Chapter 5

Properties of a Random Sample

5.1 Let $X = \#$ color blind people in a sample of size n. Then $X \sim \text{binomial}(n,p)$, where $p = .01$. The probability that a sample contains a color blind person is $P(X > 0) = 1 - P(X = 0)$, where $P(X = 0) = \binom{n}{0}(.01)^0(.99)^n = .99^n$. Thus,

$$P(X > 0) = 1 - .99^n > .95 \Leftrightarrow n > \log(.05)/\log(.99) \approx 299.$$

5.3 Note that $Y_i \sim$ Bernoulli with $p_i = P(X_i \geq \mu) = 1 - F(\mu)$ for each i. Since the Y_i's are iid Bernoulli, $\sum_{i=1}^n Y_i \sim \text{binomial}(n, p = 1 - F(\mu))$.

5.5 Let $Y = X_1 + \cdots + X_n$. Then $\bar{X} = (1/n)Y$, a scale transformation. Therefore the pdf of $\bar{X}$ is $f_{\bar{X}}(x) = \frac{1}{1/n} f_Y\left(\frac{x}{1/n}\right) = n f_Y(nx)$.

5.6 a. For $Z = X - Y$, set $W = X$. Then $Y = W - Z$, $X = W$, and $|J| = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1$. Then $f_{Z,W}(z,w) = f_X(w) f_Y(w - z) \cdot 1$, thus $f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(w - z) dw$.

b. For $Z = XY$, set $W = X$. Then $Y = Z/W$ and $|J| = \begin{vmatrix} 0 & 1 \\ 1/w & -z/w^2 \end{vmatrix} = -1/w$. Then $f_{Z,W}(z,w) = f_X(w) f_Y(z/w) \cdot |-1/w|$, thus $f_Z(z) = \int_{-\infty}^{\infty} |-1/w| \, f_X(w) f_Y(z/w) dw$.

c. For $Z = X/Y$, set $W = X$. Then Y=W/Z and $|J| = \begin{vmatrix} 0 & 1 \\ -w/z^2 & 1/z \end{vmatrix} = w/z^2$. Then $f_{Z,W}(z,w) = f_X(w) f_Y(w/z) \cdot |w/z^2|$, thus $f_Z(z) = \int_{-\infty}^{\infty} |w/z^2| f_X(w) f_Y(w/z) dw$.

5.7 It is, perhaps, easiest to recover the constants by doing the integrations. We have

$$\int_{-\infty}^{\infty} \frac{B}{1+\left(\frac{\omega}{\sigma}\right)^2} d\omega = \sigma\pi B, \qquad \int_{-\infty}^{\infty} \frac{D}{1+\left(\frac{\omega-z}{\tau}\right)^2} d\omega = \tau\pi D$$

and

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[\frac{A\omega}{1+\left(\frac{\omega}{\sigma}\right)^2} - \frac{C\omega}{1+\left(\frac{\omega-z}{\tau}\right)^2}\right] d\omega \\ &= \int_{-\infty}^{\infty} \left[\frac{A\omega}{1+\left(\frac{\omega}{\sigma}\right)^2} - \frac{C(\omega-z)}{1+\left(\frac{\omega-z}{\tau}\right)^2}\right] d\omega - Cz \int_{-\infty}^{\infty} \frac{1}{1+\left(\frac{\omega-z}{\tau}\right)^2} d\omega \\ &= A\frac{\sigma^2}{2} \log\left[1+\left(\frac{\omega}{\sigma}\right)^2\right] - \frac{C\tau^2}{2} \log\left[1+\left(\frac{\omega-z}{\tau}\right)^2\right]\Bigg|_{-\infty}^{\infty} - \tau\pi Cz. \end{aligned}$$

The integral is finite and equal to zero if $A = M\frac{2}{\sigma^2}$, $C = M\frac{2}{\tau^2}$ for some constant M. Hence

$$f_Z(z) = \frac{1}{\pi^2\sigma\tau}\left[\sigma\pi B - \tau\pi D - \frac{2\pi M z}{\tau}\right] = \frac{1}{\pi(\sigma+\tau)} \, \frac{1}{1+\left(z/(\sigma+\tau)\right)^2},$$

if $B = \frac{\tau}{\sigma+\tau}$, $D = \frac{\sigma}{\sigma+\tau)}$, $M = \frac{-\sigma\tau^2}{2z(\sigma+\tau)} \frac{1}{1+\left(\frac{z}{\sigma+\tau}\right)^2}$.

5.8 a.

$$
\begin{aligned}
&\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-X_j)^2\\
&= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-\bar{X}+\bar{X}-X_j)^2\\
&= \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(X_i-\bar{X})^2-2(X_i-\bar{X})(X_j-\bar{X})+(X_j-\bar{X})^2\right]\\
&= \frac{1}{2n(n-1)}\left[\sum_{i=1}^{n}n(X_i-\bar{X})^2-2\sum_{i=1}^{n}(X_i-\bar{X})\underbrace{\sum_{j=1}^{n}(X_j-\bar{X})}_{=0}+n\sum_{j=1}^{n}(X_j-\bar{X})^2\right]\\
&= \frac{n}{2n(n-1)}\sum_{i=1}^{n}(X_i-\bar{X})^2+\frac{n}{2n(n-1)}\sum_{j=1}^{n}(X_j-\bar{X})^2\\
&= \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2 \quad = \quad S^2.
\end{aligned}
$$

b. Although all of the calculations here are straightforward, there is a tedious amount of bookkeeping needed. It seems that induction is the easiest route. (Note: Without loss of generality we can assume $\theta_1 = 0$, so $\mathrm{E}X_i = 0$.)

(i) Prove the equation for $n = 4$. We have $S^2 = \frac{1}{24}\sum_{i=1}^{4}\sum_{j=1}^{4}(X_i - X_j)^2$, and to calculate $\mathrm{Var}(S^2)$ we need to calculate $\mathrm{E}(S^2)^2$ and $\mathrm{E}(S^2)$. The latter expectation is straightforward and we get $\mathrm{E}(S^2) = 24\theta_2$. The expected value $\mathrm{E}(S^2)^2 = \mathrm{E}(S^4)$ contains $256(= 4^4)$ terms of which $112(= 4 \times 16 + 4 \times 16 - 4^2)$ are zero, whenever $i = j$. Of the remaining terms,

- 24 are of the form $\mathrm{E}(X_i - X_j)^4 = 2(\theta_4 + 3\theta_2^2)$
- 96 are of the form $\mathrm{E}(X_i - X_j)^2(X_i - X_k)^2 = \theta_4 + 3\theta_2^2$
- 24 are of the form $\mathrm{E}(X_i - X_j)^2(X_k - X_\ell)^2 = 4\theta_2^2$

Thus,

$$
\mathrm{Var}(S^2) = \frac{1}{24^2}\left[24\times 2(\theta_4+3\theta_2^2)+96(\theta_4+3\theta_2^2)+24\times 4\theta_4-(24\theta_2)^2\right] = \frac{1}{4}\left[\theta_4-\frac{1}{3}\theta_2^2\right].
$$

(ii) Assume that the formula holds for n, and establish it for $n+1$. (Let S_n denote the variance based on n observations.) Straightforward algebra will establish

$$
\begin{aligned}
S_{n+1}^2 &= \frac{1}{2n(n+1)}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-X_j)^2+2\sum_{k=1}^{n}(X_k-X_{n+1})^2\right]\\
&\stackrel{\text{def'n}}{=} \frac{1}{2n(n+1)}[A+2B]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Var}(A) &= 4n(n-1)^2\left[\theta_4-\frac{n-3}{n-1}\theta_2^2\right] && \text{(induction hypothesis)}\\
\mathrm{Var}(B) &= n(n+1)\theta_4-n(n-3)\theta_2^2 && (X_k \text{ and } X_{n+1} \text{ are independent})\\
\mathrm{Cov}(A,B) &= 2n(n-1)\left[\theta_4-\theta_2^2\right] && \text{(some minor bookkeeping needed)}
\end{aligned}
$$

Hence,

$$\mathrm{Var}(S^2_{n+1}) = \frac{1}{4n^2(n+1)^2}\left[\mathrm{Var}(A) + 4\mathrm{Var}(B) + 4\mathrm{Cov}(A,B)\right] = \frac{1}{n+1}\left[\theta_4 - \frac{n-2}{n}\theta_2^2\right],$$

establishing the induction and verifying the result.

c. Again assume that $\theta_1 = 0$. Then

$$\mathrm{Cov}(\bar{X}, S^2) = \frac{1}{2n^2(n-1)}\mathrm{E}\left\{\sum_{k=1}^{n} X_k \sum_{i=1}^{n}\sum_{j=1}^{n}(X_i - X_j)^2\right\}.$$

The double sum over i and j has $n(n-1)$ nonzero terms. For each of these, the entire expectation is nonzero for only two values of k (when k matches either i or j). Thus

$$\mathrm{Cov}(\bar{X}, S^2) = \frac{2n(n-1)}{2n^2(n-1)}\mathrm{E}X_i(X_i - X_j)^2 = \frac{1}{n}\theta_3,$$

and $\bar{X}$ and S^2 are uncorrelated if $\theta_3 = 0$.

