore it cannot be an mgf.

2.32

$$
\frac{d}{dt}S(t)\Big|_{t=0} = \frac{d}{dt}(\log(M_x(t))\Big|_{t=0} = \frac{\frac{d}{dt}M_x(t)}{M_x(t)}\Bigg|_{t=0} = \frac{\mathrm{E}X}{1} = \mathrm{E}X \quad \big(\text{since } M_X(0) = \mathrm{E}e^0 = 1\big)
$$

$$
\begin{aligned}
\frac{d^2}{dt^2}S(t)\Big|_{t=0} &= \frac{d}{dt}\left(\frac{M_x'(t)}{M_x(t)}\right)\Big|_{t=0} = \frac{M_x(t)M_x''(t) - [M_x'(t)]^2}{[M_x(t)]^2}\Bigg|_{t=0} \\
&= \frac{1\cdot \mathrm{E}X^2 - (\mathrm{E}X)^2}{1} = \mathrm{Var}X.
\end{aligned}
$$

2.33 a. $M_X(t) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=1}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$.

$\mathrm{E}X = \frac{d}{dt}M_x(t)\big|_{t=0} = e^{\lambda(e^t-1)}\lambda e^t\Big|_{t=0} = \lambda$.

$\mathrm{E}X^2 = \frac{d^2}{dt^2}M_x(t)\Big|_{t=0} = \lambda e^t e^{\lambda(e^t-1)}\lambda e^t + \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda^2+\lambda.$

$\mathrm{Var}X = \mathrm{E}X^2 - (\mathrm{E}X)^2 = \lambda^2+\lambda-\lambda^2 = \lambda.$

b.

$$\begin{aligned}
M_x(t) &= \sum_{x=0}^{\infty} e^{tx}p(1-p)^x = p\sum_{x=0}^{\infty}((1-p)e^t)^x \\
&= p\frac{1}{1-(1-p)e^t} = \frac{p}{1-(1-p)e^t}, \quad t<-\log(1-p). \\
\mathrm{E}X &= \frac{d}{dt}M_x(t)\Big|_{t=0} = \frac{-p}{(1-(1-p)e^t)^2}\left(-(1-p)e^t\right)\Big|_{t=0} \\
&= \frac{p(1-p)}{p^2} = \frac{1-p}{p}. \\
\mathrm{E}X^2 &= \frac{d^2}{dt^2}M_x(t)\Big|_{t=0} \\
&= \frac{\left(1-(1-p)e^t\right)^2\left(p(1-p)e^t\right)+p(1-p)e^t 2\left(1-(1-p)e^t\right)(1-p)e^t}{(1-(1-p)e^t)^4}\Bigg|_{t=0} \\
&= \frac{p^3(1-p)+2p^2(1-p)^2}{p^4} = \frac{p(1-p)+2(1-p)^2}{p^2}. \\
\mathrm{Var}X &= \frac{p(1-p)+2(1-p)^2}{p^2} - \frac{(1-p)^2}{p^2} = \frac{1-p}{p^2}.
\end{aligned}$$

c. $M_x(t) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-(x^2-2\mu x-2\sigma^2 tx+\mu^2)/2\sigma^2}dx$. Now complete the square in the numerator by writing

$$\begin{aligned}
x^2-2\mu x-2\sigma^2 tx+\mu^2 &= x^2-2(\mu+\sigma^2 t)x \pm (\mu+\sigma^2 t)^2+\mu^2 \\
&= (x-(\mu+\sigma^2 t))^2-(\mu+\sigma^2 t)^2+\mu^2 \\
&= (x-(\mu+\sigma^2 t))^2-[2\mu\sigma^2 t+(\sigma^2 t)^2].
\end{aligned}$$

Then we have $M_x(t) = e^{[2\mu\sigma^2 t+(\sigma^2 t)^2]/2\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2}dx = e^{\mu t+\frac{\sigma^2 t^2}{2}}$.

$\mathrm{E}X = \frac{d}{dt}M_x(t)\Big|_{t=0} = (\mu+\sigma^2 t)e^{\mu t+\sigma^2 t^2/2}\Big|_{t=0} = \mu.$

$\mathrm{E}X^2 = \frac{d^2}{dt^2}M_x(t)\Big|_{t=0} = (\mu+\sigma^2 t)^2 e^{\mu t+\sigma^2 t^2/2}+\sigma^2 e^{\mu t+\sigma^2 t/2}\Big|_{t=0} = \mu^2+\sigma^2.$

$\mathrm{Var}X = \mu^2+\sigma^2-\mu^2 = \sigma^2.$

2.35 a.

