Chapter 6

Principles of Data Reduction

6.1 By the Factorization Theorem, $|X|$ is sufficient because the pdf of X is

$$f(x|\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}\sigma} e^{-|x|^2/2\sigma^2} = g(\,|x|\,|\,\sigma^2) \cdot \underbrace{1}_{h(x)}.$$

6.2 By the Factorization Theorem, $T(X) = \min_i(X_i/i)$ is sufficient because the joint pdf is

$$f(x_1,\ldots,x_n|\theta) = \prod_{i=1}^{n} e^{i\theta - x_i} I_{(i\theta,+\infty)}(x_i) = \underbrace{e^{in\theta} I_{(\theta,+\infty)}(T(\mathbf{x}))}_{g(T(\mathbf{x})|\theta)} \cdot \underbrace{e^{-\Sigma_i x_i}}_{h(\mathbf{x})}.$$

Notice, we use the fact that $i > 0$, and the fact that all x_is $> i\theta$ if and only if $\min_i(x_i/i) > \theta$.

6.3 Let $x_{(1)} = \min_i x_i$. Then the joint pdf is

$$f(x_1,\ldots,x_n|\mu,\sigma) = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-(x_i-\mu)/\sigma} I_{(\mu,\infty)}(x_i) = \underbrace{\left(\frac{e^{\mu/\sigma}}{\sigma}\right)^n e^{-\Sigma_i x_i/\sigma} I_{(\mu,\infty)}(x_{(1)})}_{g\left(x_{(1)},\Sigma_i x_i \middle| \mu,\sigma\right)} \cdot \underbrace{1}_{h(\mathbf{x})}.$$

Thus, by the Factorization Theorem, $\left(X_{(1)}, \sum_i X_i\right)$ is a sufficient statistic for (μ,σ).

6.4 The joint pdf is

$$\prod_{j=1}^{n} \left\{ h(x_j) c(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta}) t_i(x_j)\right)\right\} = \underbrace{c(\boldsymbol{\theta})^n \exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta}) \sum_{j=1}^{n} t_i(x_j)\right)}_{g(T(\mathbf{x})|\theta)} \cdot \underbrace{\prod_{j=1}^{n} h(x_j)}_{h(\mathbf{x})}.$$

By the Factorization Theorem, $\left(\sum_{j=1}^{n} t_1(X_j), \ldots, \sum_{j=1}^{n} t_k(X_j)\right)$ is a sufficient statistic for $\boldsymbol{\theta}$.

6.5 The sample density is given by

$$\begin{aligned}
\prod_{i=1}^{n} f(x_i|\theta) &= \prod_{i=1}^{n} \frac{1}{2i\theta} I\left(-i(\theta-1) \le x_i \le i(\theta+1)\right) \\
&= \left(\frac{1}{2\theta}\right)^n \left(\prod_{i=1}^{n} \frac{1}{i}\right) I\left(\min \frac{x_i}{i} \ge -(\theta-1)\right) I\left(\max \frac{x_i}{i} \le \theta+1\right).
\end{aligned}$$

Thus $(\min X_i/i, \max X_i/i)$ is sufficient for θ.

6.6 The joint pdf is given by

$$f(x_1,\dots,x_n|\alpha,\beta)=\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha)\beta^\alpha}{x_i}^{\alpha-1}e^{-x_i/\beta}=\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n\left(\prod_{i=1}^{n}x_i\right)^{\alpha-1}e^{-\Sigma_i x_i/\beta}.$$

By the Factorization Theorem, $(\prod_{i=1}^n X_i, \sum_{i=1}^n X_i)$ is sufficient for (α,β).

6.7 Let $x_{(1)}=\min_i\{x_1,\dots,x_n\}$, $x_{(n)}=\max_i\{x_1,\dots,x_n\}$, $y_{(1)}=\min_i\{y_1,\dots,y_n\}$ and $y_{(n)}=\max_i\{y_1,\dots,y_n\}$. Then the joint pdf is

$$\begin{aligned}
&f(\mathbf{x},\mathbf{y}|\boldsymbol{\theta})\\
&=\prod_{i=1}^{n}\frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)}I_{(\theta_1,\theta_3)}(x_i)I_{(\theta_2,\theta_4)}(y_i)\\
&=\underbrace{\left(\frac{1}{(\theta_3-\theta_1)(\theta_4-\theta_2)}\right)^n I_{(\theta_1,\infty)}(x_{(1)})I_{(-\infty,\theta_3)}(x_{(n)})I_{(\theta_2,\infty)}(y_{(1)})I_{(-\infty,\theta_4)}(y_{(n)})}_{g(T(\mathbf{x})|\theta)}\cdot\underbrace{1}_{h(\mathbf{x})}.
\end{aligned}$$

By the Factorization Theorem, $\left(X_{(1)},X_{(n)},Y_{(1)},Y_{(n)}\right)$ is sufficient for $(\theta_1,\theta_2,\theta_3,\theta_4)$.

6.9 Use Theorem 6.2.13.

a.

$$\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)}=\frac{(2\pi)^{-n/2}e^{-\Sigma_i(x_i-\theta)^2/2}}{(2\pi)^{-n/2}e^{-\Sigma_i(y_i-\theta)^2/2}}=\exp\left\{-\frac{1}{2}\left[\left(\sum_{i=1}^n x_i^2-\sum_{i=1}^n y_i^2\right)+2\theta n(\bar{y}-\bar{x})\right]\right\}.$$

This is constant as a function of θ if and only if $\bar{y}=\bar{x}$; therefore $\bar{X}$ is a minimal sufficient statistic for θ.

b. Note, for $X\sim$ location exponential(θ), the range depends on the parameter. Now

$$\begin{aligned}
\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} &= \frac{\prod_{i=1}^n\left(e^{-(x_i-\theta)}I_{(\theta,\infty)}(x_i)\right)}{\prod_{i=1}^n\left(e^{-(y_i-\theta)}I_{(\theta,\infty)}(y_i)\right)}\\
&= \frac{e^{n\theta}e^{-\Sigma_i x_i}\prod_{i=1}^n I_{(\theta,\infty)}(x_i)}{e^{n\theta}e^{-\Sigma_i y_i}\prod_{i=1}^n I_{(\theta,\infty)}(y_i)} = \frac{e^{-\Sigma_i x_i}I_{(\theta,\infty)}(\min x_i)}{e^{-\Sigma_i y_i}I_{(\theta,\infty)}(\min y_i)}.
\end{aligned}$$