5.9 To establish the Lagrange Identity consider the case when $n = 2$,

$$\begin{aligned}(a_1b_2 - a_2b_1)^2 &= a_1^2b_2^2 + a_2^2b_1^2 - 2a_1b_2a_2b_1 \\ &= a_1^2b_2^2 + a_2^2b_1^2 - 2a_1b_2a_2b_1 + a_1^2b_1^2 + a_2^2b_2^2 - a_1^2b_1^2 - a_2^2b_2^2 \\ &= (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2.\end{aligned}$$

Assume that is true for n, then

$$\begin{aligned}&\left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{i=1}^{n+1} b_i^2\right) - \left(\sum_{i=1}^{n+1} a_ib_i\right)^2 \\ &= \left(\sum_{i=1}^{n} a_i^2 + a_{n+1}^2\right)\left(\sum_{i=1}^{n} b_i^2 + b_{n+1}^2\right) - \left(\sum_{i=1}^{n} a_ib_i + a_{n+1}b_{n+1}\right)^2 \\ &= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_ib_i\right)^2 \\ &\quad + \left(\sum_{i=1}^{n} a_i^2\right) b_{n+1}^2 + a_{n+1}^2\left(\sum_{i=1}^{n} b_i^2\right) - 2\left(\sum_{i=1}^{n} a_ib_i\right) a_{n+1}b_{n+1} \\ &= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_ib_j - a_jb_i)^2 + \sum_{i=1}^{n}(a_ib_{n+1} - a_{n+1}b_i)^2 \\ &= \sum_{i=1}^{n}\sum_{j=i+1}^{n+1}(a_ib_j - a_jb_i)^2.\end{aligned}$$

If all the points lie on a straight line then $Y - \mu_y = c(X - \mu_x)$, for some constant $c \neq 0$. Let $b_i = Y - \mu_y$ and $a_i = (X - \mu_x)$, then $b_i = ca_i$. Therefore $\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}(a_ib_j - a_jb_i)^2 = 0$. Thus the correlation coefficient is equal to 1.

5.10 a.

$$\theta_1 = \mathrm{E}X_i = \mu$$

$$\begin{aligned}
\theta_2 &= \mathrm{E}(X_i-\mu)^2 \;=\; \sigma^2\\
\theta_3 &= \mathrm{E}(X_i-\mu)^3\\
&= \mathrm{E}(X_i-\mu)^2(X_i-\mu) \qquad \text{(Stein's lemma: } \mathrm{E}g(X)(X-\theta)=\sigma^2\mathrm{E}g'(X))\\
&= 2\sigma^2\mathrm{E}(X_i-\mu) \;=\; 0\\
\theta_4 &= \mathrm{E}(X_i-\mu)^4 \;=\; \mathrm{E}(X_i-\mu)^3(X_i-\mu) \;=\; 3\sigma^2\mathrm{E}(X_i-\mu)^2 \;=\; 3\sigma^4.
\end{aligned}$$

b. $\mathrm{Var}S^2 = \frac{1}{n}(\theta_4 - \frac{n-3}{n-1}\theta_2^2) = \frac{1}{n}(3\sigma^4 - \frac{n-3}{n-1}\sigma^4) = \frac{2\sigma^4}{n-1}$.

c. Use the fact that $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$ and $\mathrm{Var}\chi^2_{n-1} = 2(n-1)$ to get

$$\mathrm{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

which implies $(\frac{(n-1)^2}{\sigma^4})\mathrm{Var}S^2 = 2(n-1)$ and hence

$$\mathrm{Var}S^2 = \frac{2(n-1)}{(n-1)^2/\sigma^4} = \frac{2\sigma^4}{n-1}.$$

Remark: Another approach to b), not using the χ^2 distribution, is to use linear model theory. For any matrix A $\mathrm{Var}(X'AX) = 2\mu_2^2\mathrm{tr}A^2 + 4\mu_2\theta'A\theta$, where μ_2 is σ^2, $\theta = \mathrm{E}X = \mu 1$. Write $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}) = \frac{1}{n-1}X'(I-\bar{J}_n)X$.Where

$$I-\bar{J}_n = \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n}\\ -\frac{1}{n} & 1-\frac{1}{n} & & \vdots\\ \vdots & & \ddots & \vdots\\ -\frac{1}{n} & \cdots & \cdots & 1-\frac{1}{n}\end{pmatrix}.$$

Notice that $\mathrm{tr}A^2 = \mathrm{tr}A = n-1$, $A\theta = 0$. So

$$\mathrm{Var}S^2 = \frac{1}{(n-1)^2}\mathrm{Var}(X'AX) = \frac{1}{(n-1)^2}\left(2\sigma^4(n-1)+0\right) = \frac{2\sigma^4}{n-1}.$$

5.11 Let $g(s) = s^2$. Since $g(\cdot)$ is a convex function, we know from Jensen's inequality that $\mathrm{E}g(S) \geq g(\mathrm{E}S)$, which implies $\sigma^2 = \mathrm{E}S^2 \geq (\mathrm{E}S)^2$. Taking square roots, $\sigma \geq \mathrm{E}S$. From the proof of Jensen's Inequality, it is clear that, in fact, the inequality will be strict unless there is an interval I such that g is linear on I and $P(X \in I) = 1$. Since s^2 is "linear" only on single points, we have $\mathrm{E}T^2 > (\mathrm{E}T)^2$ for any random variable T, unless $P(T = \mathrm{E}T) = 1$.

5.13

$$\begin{aligned}
\mathrm{E}\left(c\sqrt{S^2}\right) &= c\sqrt{\frac{\sigma^2}{n-1}}\mathrm{E}\left(\sqrt{\frac{S^2(n-1)}{\sigma^2}}\right)\\
&= c\sqrt{\frac{\sigma^2}{n-1}}\int_0^\infty \sqrt{q}\frac{1}{\Gamma\left(\frac{n-1}{2}\right)2^{(n-1)/2}}q^{\left(\frac{n-1}{2}\right)-1}e^{-q/2}dq,
\end{aligned}$$

Since $\sqrt{S^2(n-1)/\sigma^2}$ is the square root of a χ^2 random variable. Now adjust the integrand to be another χ^2 pdf and get

$$\mathrm{E}\left(c\sqrt{S^2}\right) = c\sqrt{\frac{\sigma^2}{n-1}}\cdot\frac{\Gamma(n/2)2^{n/2}}{\Gamma((n-1)/2)2^{((n-1)/2}}\underbrace{\int_0^\infty \frac{1}{\Gamma(n/2)2^{n/2}}q^{(n-1)/2}-\frac{1}{2}e^{-q/2}dq}_{=1 \text{ since } \chi^2_n \text{ pdf}}.$$

So $c = \frac{\Gamma\left(\frac{n-1}{2}\right)\sqrt{n-1}}{\sqrt{2}\Gamma\left(\frac{n}{2}\right)}$ gives $\mathrm{E}(cS) = \sigma$.

5.15 a.

$$\bar{X}_{n+1} = \frac{\sum_{i=1}^{n+1} X_i}{n+1} = \frac{X_{n+1} + \sum_{i=1}^{n} X_i}{n+1} = \frac{X_{n+1} + n\bar{X}_n}{n+1}.$$

b.

$$\begin{aligned}
nS_{n+1}^2 &= \frac{n}{(n+1)-1}\sum_{i=1}^{n+1}\left(X_i - \bar{X}_{n+1}\right)^2 \\
&= \sum_{i=1}^{n+1}\left(X_i - \frac{X_{n+1}+n\bar{X}_n}{n+1}\right)^2 \qquad \text{(use (a))} \\
&= \sum_{i=1}^{n+1}\left(X_i - \frac{X_{n+1}}{n+1} - \frac{n\bar{X}_n}{n+1}\right)^2 \\
&= \sum_{i=1}^{n+1}\left[\left(X_i - \bar{X}_n\right) - \left(\frac{X_{n+1}}{n+1} - \frac{\bar{X}_n}{n+1}\right)\right]^2 \qquad (\pm\bar{X}_n) \\
&= \sum_{i=1}^{n+1}\left[\left(X_i - \bar{X}_n\right)^2 - 2\left(X_i - \bar{X}_n\right)\left(\frac{X_{n+1}-\bar{X}_n}{n+1}\right) + \frac{1}{(n+1)^2}\left(X_{n+1}-\bar{X}_n\right)^2\right] \\
&= \sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 + \left(X_{n+1}-\bar{X}_n\right)^2 - 2\frac{\left(X_{n+1}-\bar{X}_n\right)^2}{n+1} + \frac{n+1}{(n+1)^2}\left(X_{n+1}-\bar{X}_n\right)^2 \\
&\qquad \left(\text{since} \sum_{1}^{n}(X_i - \bar{X}_n) = 0\right) \\
&= (n-1)S_n^2 + \frac{n}{n+1}\left(X_{n+1}-\bar{X}_n\right)^2.
\end{aligned}$$

5.16 a. $\sum_{i=1}^{3}\left(\frac{X_i - i}{i}\right)^2 \sim \chi_3^2$

b. $\left(\frac{X_i - 1}{i}\right) \Big/ \sqrt{\sum_{i=2}^{3}\left(\frac{X_i - i}{i}\right)^2 \Big/ 2} \sim t_2$

c. Square the random variable in part b).