$$\begin{aligned}
\mathrm{E}X_1^r &= \int_0^{\infty} x^r\frac{1}{\sqrt{2\pi}x}e^{-(\log x)^2/2}dx && (f_1 \text{ is lognormal with } \mu=0,\ \sigma_2=1) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{y(r-1)}e^{-y^2/2}e^y dy && (\text{substitute } y=\log x,\ dy=(1/x)dx) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2+ry}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(y^2-2ry+r^2\right)/2}e^{r^2/2}dy \\
&= e^{r^2/2}.
\end{aligned}$$

b.

$$\begin{aligned}\int_0^\infty x^r f_1(x)\sin(2\pi\log x)dx &= \int_0^\infty x^r \frac{1}{\sqrt{2\pi}x}e^{-(\log x)^2/2}\sin(2\pi\log x)dx \\ &= \int_{-\infty}^\infty e^{(y+r)r}\frac{1}{\sqrt{2\pi}}e^{-(y+r)^2/2}\sin(2\pi y+2\pi r)dy \\ &\qquad \text{(substitute } y=\log x,\ dy=(1/x)dx) \\ &= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{(r^2-y^2)/2}\sin(2\pi y)dy \\ &\qquad (\sin(a+2\pi r)=\sin(a) \text{ if } r=0,1,2,\ldots) \\ &= 0,\end{aligned}$$

because $e^{(r^2-y^2)/2}\sin(2\pi y) = -e^{(r^2-(-y)^2)/2}\sin(2\pi(-y))$; the integrand is an odd function so the negative integral cancels the positive one.

2.36 First, it can be shown that

$$\lim_{x\to\infty} e^{tx-(\log x)^2} = \infty$$

by using l'Hôpital's rule to show

$$\lim_{x\to\infty}\frac{tx-(\log x)^2}{tx} = 1,$$

and, hence,

$$\lim_{x\to\infty} tx-(\log x)^2 = \lim_{x\to\infty} tx = \infty.$$

Then for any $k>0$, there is a constant c such that

$$\int_k^\infty \frac{1}{x}e^{tx}e^{(\log x)^2/2}dx \ge c\int_k^\infty \frac{1}{x}dx = c\log x|_k^\infty = \infty.$$

Hence $M_x(t)$ does not exist.

2.37 a. The graph looks very similar to Figure 2.3.2 except that f_1 is symmetric around 0 (since it is standard normal).

b. The functions look like $t^2/2$ – it is impossible to see any difference.

c. The mgf of f_1 is $e^{K_1(t)}$. The mgf of f_2 is $e^{K_2(t)}$.

d. Make the transformation $y=e^x$ to get the densities in Example 2.3.10.

2.39 a. $\frac{d}{dx}\int_0^x e^{-\lambda t}dt = e^{-\lambda x}$. Verify

$$\frac{d}{dx}\left[\int_0^x e^{-\lambda t}dt\right] = \frac{d}{dx}\left[-\frac{1}{\lambda}e^{-\lambda t}\Big|_0^x\right] = \frac{d}{dx}\left(-\frac{1}{\lambda}e^{-\lambda x}+\frac{1}{\lambda}\right) = e^{-\lambda x}.$$

b. $\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}dt = \int_0^\infty \frac{d}{d\lambda}e^{-\lambda t}dt = \int_0^\infty -te^{-\lambda t}dt = -\frac{\Gamma(2)}{\lambda^2} = -\frac{1}{\lambda^2}$. Verify

$$\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}dt = \frac{d}{d\lambda}\frac{1}{\lambda} = -\frac{1}{\lambda^2}.$$

c. $\frac{d}{dt}\int_t^1 \frac{1}{x^2}dx = -\frac{1}{t^2}$. Verify

$$\frac{d}{dt}\left[\int_t^1 \frac{1}{x^2}dx\right] = \frac{d}{dt}\left(-\frac{1}{x}\Big|_t^1\right) = \frac{d}{dt}\left(-1+\frac{1}{t}\right) = -\frac{1}{t^2}.$$

d. $\frac{d}{dt}\int_1^\infty \frac{1}{(x-t)^2}dx = \int_1^\infty \frac{d}{dt}\left(\frac{1}{(x-t)^2}\right)dx = \int_1^\infty 2(x-t)^{-3}dx = -(x-t)^{-2}\Big|_1^\infty = \frac{1}{(1-t)^2}$. Verify

$$\frac{d}{dt}\int_1^\infty (x-t)^{-2}dx = \frac{d}{dt}\left[-(x-t)^{-1}\Big|_1^\infty\right] = \frac{d}{dt}\frac{1}{1-t} = \frac{1}{(1-t)^2}.$$