To make the ratio independent of θ we need the ratio of indicator functions independent of θ. This will be the case if and only if $\min\{x_1,\dots,x_n\}=\min\{y_1,\dots,y_n\}$. So $T(\mathbf{X})=\min\{X_1,\dots,X_n\}$ is a minimal sufficient statistic.

c.

$$\begin{aligned}
\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} &= \frac{e^{-\Sigma_i(x_i-\theta)}}{\prod_{i=1}^n\left(1+e^{-(x_i-\theta)}\right)^2}\,\frac{\prod_{i=1}^n\left(1+e^{-(y_i-\theta)}\right)^2}{e^{-\Sigma_i(y_i-\theta)}}\\
&= e^{-\Sigma_i(y_i-x_i)}\left(\frac{\prod_{i=1}^n\left(1+e^{-(y_i-\theta)}\right)}{\prod_{i=1}^n\left(1+e^{-(x_i-\theta)}\right)}\right)^2.
\end{aligned}$$

This is constant as a function of θ if and only if $\mathbf{x}$ and $\mathbf{y}$ have the same order statistics. Therefore, the order statistics are minimal sufficient for θ.

d. This is a difficult problem. The order statistics are a minimal sufficient statistic.

e. Fix sample points $\mathbf{x}$ and $\mathbf{y}$. Define $A(\theta) = \{i : x_i \le \theta\}$, $B(\theta) = \{i : y_i \le \theta\}$, $a(\theta) =$ the number of elements in $A(\theta)$ and $b(\theta) =$ the number of elements in $B(\theta)$. Then the function $f(\mathbf{x}|\theta)/f(\mathbf{y}|\theta)$ depends on θ only through the function

$$\begin{aligned}
&\sum_{i=1}^{n} | x_i - \theta | - \sum_{i=1}^{n} | y_i - \theta | \\
&= \sum_{i\in A(\theta)} (\theta - x_i) + \sum_{i\in A(\theta)^c} (x_i - \theta) - \sum_{i\in B(\theta)} (\theta - y_i) - \sum_{i\in B(\theta)^c} (y_i - \theta) \\
&= (a(\theta) - [n - a(\theta)] - b(\theta) + [n - b(\theta)])\theta \\
&\quad + \left(- \sum_{i\in A(\theta)} x_i + \sum_{i\in A(\theta)^c} x_i + \sum_{i\in B(\theta)} y_i - \sum_{i\in B(\theta)^c} y_i \right) \\
&= 2(a(\theta) - b(\theta))\theta + \left(- \sum_{i\in A(\theta)} x_i + \sum_{i\in A(\theta)^c} x_i + \sum_{i\in B(\theta)} y_i - \sum_{i\in B(\theta)^c} y_i \right).
\end{aligned}$$

Consider an interval of θs that does not contain any x_is or y_is. The second term is constant on such an interval. The first term will be constant, on the interval if and only if $a(\theta) = b(\theta)$. This will be true for all such intervals if and only if the order statistics for $\mathbf{x}$ are the same as the order statistics for $\mathbf{y}$. Therefore, the order statistics are a minimal sufficient statistic.

6.10 To prove $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is not complete, we want to find $g[T(\mathbf{X})]$ such that $\mathrm{E}\, g[T(\mathbf{X})] = 0$ for all θ, but $g[T(\mathbf{X})] \not\equiv 0$. A natural candidate is $R = X_{(n)} - X_{(1)}$, the range of $\mathbf{X}$, because by Example 6.2.17 its distribution does not depend on θ. From Example 6.2.17, $R \sim \text{beta}(n-1, 2)$. Thus $\mathrm{E}\, R = (n-1)/(n+1)$ does not depend on θ, and $\mathrm{E}(R - \mathrm{E}\, R) = 0$ for all θ. Thus $g[X_{(n)}, X_{(1)}] = X_{(n)} - X_{(1)} - (n-1)/(n+1) = R - \mathrm{E}\, R$ is a nonzero function whose expected value is always 0. So, $(X_{(1)}, X_{(n)})$ is not complete. This problem can be generalized to show that if a function of a sufficient statistic is ancillary, then the sufficient statistic is not complete, because the expectation of that function does not depend on θ. That provides the opportunity to construct an unbiased, nonzero estimator of zero.

6.11 a. These are all location families. Let $Z_{(1)}, \dots, Z_{(n)}$ be the order statistics from a random sample of size n from the standard pdf $f(z|0)$. Then $(Z_{(1)} + \theta, \dots, Z_{(n)} + \theta)$ has the same joint distribution as $(X_{(1)}, \dots, X_{(n)})$, and $(Y_{(1)}, \dots, Y_{(n-1)})$ has the same joint distribution as $(Z_{(n)} + \theta - (Z_{(1)} + \theta), \dots, Z_{(n)} + \theta - (Z_{(n-1)} + \theta)) = (Z_{(n)} - Z_{(1)}, \dots, Z_{(n)} - Z_{(n-1)})$. The last vector depends only on $(Z_1, \dots, Z_n)$ whose distribution does not depend on θ. So, $(Y_{(1)}, \dots, Y_{(n-1)})$ is ancillary.

b. For a), Basu's lemma shows that $(Y_1, \dots, Y_{n-1})$ is independent of the complete sufficient statistic. For c), d), and e) the order statistics are sufficient, so $(Y_1, \dots, Y_{n-1})$ is not independent of the sufficient statistic. For b), $X_{(1)}$ is sufficient. Define $Y_n = X_{(1)}$. Then the joint pdf of $(Y_1, \dots, Y_n)$ is

$$f(y_1, \dots, y_n) = n! e^{-n(y_1-\theta)} e^{-(n-1)y_n} \prod_{i=2}^{n-1} e^{y_i}, \qquad \begin{array}{c} 0 < y_{n-1} < y_{n-2} < \cdots < y_1 \\ 0 < y_n < \infty. \end{array}$$

Thus, $Y_n = X_{(1)}$ is independent of $(Y_1, \dots, Y_{n-1})$.