5.17 a. Let $U \sim \chi_p^2$ and $V \sim \chi_q^2$, independent. Their joint pdf is

$$\frac{1}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)2^{(p+q)/2}} u^{\frac{p}{2}-1} v^{\frac{q}{2}-1} e^{\frac{-(u+v)}{2}}.$$

From Definition 5.3.6, the random variable $X = (U/p)/(V/q)$ has an F distribution, so we make the transformation $x = (u/p)/(v/q)$ and $y = u + v$. (Of course, many choices of y will do, but this one makes calculations easy. The choice is prompted by the exponential term in the pdf.) Solving for u and v yields

$$u = \frac{\frac{p}{q}xy}{1+\frac{q}{p}x}, \quad v = \frac{y}{1+\frac{q}{p}x}, \text{ and } |J| = \frac{\frac{q}{p}y}{\left(1+\frac{q}{p}x\right)^2}.$$

We then substitute into $f_{U,V}(u,v)$ to obtain

$$f_{X,Y}(x,y) = \frac{1}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)2^{(p+q)/2}}\left(\frac{\frac{p}{q}xy}{1+\frac{q}{p}x}\right)^{\frac{p}{2}-1}\left(\frac{y}{1+\frac{q}{p}x}\right)^{\frac{q}{2}-1} e^{\frac{-y}{2}} \frac{\frac{q}{p}y}{\left(1+\frac{q}{p}x\right)^2}.$$

Note that the pdf factors, showing that X and Y are independent, and we can read off the pdfs of each: X has the F distribution and Y is χ^2_{p+q}. If we integrate out y to recover the proper constant, we get the F pdf

$$f_X(x) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{q}{p}\right)^{p/2} \frac{x^{p/2-1}}{\left(1+\frac{q}{p}x\right)^{\frac{p+q}{2}}}.$$

b. Since $F_{p,q} = \frac{\chi^2_p/p}{\chi^2_q/q}$, let $U \sim \chi^2_p$, $V \sim \chi^2_q$ and U and V are independent. Then we have

$$\begin{aligned} \mathrm{E}F_{p,q} &= \mathrm{E}\left(\frac{U/p}{V/q}\right) = \mathrm{E}\left(\frac{U}{p}\right)\mathrm{E}\left(\frac{q}{V}\right) && \text{(by independence)} \\ &= \frac{p}{p}q\mathrm{E}\left(\frac{1}{V}\right) && (\mathrm{E}U = p). \end{aligned}$$

Then

$$\begin{aligned} \mathrm{E}\left(\frac{1}{V}\right) &= \int_0^\infty \frac{1}{v}\frac{1}{\Gamma\left(\frac{q}{2}\right)2^{q/2}} v^{\frac{q}{2}-1}e^{-\frac{v}{2}}dv = \frac{1}{\Gamma\left(\frac{q}{2}\right)2^{q/2}}\int_0^\infty v^{\frac{q-2}{2}-1}e^{-\frac{v}{2}}dv \\ &= \frac{1}{\Gamma\left(\frac{q}{2}\right)2^{q/2}}\Gamma\left(\frac{q-2}{2}\right)2^{(q-2)/2} = \frac{\Gamma\left(\frac{q-2}{2}\right)2^{(q-2)/2}}{\Gamma\left(\frac{q-2}{2}\right)\left(\frac{q-2}{2}\right)2^{q/2}} = \frac{1}{q-2}. \end{aligned}$$

Hence, $\mathrm{E}F_{p,q} = \frac{p}{p}\frac{q}{q-2} = \frac{q}{q-2}$, if $q > 2$. To calculate the variance, first calculate

$$\mathrm{E}(F^2_{p,q}) = \mathrm{E}\left(\frac{U^2}{p^2}\frac{q^2}{V^2}\right) = \frac{q^2}{p^2}\mathrm{E}(U^2)\mathrm{E}\left(\frac{1}{V^2}\right).$$

Now

$$\mathrm{E}(U^2) = \mathrm{Var}(U) + (\mathrm{E}U)^2 = 2p + p^2$$

and

$$\mathrm{E}\left(\frac{1}{V^2}\right) = \int_0^\infty \frac{1}{v^2}\frac{1}{\Gamma(q/2)\,2^{q/2}} v^{(q/2)-1}e^{-v/2}dv = \frac{1}{(q-2)(q-4)}.$$

Therefore,

$$\mathrm{E}F^2_{p,q} = \frac{q^2}{p^2}p(2+p)\frac{1}{(q-2)(q-4)} = \frac{q^2}{p}\frac{(p+2)}{(q-2)(q-4)},$$

and, hence

$$\mathrm{Var}(F_{p,q}) = \frac{q^2(p+2)}{p(q-2)(q-4)} - \frac{q^2}{(q-2)^2} = 2\left(\frac{q}{q-2}\right)^2\left(\frac{q+p-2}{p(q-4)}\right), \quad q > 4.$$

c. Write $X = \frac{U/p}{V/p}$ then $\frac{1}{X} = \frac{V/q}{U/p} \sim F_{q,p}$, since $U \sim \chi^2_p$, $V \sim \chi^2_q$ and U and V are independent.

d. Let $Y = \frac{(p/q)X}{1+(p/q)X} = \frac{pX}{q+pX}$, so $X = \frac{qY}{p(1-Y)}$ and $\left|\frac{dx}{dy}\right| = \frac{q}{p}(1-y)^{-2}$. Thus, Y has pdf

$$\begin{aligned} f_Y(y) &= \frac{\Gamma\left(\frac{q+p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{p}{q}\right)^{\frac{p}{2}} \frac{\left(\frac{qy}{p(1-y)}\right)^{\frac{p-2}{2}}}{\left(1+\frac{p}{q}\frac{qy}{p(1-y)}\right)^{\frac{p+q}{2}}}\frac{q}{p(1-y)^2} \\ &= \left[B\left(\frac{p}{2},\frac{q}{2}\right)\right]^{-1} y^{\frac{p}{2}-1}(1-y)^{\frac{q}{2}-1} \sim \mathrm{beta}\left(\frac{p}{2},\frac{q}{2}\right). \end{aligned}$$

5.18 If $X \sim t_p$, then $X = Z/\sqrt{V/p}$ where $Z \sim \mathrm{n}(0,1)$, $V \sim \chi^2_p$ and Z and V are independent.

a. $\mathrm{E}X = \mathrm{E}Z/\sqrt{V/p} = (\mathrm{E}Z)(\mathrm{E}1/\sqrt{V/p}) = 0$, since $\mathrm{E}Z = 0$, as long as the other expectation is finite. This is so if $p > 1$. From part b), $X^2 \sim F_{1,p}$. Thus $\mathrm{Var}X = \mathrm{E}X^2 = p/(p-2)$, if $p > 2$ (from Exercise 5.17b).

b. $X^2 = Z^2/(V/p)$. $Z^2 \sim \chi^2_1$, so the ratio is distributed $F_{1,p}$.

c. The pdf of X is

$$f_X(x) = \left[\frac{\Gamma(\frac{p+1}{2})}{\Gamma(p/2)\sqrt{p\pi}}\right] \frac{1}{(1+x^2/p)^{(p+1)/2}}.$$

Denote the quantity in square brackets by C_p. From an extension of Stirling's formula (Exercise 1.28) we have

$$\begin{aligned}
\lim_{p\to\infty} C_p &= \lim_{p\to\infty} \frac{\sqrt{2\pi}\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}} e^{-\frac{p-1}{2}}}{\sqrt{2\pi}\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}} e^{-\frac{p-2}{2}}} \frac{1}{\sqrt{p\pi}} \\
&= \frac{e^{-1/2}}{\sqrt{\pi}} \lim_{p\to\infty} \frac{\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}}}{\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}} \sqrt{p}} = \frac{e^{-1/2}}{\sqrt{\pi}} \frac{e^{1/2}}{\sqrt{2}},
\end{aligned}$$

by an application of Lemma 2.3.14. Applying the lemma again shows that for each x

$$\lim_{p\to\infty} \left(1+x^2/p\right)^{(p+1)/2} = e^{x^2/2},$$

establishing the result.

d. As the random variable $F_{1,p}$ is the square of a t_p, we conjecture that it would converge to the square of a $\mathrm{n}(0,1)$ random variable, a χ^2_1.

e. The random variable $qF_{q,p}$ can be thought of as the sum of q random variables, each a t_p squared. Thus, by all of the above, we expect it to converge to a χ^2_q random variable as $p \to \infty$.

5.19 a. $\chi^2_p \sim \chi^2_q + \chi^2_d$ where χ^2_q and χ^2_d are independent χ^2 random variables with q and $d = p - q$ degrees of freedom. Since χ^2_d is a positive random variable, for any $a > 0$,

$$P(\chi_p > a) = P(\chi^2_q + \chi^2_d > a) > P(\chi^2_q > a).$$

b. For $k_1 > k_2$, $k_1 \mathrm{F}_{k_1,\nu} \sim (U+V)/(W/\nu)$, where U, V and W are independent and $U \sim \chi^2_{k_2}$, $V \sim \chi^2_{k_1-k_2}$ and $W \sim \chi^2_\nu$. For any $a > 0$, because $V/(W/\nu)$ is a positive random variable, we have

$$P(k_1F_{k_1,\nu} > a) = P((U+V)/(W/\nu) > a) > P(U/(W/\nu) > a) = P(k_2F_{k_2,\nu} > a).$$

c. $\alpha = P(F_{k,\nu} > F_{\alpha,k,\nu}) = P(kF_{k,\nu} > kF_{\alpha,k,\nu})$. So, $kF_{\alpha,k,\nu}$ is the α cutoff point for the random variable $kF_{k,\nu}$. Because $kF_{k,\nu}$ is stochastically larger that $(k-1)F_{k-1,\nu}$, the α cutoff for $kF_{k,\nu}$ is larger than the α cutoff for $(k-1)F_{k-1,\nu}$, that is $kF_{\alpha,k,\nu} > (k-1)F_{\alpha,k-1,\nu}$.