6.12 a. Use Theorem 6.2.13 and write

$$\begin{aligned}
\frac{f(x, n|\theta)}{f(y, n'|\theta)} &= \frac{f(x|\theta, N = n)P(N = n)}{f(y|\theta, N = n')P(N = n')} \\
&= \frac{\binom{n}{x}\theta^x (1-\theta)^{n-x} p_n}{\binom{n'}{y}\theta^y (1-\theta)^{n'-y} p_{n'}} = \theta^{x-y}(1-\theta)^{n-n'-x+y} \frac{\binom{n}{x} p_n}{\binom{n'}{y} p_{n'}}.
\end{aligned}$$

The last ratio does not depend on θ. The other terms are constant as a function of θ if and only if $n = n'$ and $x = y$. So (X, N) is minimal sufficient for θ. Because $P(N = n) = p_n$ does not depend on θ, N is ancillary for θ. The point is that although N is independent of θ, the minimal sufficient statistic contains N in this case. A minimal sufficient statistic may contain an ancillary statistic.

b.

$$\begin{aligned}
\mathrm{E}\left(\frac{X}{N}\right) &= \mathrm{E}\left(\mathrm{E}\left(\left.\frac{X}{N}\right| N\right)\right) = \mathrm{E}\left(\frac{1}{N}\mathrm{E}(X \mid N)\right) = \mathrm{E}\left(\frac{1}{N}N\theta\right) = \mathrm{E}(\theta) = \theta. \\
\mathrm{Var}\left(\frac{X}{N}\right) &= \mathrm{Var}\left(\mathrm{E}\left(\left.\frac{X}{N}\right| N\right)\right) + \mathrm{E}\left(\mathrm{Var}\left(\left.\frac{X}{N}\right| N\right)\right) = \mathrm{Var}(\theta) + \mathrm{E}\left(\frac{1}{N^2}\mathrm{Var}(X \mid N)\right) \\
&= 0 + \mathrm{E}\left(\frac{N\theta(1-\theta)}{N^2}\right) = \theta(1-\theta)\mathrm{E}\left(\frac{1}{N}\right).
\end{aligned}$$

We used the fact that $X|N \sim \text{binomial}(N, \theta)$.

6.13 Let $Y_1 = \log X_1$ and $Y_2 = \log X_2$. Then Y_1 and Y_2 are iid and, by Theorem 2.1.5, the pdf of each is

$$f(y|\alpha) = \alpha \exp\{\alpha y - e^{\alpha y}\} = \frac{1}{1/\alpha}\exp\left\{\frac{y}{1/\alpha} - e^{y/(1/\alpha)}\right\}, \quad -\infty < y < \infty.$$

We see that the family of distributions of Y_i is a scale family with scale parameter $1/\alpha$. Thus, by Theorem 3.5.6, we can write $Y_i = \frac{1}{\alpha}Z_i$, where Z_1 and Z_2 are a random sample from $f(z|1)$. Then

$$\frac{\log X_1}{\log X_2} = \frac{Y_1}{Y_2} = \frac{(1/\alpha)Z_1}{(1/\alpha)Z_2} = \frac{Z_1}{Z_2}.$$

Because the distribution of Z_1/Z_2 does not depend on α, $(\log X_1)/(\log X_2)$ is an ancillary statistic.

6.14 Because $X_1, \ldots, X_n$ is from a location family, by Theorem 3.5.6, we can write $X_i = Z_i + \mu$, where $Z_1, \ldots, Z_n$ is a random sample from the standard pdf, $f(z)$, and μ is the location parameter. Let $M(\mathbf{X})$ denote the median calculated from $X_1, \ldots, X_n$. Then $M(\mathbf{X}) = M(\mathbf{Z}) + \mu$ and $\bar{X} = \bar{Z} + \mu$. Thus, $M(\mathbf{X}) - \bar{X} = (M(\mathbf{Z}) + \mu) - (\bar{Z} + \mu) = M(\mathbf{Z}) - \bar{Z}$. Because $M(\mathbf{X}) - \bar{X}$ is a function of only $Z_1, \ldots, Z_n$, the distribution of $M(\mathbf{X}) - \bar{X}$ does not depend on μ; that is, $M(\mathbf{X}) - \bar{X}$ is an ancillary statistic.

6.15 a. The parameter space consists only of the points (θ, ν) on the graph of the function $\nu = a\theta^2$. This quadratic graph is a line and does not contain a two-dimensional open set.

b. Use the same factorization as in Example 6.2.9 to show $(\bar{X}, S^2)$ is sufficient. $\mathrm{E}(S^2) = a\theta^2$ and $\mathrm{E}(\bar{X}^2) = \mathrm{Var}\bar{X} + (\mathrm{E}\bar{X})^2 = a\theta^2/n + \theta^2 = (a+n)\theta^2/n$. Therefore,

$$\mathrm{E}\left(\frac{n}{a+n}\bar{X}^2 - \frac{S^2}{a}\right) = \left(\frac{n}{a+n}\right)\left(\frac{a+n}{n}\theta^2\right) - \frac{1}{a}a\theta^2 = 0, \text{ for all } \theta.$$

Thus $g(\bar{X}, S^2) = \frac{n}{a+n}\bar{X}^2 - \frac{S^2}{a}$ has zero expectation so $(\bar{X}, S^2)$ not complete.

6.17 The population pmf is $f(x|\theta) = \theta(1-\theta)^{x-1} = \frac{\theta}{1-\theta}e^{\log(1-\theta)x}$, an exponential family with $t(x) = x$. Thus, $\sum_i X_i$ is a complete, sufficient statistic by Theorems 6.2.10 and 6.2.25. $\sum_i X_i - n \sim$ negative binomial(n, θ).

6.18 The distribution of $Y = \sum_i X_i$ is Poisson$(n\lambda)$. Now

$$\mathrm{E}g(Y) = \sum_{y=0}^{\infty} g(y)\frac{(n\lambda)^y e^{-n\lambda}}{y!}.$$

If the expectation exists, this is an analytic function which cannot be identically zero.