5.20 a. The given integral is

$$\begin{aligned}
&\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2x/2} \nu\sqrt{x} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} (\nu x)^{(\nu/2)-1} e^{-\nu x/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{\nu^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2}} \int_0^\infty e^{-t^2x/2} x^{((\nu+1)/2)-1} e^{-\nu x/2} dx
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}}\frac{\nu^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2}}\int_0^\infty x^{((\nu+1)/2)-1}e^{-(\nu+t^2)x/2}dx \qquad \left(\begin{array}{c}\text{integrand is kernel of}\\ \text{gamma}((\nu+1)/2, 2/(\nu+t^2))\end{array}\right) \\ &= \frac{1}{\sqrt{2\pi}}\frac{\nu^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2}}\Gamma((\nu+1)/2)\left(\frac{2}{\nu+t^2}\right)^{(\nu+1)/2} \\ &= \frac{1}{\sqrt{\nu\pi}}\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)}\frac{1}{(1+t^2/\nu)^{(\nu+1)/2}}, \end{aligned}$$

the pdf of a t_ν distribution.

b. Differentiate both sides with respect to t to obtain

$$\nu f_F(\nu t) = \int_0^\infty y f_1(ty) f_\nu(y)dy,$$

where f_F is the F pdf. Now write out the two chi-squared pdfs and collect terms to get

$$\begin{aligned} \nu f_F(\nu t) &= \frac{t^{-1/2}}{\Gamma(1/2)\Gamma(\nu/2)2^{(\nu+1)/2}}\int_0^\infty y^{(\nu-1)/2}e^{-(1+t)y/2}dy \\ &= \frac{t^{-1/2}}{\Gamma(1/2)\Gamma(\nu/2)2^{(\nu+1)/2}}\frac{\Gamma(\frac{\nu+1}{2})2^{(\nu+1)/2}}{(1+t)^{(\nu+1)/2}}. \end{aligned}$$

Now define $y = \nu t$ to get

$$f_F(y) = \frac{\Gamma(\frac{\nu+1}{2})}{\nu\Gamma(1/2)\Gamma(\nu/2)}\frac{(y/\nu)^{-1/2}}{(1+y/\nu)^{(\nu+1)/2}},$$

the pdf of an $F_{1,\nu}$.

c. Again differentiate both sides with respect to t, write out the chi-squared pdfs, and collect terms to obtain

$$(\nu/m)f_F((\nu/m)t) = \frac{t^{-m/2}}{\Gamma(m/2)\Gamma(\nu/2)2^{(\nu+m)/2}}\int_0^\infty y^{(m+\nu-2)/2}e^{-(1+t)y/2}dy.$$

Now, as before, integrate the gamma kernel, collect terms, and define $y = (\nu/m)t$ to get

$$f_F(y) = \frac{\Gamma(\frac{\nu+m}{2})}{\Gamma(m/2)\Gamma(\nu/2)}\left(\frac{m}{\nu}\right)^{m/2}\frac{y^{m/2-1}}{(1+(m/\nu)y)^{(\nu+m)/2}},$$

the pdf of an $F_{m,\nu}$.

5.21 Let m denote the median. Then, for general n we have

$$\begin{aligned} P(\max(X_1,\ldots,X_n) > m) &= 1 - P(X_i \le m \text{ for } i = 1,2,\ldots,n) \\ &= 1 - [P(X_1 \le m)]^n = 1 - \left(\frac{1}{2}\right)^n. \end{aligned}$$

5.22 Calculating the cdf of Z^2, we obtain

$$\begin{aligned} F_{Z^2}(z) &= P((\min(X,Y))^2 \le z) = P(-z \le \min(X,Y) \le \sqrt{z}) \\ &= P(\min(X,Y) \le \sqrt{z}) - P(\min(X,Y) \le -\sqrt{z}) \\ &= [1 - P(\min(X,Y) > \sqrt{z})] - [1 - P(\min(X,Y) > -\sqrt{z})] \\ &= P(\min(X,Y) > -\sqrt{z}) - P(\min(X,Y) > \sqrt{z}) \\ &= P(X > -\sqrt{z})P(Y > -\sqrt{z}) - P(X > \sqrt{z})P(Y > \sqrt{z}), \end{aligned}$$

where we use the independence of X and Y. Since X and Y are identically distributed, $P(X > a) = P(Y > a) = 1 - F_X(a)$, so

$$F_{Z^2}(z) = (1 - F_X(-\sqrt{z}))^2 - (1 - F_X(\sqrt{z}))^2 = 1 - 2F_X(-\sqrt{z}),$$

since $1 - F_X(\sqrt{z}) = F_X(-\sqrt{z})$. Differentiating and substituting gives

$$f_{Z^2}(z) = \frac{d}{dz}F_{Z^2}(z) = f_X(-\sqrt{z})\frac{1}{\sqrt{z}} = \frac{1}{\sqrt{2\pi}}e^{-z/2}z^{-1/2},$$

the pdf of a χ^2_1 random variable. Alternatively,

$$\begin{aligned}
P(Z^2 \le z) &= P\left([\min(X,Y)]^2 \le z\right) \\
&= P(-\sqrt{z} \le \min(X,Y) \le \sqrt{z}) \\
&= P(-\sqrt{z} \le X \le \sqrt{z}, X \le Y) + P(-\sqrt{z} \le Y \le \sqrt{z}, Y \le X) \\
&= P(-\sqrt{z} \le X \le \sqrt{z}|X \le Y)P(X \le Y) \\
&\quad +P(-\sqrt{z} \le Y \le \sqrt{z}|Y \le X)P(Y \le X) \\
&= \frac{1}{2}P(-\sqrt{z} \le X \le \sqrt{z}) + \frac{1}{2}P(-\sqrt{z} \le Y \le \sqrt{z}),
\end{aligned}$$

using the facts that X and Y are independent, and $P(Y \le X) = P(X \le Y) = \frac{1}{2}$. Moreover, since X and Y are identically distributed

$$P(Z^2 \le z) = P(-\sqrt{z} \le X \le \sqrt{z})$$

and

$$\begin{aligned}
f_{Z^2}(z) &= \frac{d}{dz}P(-\sqrt{z} \le X \le \sqrt{z}) = \frac{1}{\sqrt{2\pi}}(e^{-z/2}\frac{1}{2}z^{-1/2} + e^{-z/2}\frac{1}{2}z^{-1/2}) \\
&= \frac{1}{\sqrt{2\pi}}z^{-1/2}e^{-z/2},
\end{aligned}$$

the pdf of a χ^2_1.

5.23

$$\begin{aligned}
P(Z > z) &= \sum_{x=1}^{\infty} P(Z > z|x)P(X = x) = \sum_{x=1}^{\infty} P(U_1 > z, \ldots, U_x > z|x)P(X = x) \\
&= \sum_{x=1}^{\infty}\prod_{i=1}^{x} P(U_i > z)P(X = x) \qquad \text{(by independence of the } U_i\text{'s)} \\
&= \sum_{x=1}^{\infty} P(U_i > z)^x P(X = x) = \sum_{x=1}^{\infty}(1-z)^x \frac{1}{(e-1)x!} \\
&= \frac{1}{(e-1)}\sum_{x=1}^{\infty}\frac{(1-z)^x}{x!} = \frac{e^{1-z}-1}{e-1} \qquad 0 < z < 1.
\end{aligned}$$

5.24 Use $f_X(x) = 1/\theta$, $F_X(x) = x/\theta$, $0 < x < \theta$. Let $Y = X_{(n)}$, $Z = X_{(1)}$. Then, from Theorem 5.4.6,

$$f_{Z,Y}(z,y) = \frac{n!}{0!(n-2)!0!}\frac{1}{\theta}\frac{1}{\theta}\left(\frac{z}{\theta}\right)^0\left(\frac{y-z}{\theta}\right)^{n-2}\left(1-\frac{y}{\theta}\right)^0 = \frac{n(n-1)}{\theta^n}(y-z)^{n-2},\ 0 < z < y < \theta.$$

Now let $W = Z/Y$, $Q = Y$. Then $Y = Q$, $Z = WQ$, and $|J| = q$. Therefore

$$f_{W,Q}(w,q) = \frac{n(n-1)}{\theta^n}(q-wq)^{n-2}q = \frac{n(n-1)}{\theta^n}(1-w)^{n-2}q^{n-1},\ 0<w<1, 0<q<\theta.$$

The joint pdf factors into functions of w and q, and, hence, W and Q are independent.

5.25 The joint pdf of $X_{(1)},\dots,X_{(n)}$ is

$$f(u_1,\dots,u_n) = \frac{n!a^n}{\theta^{an}}u_1^{a-1}\cdots u_n^{a-1}, \quad 0<u_1<\cdots<u_n<\theta.$$

Make the one-to-one transformation to $Y_1 = X_{(1)}/X_{(2)},\dots,Y_{n-1} = X_{(n-1)}/X_{(n)}, Y_n = X_{(n)}$. The Jacobian is $J = y_2y_3^2\cdots y_n^{n-1}$. So the joint pdf of $Y_1,\dots,Y_n$ is

$$\begin{aligned} f(y_1,\dots,y_n) &= \frac{n!a^n}{\theta^{an}}(y_1\cdots y_n)^{a-1}(y_2\cdots y_n)^{a-1}\cdots(y_n)^{a-1}(y_2y_3^2\cdots y_n^{n-1}) \\ &= \frac{n!a^n}{\theta^{an}}y_1^{a-1}y_2^{2a-1}\cdots y_n^{na-1}, \quad 0<y_i<1; i=1,\dots,n-1, \quad 0<y_n<\theta. \end{aligned}$$

We see that $f(y_1,\dots,y_n)$ factors so $Y_1,\dots,Y_n$ are mutually independent. To get the pdf of Y_1, integrate out the other variables and obtain that $f_{Y_1}(y_1) = c_1y_1^{a-1}$, $0<y_1<1$, for some constant c_1. To have this pdf integrate to 1, it must be that $c_1 = a$. Thus $f_{Y_1}(y_1) = ay_1^{a-1}$, $0<y_1<1$. Similarly, for $i = 2,\dots,n-1$, we obtain $f_{Y_i}(y_i) = iay_i^{ia-1}, 0<y_i<1$. From Theorem 5.4.4, the pdf of Y_n is $f_{Y_n}(y_n) = \frac{na}{\theta^{na}}y_n^{na-1}$, $0<y_n<\theta$. It can be checked that the product of these marginal pdfs is the joint pdf given above.

5.27 a. $f_{X_{(i)}|X_{(j)}}(u|v) = f_{X_{(i)},X_{(j)}}(u,v)/f_{X_{(j)}}(v)$. Consider two cases, depending on which of i or j is greater. Using the formulas from Theorems 5.4.4 and 5.4.6, and after cancellation, we obtain the following.

(i) If $i<j$,

$$\begin{aligned} f_{X_{(i)}|X_{(j)}}(u|v) &= \frac{(j-1)!}{(i-1)!(j-1-i)!}f_X(u)F_X^{i-1}(u)[F_X(v)-F_X(u)]^{j-i-1}F_X^{1-j}(v) \\ &= \frac{(j-1)!}{(i-1)!(j-1-i)!}\frac{f_X(u)}{F_X(v)}\left[\frac{F_X(u)}{F_X(v)}\right]^{i-1}\left[1-\frac{F_X(u)}{F_X(v)}\right]^{j-i-1}, \quad u<v. \end{aligned}$$

Note this interpretation. This is the pdf of the ith order statistic from a sample of size $j-1$, from a population with pdf given by the truncated distribution, $f(u) = f_X(u)/F_X(v)$, $u<v$.

(ii) If $j<i$ and $u>v$,

$$\begin{aligned} &f_{X_{(i)}|X_{(j)}}(u|v) \\ &= \frac{(n-j)!}{(n-1)!(i-1-j)!}f_X(u)\left[1-F_X(u)\right]^{n-i}\left[F_X(u)-F_X(v)\right]^{i-1-j}\left[1-F_X(v)\right]^{j-n} \\ &= \frac{(n-j)!}{(i-j-1)!(n-i)!}\frac{f_X(u)}{1-F_X(v)}\left[\frac{F_X(u)-F_X(v)}{1-F_X(v)}\right]^{i-j-1}\left[1-\frac{F_X(u)-F_X(v)}{1-F_X(v)}\right]^{n-i}. \end{aligned}$$

This is the pdf of the $(i-j)$th order statistic from a sample of size $n-j$, from a population with pdf given by the truncated distribution, $f(u) = f_X(u)/(1-F_X(v))$, $u>v$.

b. From Example 5.4.7,

$$f_{V|R}(v|r) = \frac{n(n-1)r^{n-2}/a^n}{n(n-1)r^{n-2}(a-r)/a^n} = \frac{1}{a-r}, \quad r/2<v<a-r/2.$$

5.29 Let X_i = weight of ith booklet in package. The X_is are iid with $EX_i = 1$ and $\text{Var}X_i = .05^2$. We want to approximate $P\left(\sum_{i=1}^{100} X_i > 100.4\right) = P\left(\sum_{i=1}^{100} X_i/100 > 1.004\right) = P(\bar{X} > 1.004)$. By the CLT, $P(\bar{X} > 1.004) \approx P(Z > (1.004 - 1)/(.05/10)) = P(Z > .8) = .2119$.

5.30 From the CLT we have, approximately, $\bar{X}_1 \sim \text{n}(\mu, \sigma^2/n)$, $\bar{X}_2 \sim \text{n}(\mu, \sigma^2/n)$. Since $\bar{X}_1$ and $\bar{X}_2$ are independent, $\bar{X}_1 - \bar{X}_2 \sim \text{n}(0, 2\sigma^2/n)$. Thus, we want

$$\begin{aligned} .99 &\approx P\left(\left|\bar{X}_1 - \bar{X}_2\right| < \sigma/5\right) \\ &= P\left(\frac{-\sigma/5}{\sigma/\sqrt{n/2}} < \frac{\bar{X}_1 - \bar{X}_2}{\sigma/\sqrt{n/2}} < \frac{\sigma/5}{\sigma/\sqrt{n/2}}\right) \\ &\approx P\left(-\frac{1}{5}\sqrt{\frac{n}{2}} < Z < \frac{1}{5}\sqrt{\frac{n}{2}}\right), \end{aligned}$$

where $Z \sim \text{n}(0,1)$. Thus we need $P(Z \geq \sqrt{n}/5(\sqrt{2})) \approx .005$. From Table 1, $\sqrt{n}/5\sqrt{2} = 2.576$, which implies $n = 50(2.576)^2 \approx 332$.

5.31 We know that $\sigma^2_{\bar{X}} = 9/100$. Use Chebyshev's Inequality to get

$$P\left(-3k/10 < \bar{X} - \mu < 3k/10\right) \geq 1 - 1/k^2.$$

We need $1 - 1/k^2 \geq .9$ which implies $k \geq \sqrt{10} = 3.16$ and $3k/10 = .9487$. Thus

$$P(-.9487 < \bar{X} - \mu < .9487) \geq .9$$

by Chebychev's Inequality. Using the CLT, $\bar{X}$ is approximately $\text{n}\left(\mu, \sigma^2_{\bar{X}}\right)$ with $\sigma_{\bar{X}} = \sqrt{.09} = .3$ and $(\bar{X} - \mu)/.3 \sim \text{n}(0,1)$. Thus

$$.9 = P\left(-1.645 < \frac{\bar{X} - \mu}{.3} < 1.645\right) = P(-.4935 < \bar{X} - \mu < .4935).$$

Thus, we again see the conservativeness of Chebychev's Inequality, yielding bounds on $\bar{X} - \mu$ that are almost twice as big as the normal approximation. Moreover, with a sample of size 100, $\bar{X}$ is probably very close to normally distributed, even if the underlying X distribution is not close to normal.

5.32 a. For any $\epsilon > 0$,

$$\begin{aligned} P\left(\left|\sqrt{X_n} - \sqrt{a}\right| > \epsilon\right) &= P\left(\left|\sqrt{X_n} - \sqrt{a}\right|\left|\sqrt{X_n} + \sqrt{a}\right| > \epsilon\left|\sqrt{X_n} + \sqrt{a}\right|\right) \\ &= P\left(|X_n - a| > \epsilon\left|\sqrt{X_n} + \sqrt{a}\right|\right) \\ &\leq P\left(|X_n - a| > \epsilon\sqrt{a}\right) \to 0, \end{aligned}$$

as $n \to \infty$, since $X_n \to a$ in probability. Thus $\sqrt{X_n} \to \sqrt{a}$ in probability.

b. For any $\epsilon > 0$,

$$\begin{aligned} P\left(\left|\frac{a}{X_n} - 1\right| \leq \epsilon\right) &= P\left(\frac{a}{1+\epsilon} \leq X_n \leq \frac{a}{1-\epsilon}\right) \\ &= P\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1-\epsilon}\right) \\ &\geq P\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1+\epsilon}\right) \qquad \left(a + \frac{a\epsilon}{1+\epsilon} < a + \frac{a\epsilon}{1-\epsilon}\right) \\ &= P\left(|X_n - a| \leq \frac{a\epsilon}{1+\epsilon}\right) \to 1, \end{aligned}$$

as $n \to \infty$, since $X_n \to a$ in probability. Thus $a/X_n \to 1$ in probability.

c. $S_n^2 \to \sigma^2$ in probability. By a), $S_n = \sqrt{S_n^2} \to \sqrt{\sigma^2} = \sigma$ in probability. By b), $\sigma/S_n \to 1$ in probability.