6.19 To check if the family of distributions of X is complete, we check if $\mathrm{E}_p\, g(X) = 0$ for all p, implies that $g(X) \equiv 0$. For Distribution 1,

$$\mathrm{E}_p\, g(X) = \sum_{x=0}^{2} g(x)P(X = x) = pg(0) + 3pg(1) + (1 - 4p)g(2).$$

Note that if $g(0) = -3g(1)$ and $g(2) = 0$, then the expectation is zero for all p, but $g(x)$ need not be identically zero. Hence the family is not complete. For Distribution 2 calculate

$$\mathrm{E}_p\, g(X) = g(0)p + g(1)p^2 + g(2)(1 - p - p^2) = [g(1) - g(2)]p^2 + [g(0) - g(2)]p + g(2).$$

This is a polynomial of degree 2 in p. To make it zero for all p each coefficient must be zero. Thus, $g(0) = g(1) = g(2) = 0$, so the family of distributions is complete.

6.20 The pdfs in b), c), and e) are exponential families, so they have complete sufficient statistics from Theorem 6.2.25. For a), $Y = \max\{X_i\}$ is sufficient and

$$f(y) = \frac{2n}{\theta^{2n}} y^{2n-1}, \quad 0 < y < \theta.$$

For a function $g(y)$,

$$\mathrm{E}\, g(Y) = \int_0^\theta g(y) \frac{2n}{\theta^{2n}} y^{2n-1}\, dy = 0 \text{ for all } \theta \text{ implies } g(\theta)\frac{2n\theta^{2n-1}}{\theta^{2n}} = 0 \text{ for all } \theta$$

by taking derivatives. This can only be zero if $g(\theta) = 0$ for all θ, so $Y = \max\{X_i\}$ is complete. For d), the order statistics are minimal sufficient. This is a location family. Thus, by Example 6.2.18 the range $R = X_{(n)} - X_{(1)}$ is ancillary, and its expectation does not depend on θ. So this sufficient statistic is not complete.

6.21 a. X is sufficient because it is the data. To check completeness, calculate

$$\mathrm{E}g(X) = \frac{\theta}{2} g(-1) + (1 - \theta)g(0) + \frac{\theta}{2} g(1).$$

If $g(-1) = g(1)$ and $g(0) = 0$, then $\mathrm{E}g(X) = 0$ for all θ, but $g(x)$ need not be identically 0. So the family is not complete.

b. $|X|$ is sufficient by Theorem 6.2.6, because $f(x|\theta)$ depends on x only through the value of $|x|$. The distribution of $|X|$ is Bernoulli, because $P(|X| = 0) = 1 - \theta$ and $P(|X| = 1) = \theta$. By Example 6.2.22, a binomial family (Bernoulli is a special case) is complete.

c. Yes, $f(x|\theta) = (1 - \theta)(\theta/(2(1 - \theta))^{|x|} = (1 - \theta)e^{|x| \log[\theta/(2(1-\theta)]}$, the form of an exponential family.

6.22 a. The sample density is $\prod_i \theta x_i^{\theta-1} = \theta^n (\prod_i x_i)^{\theta-1}$, so $\prod_i X_i$ is sufficient for θ, not $\sum_i X_i$.

b. Because $\prod_i f(x_i|\theta) = \theta^n e^{(\theta-1)\log(\Pi_i x_i)}$, $\log(\prod_i X_i)$ is complete and sufficient by Theorem 6.2.25. Because $\prod_i X_i$ is a one-to-one function of $\log(\prod_i X_i)$, $\prod_i X_i$ is also a complete sufficient statistic.

6.23 Use Theorem 6.2.13. The ratio

$$\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} = \frac{\theta^{-n} I_{(x_{(n)}/2, x_{(1)})}(\theta)}{\theta^{-n} I_{(y_{(n)}/2, y_{(1)})}(\theta)}$$

is constant (in fact, one) if and only if $x_{(1)} = y_{(1)}$ and $x_{(n)} = y_{(n)}$. So $(X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ. From Exercise 6.10, we know that if a function of the sufficient statistics is ancillary, then the sufficient statistic is not complete. The uniform$(\theta, 2\theta)$ family is a scale family, with standard pdf $f(z) \sim$ uniform$(1, 2)$. So if $Z_1, \ldots, Z_n$ is a random sample from a uniform$(1,2)$ population, then $X_1 = \theta Z_1, \ldots, X_n = \theta Z_n$ is a random sample from a uniform$(\theta, 2\theta)$ population, and $X_{(1)} = \theta Z_{(1)}$ and $X_{(n)} = \theta Z_{(n)}$. So $X_{(1)}/X_{(n)} = Z_{(1)}/Z_{(n)}$, a statistic whose distribution does not depend on θ. Thus, as in Exercise 6.10, $(X_{(1)}, X_{(n)})$ is not complete.

6.24 If $\lambda = 0$, $\mathrm{E}h(X) = h(0)$. If $\lambda = 1$,

$$\mathrm{E}h(X) = e^{-1}h(0) + e^{-1}\sum_{x=1}^{\infty} \frac{h(x)}{x!}.$$

Let $h(0) = 0$ and $\sum_{x=1}^{\infty} \frac{h(x)}{x!} = 0$, so $\mathrm{E}h(X) = 0$ but $h(x) \not\equiv 0$. (For example, take $h(0) = 0$, $h(1) = 1$, $h(2) = -2$, $h(x) = 0$ for $x \geq 3$.)

6.25 Using the fact that $(n-1)s_{\mathbf{x}}^2 = \sum_i x_i^2 - n\bar{x}^2$, for any (μ, σ^2) the ratio in Example 6.2.14 can be written as

$$\frac{f(\mathbf{x}|\mu,\sigma^2)}{f(\mathbf{y}|\mu,\sigma^2)} = \exp\left[\frac{\mu}{\sigma^2}\left(\sum_i x_i - \sum_i y_i\right) - \frac{1}{2\sigma^2}\left(\sum_i x_i^2 - \sum_i y_i^2\right)\right].$$

a. Do part b) first showing that $\sum_i X_i^2$ is a minimal sufficient statistic. Because $\left(\sum_i X_i, \sum_i X_i^2\right)$ is not a function of $\sum_i X_i^2$, by Definition 6.2.11 $\left(\sum_i X_i, \sum_i X_i^2\right)$ is not minimal.

b. Substituting $\sigma^2 = \mu$ in the above expression yields

$$\frac{f(\mathbf{x}|\mu,\mu)}{f(\mathbf{y}|\mu,\mu)} = \exp\left[\sum_i x_i - \sum_i y_i\right] \exp\left[-\frac{1}{2\mu}\left(\sum_i x_i^2 - \sum_i y_i^2\right)\right].$$

This is constant as a function of μ if and only if $\sum_i x_i^2 = \sum_i y_i^2$. Thus, $\sum_i X_i^2$ is a minimal sufficient statistic.

c. Substituting $\sigma^2 = \mu^2$ in the first expression yields

$$\frac{f(\mathbf{x}|\mu,\mu^2)}{f(\mathbf{y}|\mu,\mu^2)} = \exp\left[\frac{1}{\mu}\left(\sum_i x_i - \sum_i y_i\right) - \frac{1}{2\mu^2}\left(\sum_i x_i^2 - \sum_i y_i^2\right)\right].$$

This is constant as a function of μ if and only if $\sum_i x_i = \sum_i y_i$ and $\sum_i x_i^2 = \sum_i y_i^2$. Thus, $\left(\sum_i X_i, \sum_i X_i^2\right)$ is a minimal sufficient statistic.

d. The first expression for the ratio is constant a function of μ and σ^2 if and only if $\sum_i x_i = \sum_i y_i$ and $\sum_i x_i^2 = \sum_i y_i^2$. Thus, $\left(\sum_i X_i, \sum_i X_i^2\right)$ is a minimal sufficient statistic.

6.27 a. This pdf can be written as

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{1/2}\left(\frac{1}{x^3}\right)^{1/2} \exp\left(\frac{\lambda}{\mu}\right) \exp\left(-\frac{\lambda}{2\mu^2}x - \frac{\lambda}{2}\frac{1}{x}\right).$$

This is an exponential family with $t_1(x) = x$ and $t_2(x) = 1/x$. By Theorem 6.2.25, the statistic $(\sum_i X_i, \sum_i (1/X_i))$ is a complete sufficient statistic. $(\bar{X}, T)$ given in the problem is a one-to-one function of $(\sum_i X_i, \sum_i (1/X_i))$. Thus, $(\bar{X}, T)$ is also a complete sufficient statistic.

b. This can be accomplished using the methods from Section 4.3 by a straightforward but messy two-variable transformation $U = (X_1 + X_2)/2$ and $V = 2\lambda/T = \lambda[(1/X_1) + (1/X_2) - (2/[X_1 + X_2])]$. This is a two-to-one transformation.

6.29 Let $f_j = \text{logistic}(\alpha_j, \beta_j)$, $j = 0, 1, \ldots, k$. From Theorem 6.6.5, the statistic

$$T(\mathbf{x}) = \left(\frac{\prod_{i=1}^n f_1(x_i)}{\prod_{i=1}^n f_0(x_i)}, \ldots, \frac{\prod_{i=1}^n f_k(x_i)}{\prod_{i=1}^n f_0(x_i)} \right) = \left(\frac{\prod_{i=1}^n f_1(x_{(i)})}{\prod_{i=1}^n f_0(x_{(i)})}, \ldots, \frac{\prod_{i=1}^n f_k(x_{(i)})}{\prod_{i=1}^n f_0(x_{(i)})} \right)$$

is minimal sufficient for the family $\{f_0, f_1, \ldots, f_k\}$. As T is a $1-1$ function of the order statistics, the order statistics are also minimal sufficient for the family $\{f_0, f_1, \ldots, f_k\}$. If $\mathcal{F}$ is a nonparametric family, $f_j \in \mathcal{F}$, so part (b) of Theorem 6.6.5 can now be directly applied to show that the order statistics are minimal sufficient for $\mathcal{F}$.

6.30 a. From Exercise 6.9b, we have that $X_{(1)}$ is a minimal sufficient statistic. To check completeness compute $f_{Y_1}(y)$, where $Y_1 = X_{(1)}$. From Theorem 5.4.4 we have

$$f_{Y_1}(y) = f_X(y)\,(1-F_X(y))^{n-1}\, n = e^{-(y-\mu)} \left[e^{-(y-\mu)}\right]^{n-1} n = ne^{-n(y-\mu)}, \quad y > \mu.$$

Now, write $\mathrm{E}_\mu\, g(Y_1) = \int_\mu^\infty g(y) n e^{-n(y-\mu)}\, dy$. If this is zero for all μ, then $\int_\mu^\infty g(y)e^{-ny}\,dy = 0$ for all μ (because $ne^{n\mu} > 0$ for all μ and does not depend on y). Moreover,

$$0 = \frac{d}{d\mu}\left[\int_\mu^\infty g(y)e^{-ny}\,dy\right] = -g(\mu)e^{-n\mu}$$

for all μ. This implies $g(\mu) = 0$ for all μ, so $X_{(1)}$ is complete.

b. Basu's Theorem says that if $X_{(1)}$ is a complete sufficient statistic for μ, then $X_{(1)}$ is independent of any ancillary statistic. Therefore, we need to show only that S^2 has distribution independent of μ; that is, S^2 is ancillary. Recognize that $f(x|\mu)$ is a location family. So we can write $X_i = Z_i + \mu$, where $Z_1, \ldots, Z_n$ is a random sample from $f(x|0)$. Then

$$S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2 = \frac{1}{n-1}\sum((Z_i+\mu) - (\bar{Z}+\mu))^2 = \frac{1}{n-1}\sum(Z_i - \bar{Z})^2.$$

Because S^2 is a function of only $Z_1, \ldots, Z_n$, the distribution of S^2 does not depend on μ; that is, S^2 is ancillary. Therefore, by Basu's theorem, S^2 is independent of $X_{(1)}$.

6.31 a. (i) By Exercise 3.28 this is a one-dimensional exponential family with $t(x) = x$. By Theorem 6.2.25, $\sum_i X_i$ is a complete sufficient statistic. $\bar{X}$ is a one-to-one function of $\sum_i X_i$, so $\bar{X}$ is also a complete sufficient statistic. From Theorem 5.3.1 we know that $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1} = \text{gamma}((n-1)/2, 2)$. $S^2 = [\sigma^2/(n-1)][(n-1)S^2/\sigma^2]$, a simple scale transformation, has a $\text{gamma}((n-1)/2, 2\sigma^2/(n-1))$ distribution, which does not depend on μ; that is, S^2 is ancillary. By Basu's Theorem, $\bar{X}$ and S^2 are independent.