5.33 For all $\epsilon > 0$ there exist N such that if $n > N$, then $P(X_n + Y_n > c) > 1 - \epsilon$. Choose N_1 such that $P(X_n > -m) > 1 - \epsilon/2$ and N_2 such that $P(Y_n > c + m) > 1 - \epsilon/2$. Then

$$P(X_n + Y_n > c) \geq P(X_n > -m, +Y_n > c + m) \geq P(X_n > -m) + P(Y_n > c + m) - 1 = 1 - \epsilon.$$

5.34 Using $\mathrm{E}\bar{X}_n = \mu$ and $\mathrm{Var}\bar{X}_n = \sigma^2/n$, we obtain

$$\mathrm{E}\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}}{\sigma}\mathrm{E}(\bar{X}_n - \mu) = \frac{\sqrt{n}}{\sigma}(\mu - \mu) = 0.$$

$$\mathrm{Var}\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{n}{\sigma^2}\mathrm{Var}(\bar{X}_n - \mu) = \frac{n}{\sigma^2}\mathrm{Var}\bar{X} = \frac{n}{\sigma^2}\frac{\sigma^2}{n} = 1.$$

5.35 a. $X_i \sim$ exponential(1). $\mu_X = 1$, $\mathrm{Var}X = 1$. From the CLT, $\bar{X}_n$ is approximately $\mathrm{n}(1, 1/n)$. So

$$\frac{\bar{X}_n - 1}{1/\sqrt{n}} \to Z \sim \mathrm{n}(0,1) \quad \text{and} \quad P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) \to P(Z \leq x).$$

b.

$$\frac{d}{dx}P(Z \leq x) = \frac{d}{dx}F_Z(x) = f_Z(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

$$\begin{aligned} &\frac{d}{dx}P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \leq x\right) \\ &= \frac{d}{dx}\left(\sum_{i=1}^n X_i \leq x\sqrt{n} + n\right) \qquad \left(W = \sum_{i=1}^n X_i \sim \mathrm{gamma}(n,1)\right) \\ &= \frac{d}{dx}F_W(x\sqrt{n} + n) = f_W(x\sqrt{n} + n) \cdot \sqrt{n} = \frac{1}{\Gamma(n)}(x\sqrt{n} + n)^{n-1}e^{-(x\sqrt{n}+n)}\sqrt{n}. \end{aligned}$$

Therefore, $(1/\Gamma(n))(x\sqrt{n} + n)^{n-1}e^{-(x\sqrt{n}+n)}\sqrt{n} \approx \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ as $n \to \infty$. Substituting $x = 0$ yields $n! \approx n^{n+1/2}e^{-n}\sqrt{2\pi}$.

5.37 a. For the exact calculations, use the fact that V_n is itself distributed negative binomial$(10r, p)$. The results are summarized in the following table. Note that the recursion relation of problem 3.48 can be used to simplify calculations.

	$P(V_n = v)$		
	(a)	(b)	(c)
v	Exact	Normal App.	Normal w/cont.
0	.0008	.0071	.0056
1	.0048	.0083	.0113
2	.0151	.0147	.0201
3	.0332	.0258	.0263
4	.0572	.0392	.0549
5	.0824	.0588	.0664
6	.1030	.0788	.0882
7	.1148	.0937	.1007
8	.1162	.1100	.1137
9	.1085	.1114	.1144
10	.0944	.1113	.1024

b. Using the normal approximation, we have $\mu_v = r(1-p)/p = 20(.3)/.7 = 8.57$ and

$$\sigma_v = \sqrt{r(1-p)/p^2} = \sqrt{(20)(.3)/.49} = 3.5.$$

Then,

$$P(V_n = 0) = 1 - P(V_n \ge 1) = 1 - P\left(\frac{V_n - 8.57}{3.5} \ge \frac{1 - 8.57}{3.5}\right) = 1 - P(Z \ge -2.16) = .0154.$$

Another way to approximate this probability is

$$P(V_n = 0) = P(V_n \le 0) = P\left(\frac{V - 8.57}{3.5} \le \frac{0 - 8.57}{3.5}\right) = P(Z \le -2.45) = .0071.$$

Continuing in this way we have $P(V = 1) = P(V \le 1) - P(V \le 0) = .0154 - .0071 = .0083$, etc.

c. With the continuity correction, compute $P(V = k)$ by $P\left(\frac{(k-.5)-8.57}{3.5} \le Z \le \frac{(k+.5)-8.57}{3.5}\right)$, so $P(V = 0) = P(-9.07/3.5 \le Z \le -8.07/3.5) = .0104 - .0048 = .0056$, etc. Notice that the continuity correction gives some improvement over the uncorrected normal approximation.

5.39 a. If h is continuous given $\epsilon > 0$ there exits δ such that $|h(x_n) - h(x)| < \epsilon$ for $|x_n - x| < \delta$. Since $X_1, \ldots, X_n$ converges in probability to the random variable X, then $\lim_{n\to\infty} P(|X_n - X| < \delta) = 1$. Thus $\lim_{n\to\infty} P(|h(X_n) - h(X)| < \epsilon) = 1$.

b. Define the subsequence $X_j(s) = s + I_{[a,b]}(s)$ such that in $I_{[a,b]}$, a is always 0, i.e, the subsequence $X_1, X_2, X_4, X_7, \ldots$. For this subsequence

$$X_j(s) \to \begin{cases} s & \text{if } s > 0 \\ s+1 & \text{if } s = 0. \end{cases}$$

5.41 a. Let $\epsilon = |x - \mu|$.

(i) For $x - \mu \ge 0$

$$\begin{aligned} P(|X_n - \mu| > \epsilon) &= P(|X_n - \mu| > x - \mu) \\ &= P(X_n - \mu < -(x-\mu)) + P(X_n - \mu > x - \mu) \\ &\ge P(X_n - \mu > x - \mu) \\ &= P(X_n > x) \quad = \quad 1 - P(X_n \le x). \end{aligned}$$

Therefore, $0 = \lim_{n\to\infty} P(|X_n - \mu| > \epsilon) \ge \lim_{n\to\infty} 1 - P(X_n \le x)$. Thus $\lim_{n\to\infty} P(X_n \le x) \ge 1$.

(ii) For $x - \mu < 0$

$$\begin{aligned} P(|X_n - \mu| > \epsilon) &= P(|X_n - \mu| > -(x-\mu)) \\ &= P(X_n - \mu < x - \mu) + P(X_n - \mu > -(x - \mu)) \\ &\ge P(X_n - \mu < x - \mu) \\ &= P(X_n \le x). \end{aligned}$$

Therefore, $0 = \lim_{n\to\infty} P(|X_n - \mu| > \epsilon) \ge \lim_{n\to\infty} P(X_n \le x)$.

By (i) and (ii) the results follows.

b. For every $\epsilon > 0$,

$$\begin{aligned} P(|X_n - \mu| > \epsilon) &\le P(X_n - \mu < -\epsilon) + P(X_n - \mu > \epsilon) \\ &= P(X_n < \mu - \epsilon) + 1 - P(X_n \le \mu + \epsilon) \quad \to \quad 0 \text{ as } n \to \infty. \end{aligned}$$

5.43 a. $P(|Y_n-\theta|<\epsilon) = P\left(\left|\sqrt(n)(Y_n-\theta)\right| < \sqrt(n)\epsilon\right)$. Therefore,

$$\lim_{n\to\infty} P(|Y_n-\theta|<\epsilon) = \lim_{n\to\infty} P\left(\left|\sqrt(n)(Y_n-\theta)\right| < \sqrt(n)\epsilon\right) = P(|Z|<\infty) = 1,$$

where $Z \sim \mathrm{n}(0,\sigma^2)$. Thus $Y_n \to \theta$ in probability.

b. By Slutsky's Theorem (a), $g'(\theta)\sqrt{n}(Y_n-\theta) \to g'(\theta)X$ where $X \sim \mathrm{n}(0,\sigma^2)$. Therefore $\sqrt{n}[g(Y_n)-g(\theta)] = g'(\theta)\sqrt{n}(Y_n-\theta) \to \mathrm{n}(0,\sigma^2[g'(\theta)]^2)$.

5.45 We do part (a), the other parts are similar. Using Mathematica, the exact calculation is

```
In[120]:=
f1[x_]=PDF[GammaDistribution[4,25],x]
p1=Integrate[f1[x],{x,100,\[Infinity]}]//N
1-CDF[BinomialDistribution[300,p1],149]

Out[120]=
e^(-x/25) x^3/2343750

Out[121]=
0.43347

Out[122]=
0.0119389.
```

The answer can also be simulated in Mathematica or in R. Here is the R code for simulating the same probability

```
p1<-mean(rgamma(10000,4,scale=25)>100)
mean(rbinom(10000, 300, p1)>149)
```

In each case 10,000 random variables were simulated. We obtained $p1 = 0.438$ and a binomial probability of 0.0108.

5.47 a. $-2\log(U_j) \sim \text{exponential}(2) \sim \chi^2_2$. Thus Y is the sum of ν independent χ^2_2 random variables. By Lemma 5.3.2(b), $Y \sim \chi^2_{2\nu}$.

b. $\beta\log(U_j) \sim \text{exponential}(2) \sim \text{gamma}(1,\beta)$. Thus Y is the sum of independent gamma random variables. By Example 4.6.8, $Y \sim \text{gamma}(a,\beta)$

c. Let $V = \sum_{j=1}^a \log(U_j) \sim \text{gamma}(a,1)$. Similarly $W = \sum_{j=1}^b \log(U_j) \sim \text{gamma}(b,1)$. By Exercise 4.24, $\frac{V}{V+W} \sim \text{beta}(a,b)$.

5.49 a. See Example 2.1.4.

b. $X = g(U) = -\log\frac{1-U}{U}$. Then $g^{-1}(x) = \frac{1}{1+e^{-y}}$. Thus

$$f_X(x) = 1 \times \left|\frac{e^{-y}}{(1+e^{-y})^2}\right| = \frac{e^{-y}}{(1+e^{-y})^2} \qquad -\infty < y < \infty,$$

which is the density of a logistic$(0,1)$ random variable.

c. Let $Y \sim \text{logistic}(\mu,\beta)$ then $f_Y(y) = \frac{1}{\beta}f_Z(\frac{-(y-\mu)}{\beta})$ where f_Z is the density of a logistic$(0,1)$. Then $Y = \beta Z + \mu$. To generate a logistic(μ,β) random variable generate (i) generate $U \sim \text{uniform}(0,1)$, (ii) Set $Y = \beta\log\frac{U}{1-U} + \mu$.