(ii) The independence of $\bar{X}$ and S^2 is determined by the joint distribution of $(\bar{X}, S^2)$ for each value of (μ, σ^2). By part (i), for each value of (μ, σ^2), $\bar{X}$ and S^2 are independent.

b. (i) μ is a location parameter. By Exercise 6.14, $M - \bar{X}$ is ancillary. As in part (a) $\bar{X}$ is a complete sufficient statistic. By Basu's Theorem, $\bar{X}$ and $M - \bar{X}$ are independent. Because they are independent, by Theorem 4.5.6 $\mathrm{Var}\, M = \mathrm{Var}(M - \bar{X} + \bar{X}) = \mathrm{Var}(M - \bar{X}) + \mathrm{Var}\,\bar{X}$.

(ii) If S^2 is a sample variance calculated from a normal sample of size N, $(N-1)S^2/\sigma^2 \sim \chi^2_{N-1}$. Hence, $(N-1)^2 \mathrm{Var}\, S^2/(\sigma^2)^2 = 2(N-1)$ and $\mathrm{Var}\, S^2 = 2(\sigma^2)^2/(N-1)$. Both M and $M - \bar{X}$ are asymptotically normal, so, $M_1, \ldots, M_N$ and $M_1 - \bar{X}_1, \ldots, M_N - \bar{X}_N$ are each approximately normal samples if n is reasonable large. Thus, using the above expression we get the two given expressions where in the straightforward case σ^2 refers to $\mathrm{Var}\, M$, and in the swindle case σ^2 refers to $\mathrm{Var}(M - \bar{X})$.

c. (i)

$$\mathrm{E}(X^k) = \mathrm{E}\left(\frac{X}{Y}Y\right)^k = \mathrm{E}\left[\left(\frac{X}{Y}\right)^k (Y^k)\right] \overset{\text{indep.}}{=} \mathrm{E}\left(\frac{X}{Y}\right)^k \mathrm{E}\left(Y^k\right).$$

Divide both sides by $\mathrm{E}\left(Y^k\right)$ to obtain the desired equality.

(ii) If α is fixed, $T = \sum_i X_i$ is a complete sufficient statistic for β by Theorem 6.2.25. Because β is a scale parameter, if $Z_1, \ldots, Z_n$ is a random sample from a gamma$(\alpha, 1)$ distribution, then $X_{(i)}/T$ has the same distribution as $(\beta Z_{(i)})/\left(\beta \sum_i Z_i\right) = Z_{(i)}/\left(\sum_i Z_i\right)$, and this distribution does not depend on β. Thus, $X_{(i)}/T$ is ancillary, and by Basu's Theorem, it is independent of T. We have

$$\mathrm{E}(X_{(i)}|T) = \mathrm{E}\left(\frac{X_{(i)}}{T}T\middle|T\right) = T\mathrm{E}\left(\frac{X_{(i)}}{T}\middle|T\right) \stackrel{\text{indep.}}{=} T\mathrm{E}\left(\frac{X_{(i)}}{T}\right) \stackrel{\text{part (i)}}{=} T\frac{\mathrm{E}(X_{(i)})}{\mathrm{E}T}.$$

Note, this expression is correct for each fixed value of (α, β), regardless whether α is "known" or not.

6.32 In the Formal Likelihood Principle, take $E_1 = E_2 = E$. Then the conclusion is $\mathrm{Ev}(E, x_1) = \mathrm{Ev}(E, x_2)$ if $L(\theta|x_1)/L(\theta|x_2) = c$. Thus evidence is equal whenever the likelihood functions are equal, and this follows from Formal Sufficiency and Conditionality.

6.33 a. For all sample points except $(2, \mathbf{x}_2^*)$ (but including $(1, \mathbf{x}_1^*)$), $T(j, \mathbf{x}_j) = (j, \mathbf{x}_j)$. Hence,

$$g(T(j, \mathbf{x}_j)|\theta)h(j, \mathbf{x}_j) = g((j, \mathbf{x}_j)|\theta)1 = f^*((j, \mathbf{x}_j)|\theta).$$

For $(2, \mathbf{x}_2^*)$ we also have

$$\begin{aligned} g(T(2, \mathbf{x}_2^*)|\theta)h(2, \mathbf{x}_2^*) &= g((1, \mathbf{x}_1^*)|\theta)C = f^*((1, \mathbf{x}_1^*)|\theta)C = C\frac{1}{2}f_1(\mathbf{x}_1^*|\theta) \\ &= C\frac{1}{2}L(\theta|\mathbf{x}_1^*) = \frac{1}{2}L(\theta|\mathbf{x}_2^*) = \frac{1}{2}f_2(\mathbf{x}_2^*|\theta) = f^*((2, \mathbf{x}_2^*)|\theta). \end{aligned}$$

By the Factorization Theorem, $T(J, \mathbf{X}_J)$ is sufficient.

b. Equations 6.3.4 and 6.3.5 follow immediately from the two Principles. Combining them we have $\mathrm{Ev}(E_1, \mathbf{x}_1^*) = \mathrm{Ev}(E_2, \mathbf{x}_2^*)$, the conclusion of the Formal Likelihood Principle.

c. To prove the Conditionality Principle. Let one experiment be the E^* experiment and the other E_j. Then

$$L(\theta|(j, \mathbf{x}_j)) = f^*((j, \mathbf{x}_j)|\theta) = \frac{1}{2}f_j(\mathbf{x}_j|\theta) = \frac{1}{2}L(\theta|\mathbf{x}_j).$$

Letting $(j, \mathbf{x}_j)$ and $\mathbf{x}_j$ play the roles of $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$ in the Formal Likelihood Principle we can conclude $\mathrm{Ev}(E^*, (j, \mathbf{x}_j)) = \mathrm{Ev}(E_j, \mathbf{x}_j)$, the Conditionality Principle. Now consider the Formal Sufficiency Principle. If $T(\mathbf{X})$ is sufficient and $T(\mathbf{x}) = T(\mathbf{y})$, then $L(\theta|\mathbf{x}) = CL(\theta|\mathbf{y})$, where $C = h(\mathbf{x})/h(\mathbf{y})$ and h is the function from the Factorization Theorem. Hence, by the Formal Likelihood Principle, $\mathrm{Ev}(E, \mathbf{x}) = \mathrm{Ev}(E, \mathbf{y})$, the Formal Sufficiency Principle.