5.51 a. For $U_i \sim \text{uniform}(0,1)$, $\mathrm{E}U_i = 1/2$, $\mathrm{Var}U_i = 1/12$. Then

$$X = \sum_{i=1}^{12} U_i - 6 = 12\bar{U} - 6 = \sqrt{12}\left(\frac{\bar{U}-1/2}{1/\sqrt{12}}\right)$$

is in the form $\sqrt{n}\left((\bar{U}-\mathrm{E}U)/\sigma\right)$ with $n = 12$, so X is approximately $\mathrm{n}(0,1)$ by the Central Limit Theorem.

b. The approximation does not have the same range as $Z \sim \mathrm{n}(0,1)$ where $-\infty < Z < +\infty$, since $-6 < X < 6$.

c.

$$\mathrm{E}X = \mathrm{E}\left(\sum_{i=1}^{12} U_i - 6\right) = \sum_{i=1}^{12} \mathrm{E}U_i - 6 = \left(\sum_{i=1}^{12} \frac{1}{2}\right) - 6 = 6 - 6 = 0.$$

$$\mathrm{Var}X = \mathrm{Var}\left(\sum_{i=1}^{12} U_i - 6\right) = \mathrm{Var}\sum_{i=1}^{12} U_i = 12\mathrm{Var}U_1 = 1$$

$\mathrm{E}X^3 = 0$ since X is symmetric about 0. (In fact, all odd moments of X are 0.) Thus, the first three moments of X all agree with the first three moments of a $\mathrm{n}(0,1)$. The fourth moment is not easy to get, one way to do it is to get the mgf of X. Since $\mathrm{E}e^{tU} = (e^t - 1)/t$,

$$\mathrm{E}\left[e^{t\left(\sum_{i=1}^{12} U_i - 6\right)}\right] = e^{-6t}\left(\frac{e^t - 1}{t}\right)^{12} = \left(\frac{e^{t/2} - e^{-t/2}}{t}\right)^{12}.$$

Computing the fourth derivative and evaluating it at $t = 0$ gives us $\mathrm{E}X^4$. This is a lengthy calculation. The answer is $\mathrm{E}X^4 = 29/10$, slightly smaller than $\mathrm{E}Z^4 = 3$, where $Z \sim \mathrm{n}(0,1)$.

5.53 The R code is the following:

a.
```
obs <- rbinom(1000,8,2/3)
meanobs <- mean(obs)
variance <- var(obs)
hist(obs)
Output:
> meanobs
[1] 5.231
> variance
[1] 1.707346
```

b.
```
obs<- rhyper(1000,8,2,4)
meanobs <- mean(obs)
variance <- var(obs)
hist(obs)
Output:
> meanobs
[1] 3.169
> variance
[1] 0.4488879
```

c.
```
obs <- rnbinom(1000,5,1/3)
meanobs <- mean(obs)
variance <- var(obs)
hist(obs)
Output:
> meanobs
[1] 10.308
> variance
[1] 29.51665
```

5.55 Let X denote the number of comparisons. Then

$$\begin{aligned} \mathrm{E}X &= \sum_{k=0}^{\infty} P(X>k) = 1+\sum_{k=1}^{\infty} P(U>F_y(y_{k-1})) \\ &= 1+\sum_{k=1}^{\infty}(1-F_y(y_{k-1})) = 1+\sum_{k=0}^{\infty}(1-F_y(y_i)) = 1+\mathrm{E}Y \end{aligned}$$

5.57 a. $\mathrm{Cov}(Y_1,Y_2) = \mathrm{Cov}(X_1+X_3, X_2+X_3) = \mathrm{Cov}(X_3,X_3) = \lambda_3$ since X_1, X_2 and X_3 are independent.

b.

$$Z_i = \begin{cases} 1 & \text{if } X_i = X_3 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$p_i = P(Z_i=0) = P(Y_i=0) = P(X_i=0, X_3=0) = e^{-(\lambda_i+\lambda_3)}$. Therefore Z_i are Bernoulli(p_i) with $\mathrm{E}[Z_i]=p_i$, $\mathrm{Var}(Z_i)=p_i(1-p_i)$ and

$$\begin{aligned} \mathrm{E}[Z_1Z_2] &= P(Z_1=1,Z_2=1) = P(Y_1=0,Y_2=0) \\ &= P(X_1+X_3=0, X_2+X_3=0) = P(X_1=0)P(X_2=0)P(X_3=0) \\ &= e^{-\lambda_1}e^{-\lambda_2}e^{-\lambda_3}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{Cov}(Z_1,Z_2) &= \mathrm{E}[Z_1Z_2]-\mathrm{E}[Z_1]\mathrm{E}[Z_2] \\ &= e^{-\lambda_1}e^{-\lambda_2}e^{-\lambda_3} - e^{-(\lambda_i+\lambda_3)}e^{-(\lambda_2+\lambda_3)} = e^{-(\lambda_i+\lambda_3)}e^{-(\lambda_2+\lambda_3)}(e^{\lambda_3}-1) \\ &= p_1p_2(e^{\lambda_3}-1). \end{aligned}$$

Thus $\mathrm{Corr}(Z_1,Z_2) = \frac{p_1p_2(e^{\lambda_3}-1)}{\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}}$.

c. $\mathrm{E}[Z_1Z_2] \le p_i$, therefore

$$\begin{aligned} \mathrm{Cov}(Z_1,Z_2) &= \mathrm{E}[Z_1Z_2]-\mathrm{E}[Z_1]\mathrm{E}[Z_2] \le p_1 - p_1p_2 = p_1(1-p_2), \quad \text{and} \\ \mathrm{Cov}(Z_1,Z_2) &\le p_2(1-p_1). \end{aligned}$$

Therefore,

$$\mathrm{Corr}(Z_1,Z_2) \le \frac{p_1(1-p_2)}{\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}} = \frac{\sqrt{p_1(1-p_2)}}{\sqrt{p_2(1-p_1)}}$$

and

$$\mathrm{Corr}(Z_1,Z_2) \le \frac{p_2(1-p_1)}{\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}} = \frac{\sqrt{p_2(1-p_1)}}{\sqrt{p_1(1-p_2)}}$$

which implies the result.

5.59

$$\begin{aligned} P(Y\le y) &= P(V\le y|U<\frac{1}{c}f_Y(V)) = \frac{P(V\le y, U<\frac{1}{c}f_Y(V))}{P(U<\frac{1}{c}f_Y(V))} \\ &= \frac{\int_0^y\int_0^{\frac{1}{c}f_Y(v)} du dv}{\frac{1}{c}} = \frac{\frac{1}{c}\int_0^y f_Y(v)dv}{\frac{1}{c}} = \int_0^y f_Y(v)dv \end{aligned}$$

5.61 a. $M = \sup_y \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}y^{a-1}(1-y)^{b-1}}{\frac{\Gamma([a]+[b])}{\Gamma([a])\Gamma([b])}y^{[a]-1}(1-y)^{[b]-1}} < \infty$, since $a-[a]>0$ and $b-[b]>0$ and $y\in(0,1)$.

b. $M = \sup_y \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}}{\frac{\Gamma([a]+b)}{\Gamma([a])\Gamma(b)} y^{[a]-1}(1-y)^{b-1}} < \infty$, since $a - [a] > 0$ and $y \in (0, 1)$.

c. $M = \sup_y \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}}{\frac{\Gamma([a]+1+\beta)}{\Gamma([a]+1)\Gamma(b')} y^{[a]+1-1}(1-y)^{b'-1}} < \infty$, since $a - [a] - 1 < 0$ and $y \in (0, 1)$. $b - b' > 0$ when $b' = [b]$ and will be equal to zero when $b' = b$, thus it does not affect the result.

d. Let $f(y) = y^\alpha(1-y)^\beta$. Then

$$\frac{df(y)}{dy} = \alpha y^{\alpha-1}(1-y)^\beta - y^\alpha \beta(1-y)^{\beta-1} = y^{\alpha-1}(1-y)^{\beta-1}[\alpha(1-y) + \beta y]$$

which is maximize at $y = \frac{\alpha}{\alpha+\beta}$. Therefore for, $\alpha = a - a'$ and $\beta = b - b'$

$$M = \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}{\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}} \left(\frac{a-a'}{a-a'+b-b'}\right)^{a-a'} \left(\frac{b-b'}{a-a'+b-b'}\right)^{b-b'}.$$

We need to minimize M in a' and b'. First consider $\left(\frac{a-a'}{a-a'+b-b'}\right)^{a-a'} \left(\frac{b-b'}{a-a'+b-b'}\right)^{b-b'}$. Let $c = \alpha + \beta$, then this term becomes $\left(\frac{\alpha}{c}\right)^\alpha \left(\frac{c-\alpha}{c}\right)^{c-\alpha}$. This term is maximize at $\frac{\alpha}{c} = \frac{1}{2}$, this is at $\alpha = \frac{1}{2}c$. Then $M = (\frac{1}{2})^{(a-a'+b-b')} \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}{\frac{\Gamma(a'+b')}{\Gamma(a')\Gamma(b')}}$. Note that the minimum that M could be is one, which it is attain when $a = a'$ and $b = b'$. Otherwise the minimum will occur when $a - a'$ and $b - b'$ are minimum but greater or equal than zero, this is when $a' = [a]$ and $b' = [b]$ or $a' = a$ and $b' = [b]$ or $a' = [a]$ and $b' = b$.