6.35 Let 1 = success and 0 = failure. The four sample points are $\{0, 10, 110, 111\}$. From the likelihood principle, inference about p is only through $L(p|\mathbf{x})$. The values of the likelihood are 1, p, p^2, and p^3, and the sample size does not directly influence the inference.

6.37 a. For one observation (X, Y) we have

$$I(\theta) = -\mathrm{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(X, Y|\theta)\right) = -\mathrm{E}\left(-\frac{2Y}{\theta^3}\right) = \frac{2\mathrm{E}Y}{\theta^3}.$$

But, $Y \sim$ exponential(θ), and $\mathrm{E}Y = \theta$. Hence, $I(\theta) = 2/\theta^2$ for a sample of size one, and $I(\theta) = 2n/\theta^2$ for a sample of size n.

b. (i) The cdf of T is

$$P(T \le t) = P\left(\frac{\sum_i Y_i}{\sum_i X_i} \le t^2\right) = P\left(\frac{2\sum_i Y_i/\theta}{2\sum_i X_i\theta} \le t^2/\theta^2\right) = P(F_{2n,2n} \le t^2/\theta^2)$$

where $F_{2n,2n}$ is an F random variable with $2n$ degrees of freedom in the numerator and denominator. This follows since $2Y_i/\theta$ and $2X_i\theta$ are all independent exponential(1), or χ^2_2. Differentiating (in t) and simplifying gives the density of T as

$$f_T(t) = \frac{\Gamma(2n)}{\Gamma(n)^2}\frac{2}{t}\left(\frac{t^2}{t^2+\theta^2}\right)^n\left(\frac{\theta^2}{t^2+\theta^2}\right)^n,$$

and the second derivative (in θ) of the log density is

$$2n\frac{t^4+2t^2\theta^2-\theta^4}{\theta^2(t^2+\theta^2)^2} = \frac{2n}{\theta^2}\left(1-\frac{2}{(t^2/\theta^2+1)^2}\right),$$

and the information in T is

$$\frac{2n}{\theta^2}\left[1-2\mathrm{E}\left(\frac{1}{T^2/\theta^2+1}\right)^2\right] = \frac{2n}{\theta^2}\left[1-2\mathrm{E}\left(\frac{1}{F^2_{2n,2n}+1}\right)^2\right].$$

The expected value is

$$\mathrm{E}\left(\frac{1}{F^2_{2n,2n}+1}\right)^2 = \frac{\Gamma(2n)}{\Gamma(n)^2}\int_0^\infty \frac{1}{(1+w)^2}\frac{w^{n-1}}{(1+w)^{2n}} = \frac{\Gamma(2n)}{\Gamma(n)^2}\frac{\Gamma(n)\Gamma(n+2)}{\Gamma(2n+2)} = \frac{n+1}{2(2n+1)}.$$

Substituting this above gives the information in T as

$$\frac{2n}{\theta^2}\left[1-2\frac{n+1}{2(2n+1)}\right] = I(\theta)\frac{n}{2n+1},$$

which is not the answer reported by Joshi and Nabar.

(ii) Let $W=\sum_i X_i$ and $V=\sum_i Y_i$. In each pair, X_i and Y_i are independent, so W and V are independent. $X_i \sim$ exponential$(1/\theta)$; hence, $W \sim$ gamma$(n, 1/\theta)$. $Y_i \sim$ exponential(θ); hence, $V \sim$ gamma(n,θ). Use this joint distribution of (W,V) to derive the joint pdf of (T,U) as

$$f(t,u|\theta) = \frac{2}{[\Gamma(n)]^2 t}u^{2n-1}\exp\left(-\frac{u\theta}{t}-\frac{ut}{\theta}\right), \quad u>0, \quad t>0.$$

Now, the information in (T,U) is

$$-\mathrm{E}\left(\frac{\partial^2}{\partial\theta^2}\log f(T,U|\theta)\right) = -\mathrm{E}\left(-\frac{2UT}{\theta^3}\right) = \mathrm{E}\left(\frac{2V}{\theta^3}\right) = \frac{2n\theta}{\theta^3} = \frac{2n}{\theta^2}.$$

(iii) The pdf of the sample is $f(\mathbf{x},\mathbf{y}) = \exp\left[-\theta\left(\sum_i x_i\right) - \left(\sum_i y_i\right)/\theta\right]$. Hence, (W,V) defined as in part (ii) is sufficient. (T,U) is a one-to-one function of (W,V), hence (T,U) is also sufficient. But, $\mathrm{E}\,U^2 = \mathrm{E}\,WV = (n/\theta)(n\theta) = n^2$ does not depend on θ. So $\mathrm{E}(U^2-n^2)=0$ for all θ, and (T,U) is not complete.

6.39 a. The transformation from Celsius to Fahrenheit is $y = 9x/5+32$. Hence,

$$\begin{aligned}\frac{5}{9}(T^*(y)-32) &= \frac{5}{9}((.5)(y)+(.5)(212)-32)\\ &= \frac{5}{9}((.5)(9x/5+32)+(.5)(212)-32) = (.5)x+50 = T(x).\end{aligned}$$

b. $T(x) = (.5)x+50 \neq (.5)x+106 = T^*(x)$. Thus, we do not have equivariance.