5.63 $M = \sup_y \frac{\frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}}{\frac{1}{2\lambda} e^{\frac{-|y|}{\lambda}}}$. Let $f(y) = \frac{-y^2}{2} + \frac{|y|}{\lambda}$. Then $f(y)$ is maximize at $y = \frac{1}{\lambda}$ when $y \geq 0$ and at $y = \frac{-1}{\lambda}$ when $y < 0$. Therefore in both cases $M = \frac{\frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2\lambda^2}}}{\frac{1}{2\lambda} e^{\frac{-1}{\lambda^2}}}$. To minimize M let $M' = \lambda e^{\frac{1}{2\lambda^2}}$. Then $\frac{d \log M'}{d\lambda} = \frac{1}{\lambda} - \frac{1}{\lambda^3}$, therefore M is minimize at $\lambda = 1$ or $\lambda = -1$. Thus the value of λ that will optimize the algorithm is $\lambda = 1$.

5.65

$$\begin{aligned} P(X^* \leq x) &= \sum_{i=1}^m P(X^* \leq x | q_i) q_i = \sum_{i=1}^m I(Y_i \leq x) q_i = \frac{\frac{1}{m}\sum_{i=1}^m \frac{f(Y_i)}{g(Y_i)} I(Y_i \leq x)}{\frac{1}{m}\sum_{i=1}^m \frac{f(Y_i)}{g(Y_i)}} \\ &\underset{m\to\infty}{\longrightarrow} \frac{E_g \frac{f(Y)}{g(Y)} I(Y \leq x)}{E_g \frac{f(Y)}{g(Y)}} = \frac{\int_{-\infty}^x \frac{f(y)}{g(y)} g(y)dy}{\int_{-\infty}^\infty \frac{f(y)}{g(y)} g(y)dy} = \int_{-\infty}^x f(y)dy. \end{aligned}$$

5.67 An R code to generate the sample of size 100 from the specified distribution is shown for part c). The Metropolis Algorithm is used to generate 2000 variables. Among other options one can choose the 100 variables in positions 1001 to 1100 or the ones in positions $1010, 1020, ..., 2000$.

a. We want to generate $X = \sigma Z + \mu$ where $Z \sim$ Student's t with ν degrees of freedom. Therefore we first can generate a sample of size 100 from a Student's t distribution with ν degrees of freedom and then make the transformation to obtain the X's. Thus $f_Z(z) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}} \frac{1}{\left(1+\left(\frac{z^2}{\nu}\right)\right)^{(v+1)/2}}$. Let $V \sim \mathrm{n}(0, \frac{\nu}{\nu-2})$ since given ν we can set

$$\mathrm{E}V = \mathrm{E}Z = 0, \quad \text{and} \quad \mathrm{Var}(V) = \mathrm{Var}(Z) = \frac{\nu}{\nu-2}.$$

Now, follow the algorithm on page 254 and generate the sample $Z_1, Z_2 \ldots, Z_{100}$ and then calculate $X_i = \sigma Z_i + \mu$.

b. $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\frac{e^{-(\log x-\mu)^2/2\sigma^2}}{x}$. Let $V \sim \text{gamma}(\alpha,\beta)$ where

$$\alpha = \frac{(e^{\mu+(\sigma^2/2)})^2}{e^{2(\mu+\sigma^2)}-e^{2\mu+\sigma^2}}, \quad \text{and} \quad \beta = \frac{e^{2(\mu+\sigma^2)}-e^{2\mu+\sigma^2}}{e^{\mu+(\sigma^2/2)}},$$

since given μ and σ^2 we can set

$$\mathrm{E}V = \alpha\beta = e^{\mu+(\sigma^2/2)} = \mathrm{E}X$$

and

$$\mathrm{Var}(V) = \alpha\beta^2 = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2} = \mathrm{Var}(X).$$

Now, follow the algorithm on page 254.

c. $f_X(x) = \frac{\alpha}{\beta}e^{\frac{-x^\alpha}{\beta}}x^{\alpha-1}$. Let $V \sim \text{exponential}(\beta)$. Now, follow the algorithm on page 254 where

$$\rho_i = \min\left\{\frac{V_i^{\alpha-1}}{Z_{i-1}^{\alpha-1}}e^{\frac{-V_i^\alpha+V_i-Z_{i-1}+Z_{i-1}^\alpha}{\beta}}, 1\right\}$$

An R code to generate a sample size of 100 from a Weibull(3,2) is:

```
#initialize a and b
b <- 2
a <- 3
Z <- rexp(1,1/b)
ranvars <- matrix(c(Z),byrow=T,ncol=1)
for( i in seq(2000))
{
U <- runif(1,min=0,max=1)
V <- rexp(1,1/b)
p <- pmin((V/Z)^(a-1)*exp((-V^a+V-Z+Z^a)/b),1)
if (U <= p)
  Z <- V
ranvars <- cbind(ranvars,Z)
}
#One option: choose elements in position 1001,1002,...,1100
to be the sample
vector.1 <- ranvars[1001:1100]
mean(vector.1)
var(vector.1)
#Another option: choose elements in position 1010,1020,...,2000
to be the sample
vector.2 <- ranvars[seq(1010,2000,10)]
mean(vector.2)
var(vector.2)
Output:
[1] 1.048035
[1] 0.1758335
[1] 1.130649
[1] 0.1778724
```

5.69 Let $w(v,z) = \frac{f_Y(v)f_V(z)}{f_V(v)f_Y(z)}$, and then $\rho(v,z) = \min\{w(v,z),1\}$. We will show that

$$Z_i \sim f_Y \Rightarrow P(Z_{i+1} \le a) = P(Y \le a).$$

Write

$$P(Z_{i+1} \le a) = P(V_{i+1} \le a \text{ and } U_{i+1} \le \rho_{i+1}) + P(Z_i \le a \text{ and } U_{i+1} > \rho_{i+1}).$$

Since $Z_i \sim f_Y$, suppressing the unnecessary subscripts we can write

$$P(Z_{i+1} \le a) = P(V \le a \text{ and } U \le \rho(V,Y)) + P(Y \le a \text{ and } U > \rho(V,Y)).$$

Add and subtract $P(Y \le a \text{ and } U \le \rho(V,Y))$ to get

$$\begin{aligned} P(Z_{i+1} \le a) &= P(Y \le a) + P(V \le a \text{ and } U \le \rho(V,Y)) \\ &\quad - P(Y \le a \text{ and } U \le \rho(V,Y)). \end{aligned}$$

Thus we need to show that

$$P(V \le a \text{ and } U \le \rho(V,Y)) = P(Y \le a \text{ and } U \le \rho(V,Y)).$$

Write out the probability as

$$\begin{aligned} &P(V \le a \text{ and } U \le \rho(V,Y)) \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} \rho(v,y) f_Y(y) f_V(v)\,dy\,dv \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(v,y) \le 1)\left(\frac{f_Y(v)f_V(y)}{f_V(v)f_Y(y)}\right) f_Y(y) f_V(v)\,dy\,dv \\ &\quad + \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(v,y) \ge 1) f_Y(y) f_V(v)\,dy\,dv \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(v,y) \le 1) f_Y(v) f_V(y)\,dy\,dv \\ &\quad + \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(v,y) \ge 1) f_Y(y) f_V(v)\,dy\,dv. \end{aligned}$$

Now, notice that $w(v,y) = 1/w(y,v)$, and thus first term above can be written

$$\begin{aligned} &\int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(v,y) \le 1) f_Y(v) f_V(y)\,dy\,dv \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(y,v) > 1) f_Y(v) f_V(y)\,dy\,dv \\ &= P(Y \le a, \rho(V,Y) = 1, U \le \rho(V,Y)). \end{aligned}$$

The second term is

$$\begin{aligned} &\int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(v,y) \ge 1) f_Y(y) f_V(v)\,dy\,dv \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(y,v) \le 1) f_Y(y) f_V(v)\,dy\,dv \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(y,v) \le 1)\left(\frac{f_V(y)f_Y(v)}{f_V(y)f_Y(v)}\right) f_Y(y) f_V(v)\,dy\,dv \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(y,v) \le 1)\left(\frac{f_Y(y)f_V(v)}{f_V(y)f_Y(v)}\right) f_V(y) f_Y(v)\,dy\,dv \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} I(w(y,v) \le 1) w(y,v) f_V(y) f_Y(v)\,dy\,dv \\ &= P(Y \le a, U \le \rho(V,Y), \rho(V,Y) \le 1). \end{aligned}$$

Putting it all together we have

$$\begin{aligned} P(V \le a \text{ and } U \le \rho(V,Y)) &= P(Y \le a, \rho(V,Y) = 1, U \le \rho(V,Y)) \\ &\quad + P(Y \le a, U \le \rho(V,Y), \rho(V,Y) \le 1) \\ &= P(Y \le a \text{ and } U \le \rho(V,Y)), \end{aligned}$$

and hence

$$P(Z_{i+1} \le a) = P(Y \le a),$$

so f_Y is the stationary density.