6.40 a. Because $X_1,\ldots,X_n$ is from a location scale family, by Theorem 3.5.6, we can write $X_i = \sigma Z_i + \mu$, where $Z_1,\ldots,Z_n$ is a random sample from the standard pdf $f(z)$. Then

$$\frac{T_1(X_1,\ldots,X_n)}{T_2(X_1,\ldots,X_n)} = \frac{T_1(\sigma Z_1+\mu,\ldots,\sigma Z_n+\mu)}{T_2(\sigma Z_1+\mu,\ldots,\sigma Z_n+\mu)} = \frac{\sigma T_1(Z_1,\ldots,Z_n)}{\sigma T_2(Z_1,\ldots,Z_n)} = \frac{T_1(Z_1,\ldots,Z_n)}{T_2(Z_1,\ldots,Z_n)}.$$

Because T_1/T_2 is a function of only $Z_1,\ldots,Z_n$, the distribution of T_1/T_2 does not depend on μ or σ; that is, T_1/T_2 is an ancillary statistic.

b. $R(x_1,\ldots,x_n) = x_{(n)} - x_{(1)}$. Because $a > 0$, $\max\{ax_1+b,\ldots,ax_n+b\} = ax_{(n)}+b$ and $\min\{ax_1+b,\ldots,ax_n+b\} = ax_{(1)}+b$. Thus, $R(ax_1+b,\ldots,ax_n+b) = (ax_{(n)}+b)-(ax_{(1)}+b) = a(x_{(n)}-x_{(1)}) = aR(x_1,\ldots,x_n)$. For the sample variance we have

$$\begin{aligned} S^2(ax_1+b,\ldots,ax_n+b) &= \frac{1}{n-1}\sum((ax_i+b)-(a\bar{x}+b))^2 \\ &= a^2\frac{1}{n-1}\sum(x_i-\bar{x})^2 \;=\; a^2S^2(x_1,\ldots,x_n). \end{aligned}$$

Thus, $S(ax_1+b,\ldots,ax_n+b) = aS(x_1,\ldots,x_n)$. Therefore, R and S both satisfy the above condition, and R/S is ancillary by a).

6.41 a. Measurement equivariance requires that the estimate of μ based on $\mathbf{y}$ be the same as the estimate of μ based on $\mathbf{x}$; that is, $T^*(x_1+a,\ldots,x_n+a)-a = T^*(\mathbf{y})-a = T(\mathbf{x})$.

b. The formal structures for the problem involving $\mathbf{X}$ and the problem involving $\mathbf{Y}$ are the same. They both concern a random sample of size n from a normal population and estimation of the mean of the population. Thus, formal invariance requires that $T(\mathbf{x}) = T^*(\mathbf{x})$ for all $\mathbf{x}$. Combining this with part (a), the Equivariance Principle requires that $T(x_1+a,\ldots,x_n+a)-a = T^*(x_1+a,\ldots,x_n+a)-a = T(x_1,\ldots,x_n)$, i.e., $T(x_1+a,\ldots,x_n+a) = T(x_1,\ldots,x_n)+a$.

c. $W(x_1+a,\ldots,x_n+a) = \sum_i(x_i+a)/n = \left(\sum_i x_i\right)/n + a = W(x_1,\ldots,x_n)+a$, so $W(\mathbf{x})$ is equivariant. The distribution of $(X_1,\ldots,X_n)$ is the same as the distribution of $(Z_1+\theta,\ldots,Z_n+\theta)$, where $Z_1,\ldots,Z_n$ are a random sample from $f(x-0)$ and $\mathrm{E}\,Z_i = 0$. Thus, $\mathrm{E}_\theta W = \mathrm{E}\sum_i(Z_i+\theta)/n = \theta$, for all θ.

6.43 a. For a location-scale family, if $X \sim f(x|\theta,\sigma^2)$, then $Y = g_{a,c}(X) \sim f(y|c\theta+a, c^2\sigma^2)$. So for estimating σ^2, $\bar{g}_{a,c}(\sigma^2) = c^2\sigma^2$. An estimator of σ^2 is invariant with respect to $\mathcal{G}_1$ if $W(cx_1+a,\ldots,cx_n+a) = c^2W(x_1,\ldots,x_n)$. An estimator of the form kS^2 is invariant because

$$\begin{aligned} kS^2(cx_1+a,\ldots,cx_n+a) &= \frac{k}{n-1}\sum_{i=1}^n\left((cx_i+a)-\sum_{i=1}^n(cx_i+a)/n\right)^2 \\ &= \frac{k}{n-1}\sum_{i=1}^n((cx_i+a)-(c\bar{x}+a))^2 \\ &= c^2\frac{k}{n-1}\sum_{i=1}^n(x_i-\bar{x})^2 \;=\; c^2kS^2(x_1,\ldots,x_n). \end{aligned}$$

To show invariance with respect to $\mathcal{G}_2$, use the above argument with $c = 1$. To show invariance with respect to $\mathcal{G}_3$, use the above argument with $a = 0$. ($\mathcal{G}_2$ and $\mathcal{G}_3$ are both subgroups of $\mathcal{G}_1$. So invariance with respect to $\mathcal{G}_1$ implies invariance with respect to $\mathcal{G}_2$ and $\mathcal{G}_3$.)

b. The transformations in $\mathcal{G}_2$ leave the scale parameter unchanged. Thus, $\bar{g}_a(\sigma^2) = \sigma^2$. An estimator of σ^2 is invariant with respect to this group if

$$W(x_1+a,\ldots,x_n+a) = W(g_a(\mathbf{x})) = \bar{g}_a(W(\mathbf{x})) = W(x_1,\ldots,x_n).$$

An estimator of the given form is invariant if, for all a and $(x_1, \ldots, x_n)$,

$$W(x_1+a, \ldots, x_n+a) = \phi\left(\frac{\bar{x}+a}{s}\right)s^2 = \phi\left(\frac{\bar{x}}{s}\right)s^2 = W(x_1, \ldots, x_n).$$

In particular, for a sample point with $s = 1$ and $\bar{x} = 0$, this implies we must have $\phi(a) = \phi(0)$, for all a; that is, ϕ must be constant. On the other hand, if ϕ is constant, then the estimators are invariant by part a). So we have invariance if and only if ϕ is constant. Invariance with respect to $\mathcal{G}_1$ also requires ϕ to be constant because $\mathcal{G}_2$ is a subgroup of $\mathcal{G}_1$. Finally, an estimator of σ^2 is invariant with respect to $\mathcal{G}_3$ if $W(cx_1, \ldots, cx_n) = c^2 W(x_1, \ldots, x_n)$. Estimators of the given form are invariant because

$$W(cx_1, \ldots, cx_n) = \phi\left(\frac{c\bar{x}}{cs}\right)c^2 s^2 = c^2\phi\left(\frac{\bar{x}}{s}\right)s^2 = c^2 W(x_1, \ldots, x_n).$$