Chapter 8

Hypothesis Testing

8.1 Let $X = \#$ of heads out of 1000. If the coin is fair, then $X \sim \text{binomial}(1000, 1/2)$. So

$$P(X \geq 560) = \sum_{x=560}^{1000} \binom{1000}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x} \approx .0000825,$$

where a computer was used to do the calculation. For this binomial, $\mathrm{E}\,X = 1000p = 500$ and $\mathrm{Var}\,X = 1000p(1-p) = 250$. A normal approximation is also very good for this calculation.

$$P\left\{X \geq 560\right\} = P\left\{\frac{X-500}{\sqrt{250}} \geq \frac{559.5{-}500}{\sqrt{250}}\right\} \approx P\left\{Z \geq 3.763\right\} \approx .0000839.$$

Thus, if the coin is fair, the probability of observing 560 or more heads out of 1000 is very small. We might tend to believe that the coin is not fair, and $p > 1/2$.

8.2 Let $X \sim \text{Poisson}(\lambda)$, and we observed $X = 10$. To assess if the accident rate has dropped, we could calculate

$$P(\,X \leq 10|\,\lambda = 15) = \sum_{i=0}^{10} \frac{e^{-15}\,15^i}{i!} = e^{-15}\,\left[1{+}15{+}\frac{15^2}{2!}{+}\cdots{+}\frac{15^{10}}{10!}\right] \approx .11846.$$

This is a fairly large value, not overwhelming evidence that the accident rate has dropped. (A normal approximation with continuity correction gives a value of .12264.)

8.3 The LRT statistic is

$$\lambda(y) = \frac{\sup_{\theta \leq \theta_0} L(\theta|y_1, \ldots, y_m)}{\sup_{\Theta} L(\theta|y_1, \ldots, y_m)}.$$

Let $y = \sum_{i=1}^m y_i$, and note that the MLE in the numerator is $\min\{y/m,\theta_0\}$ (see Exercise 7.12) while the denominator has y/m as the MLE (see Example 7.2.7). Thus

$$\lambda(y) = \begin{cases} 1 & \text{if } y/m \leq \theta_0 \\ \frac{(\theta_0)^y (1-\theta_0)^{m-y}}{(y/m)^y (1-y/m)^{m-y}} & \text{if } y/m > \theta_0, \end{cases}$$

and we reject H_0 if

$$\frac{(\theta_0)^y (1{-}\theta_0)^{m-y}}{(y/m)^y (1-y/m)^{m-y}} < c.$$

To show that this is equivalent to rejecting if $y > b$, we could show $\lambda(y)$ is decreasing in y so that $\lambda(y) < c$ occurs for $y > b > m\theta_0$. It is easier to work with $\log \lambda(y)$, and we have

$$\log \lambda(y) = y \log \theta_0 + (m-y) \log(1-\theta_0) - y \log\left(\frac{y}{m}\right) - (m-y) \log\left(\frac{m-y}{m}\right),$$

and

$$\begin{aligned}\frac{d}{dy}\log\lambda(y) &= \log\theta_0 - \log(1-\theta_0) - \log\left(\frac{y}{m}\right) - y\frac{1}{y} + \log\left(\frac{m-y}{m}\right) + (m-y)\frac{1}{m-y} \\ &= \log\left(\frac{\theta_0}{y/m}\frac{\left(\frac{m-y}{m}\right)}{1-\theta_0}\right).\end{aligned}$$

For $y/m > \theta_0$, $1 - y/m = (m-y)/m < 1-\theta_0$, so each fraction above is less than 1, and the log is less than 0. Thus $\frac{d}{dy}\log\lambda < 0$ which shows that λ is decreasing in y and $\lambda(y) < c$ if and only if $y > b$.

8.4 For discrete random variables, $L(\theta|\mathbf{x}) = f(\mathbf{x}|\theta) = P(\mathbf{X} = \mathbf{x}|\theta)$. So the numerator and denominator of $\lambda(\mathbf{x})$ are the supremum of this probability over the indicated sets.

8.5 a. The log-likelihood is

$$\log L(\theta,\nu|\mathbf{x}) = n\log\theta + n\theta\log\nu - (\theta+1)\log\left(\prod_i x_i\right), \quad \nu \le x_{(1)},$$

where $x_{(1)} = \min_i x_i$. For any value of θ, this is an increasing function of ν for $\nu \le x_{(1)}$. So both the restricted and unrestricted MLEs of ν are $\hat{\nu} = x_{(1)}$. To find the MLE of θ, set

$$\frac{\partial}{\partial\theta}\log L(\theta, x_{(1)}|\mathbf{x}) = \frac{n}{\theta} + n\log x_{(1)} - \log\left(\prod_i x_i\right) = 0,$$

and solve for θ yielding

$$\hat{\theta} = \frac{n}{\log(\prod_i x_i/x_{(1)}^n)} = \frac{n}{T}.$$

$(\partial^2/\partial\theta^2)\log L(\theta, x_{(1)}|\mathbf{x}) = -n/\theta^2 < 0$, for all θ. So $\hat{\theta}$ is a maximum.

b. Under H_0, the MLE of θ is $\hat{\theta}_0 = 1$, and the MLE of ν is still $\hat{\nu} = x_{(1)}$. So the likelihood ratio statistic is

$$\lambda(\mathbf{x}) = \frac{x_{(1)}^n/\left(\prod_i x_i\right)^2}{(n/T)^n x_{(1)}^{n^2/T}\Big/\left(\prod_i x_i\right)^{n/T+1}} = \left(\frac{T}{n}\right)^n \frac{e^{-T}}{\left(e^{-T}\right)^{n/T}} = \left(\frac{T}{n}\right)^n e^{-T+n}.$$

$(\partial/\partial T)\log\lambda(\mathbf{x}) = (n/T) - 1$. Hence, $\lambda(\mathbf{x})$ is increasing if $T \le n$ and decreasing if $T \ge n$. Thus, $T \le c$ is equivalent to $T \le c_1$ or $T \ge c_2$, for appropriately chosen constants c_1 and c_2.

c. We will not use the hint, although the problem can be solved that way. Instead, make the following three transformations. First, let $Y_i = \log X_i$, $i = 1,\ldots,n$. Next, make the n-to-1 transformation that sets $Z_1 = \min_i Y_i$ and sets $Z_2,\ldots,Z_n$ equal to the remaining Y_is, with their order unchanged. Finally, let $W_1 = Z_1$ and $W_i = Z_i - Z_1$, $i = 2,\ldots,n$. Then you find that the W_is are independent with $W_1 \sim f_{W_1}(w) = n\nu^n e^{-nw}$, $w > \log\nu$, and $W_i \sim$ exponential(1), $i = 2,\ldots,n$. Now $T = \sum_{i=2}^n W_i \sim$ gamma$(n-1,1)$, and, hence, $2T \sim$ gamma$(n-1,2) = \chi^2_{2(n-1)}$.

8.6 a.

$$\begin{aligned}\lambda(\mathbf{x},\mathbf{y}) &= \frac{\sup_{\Theta_0} L(\theta|\mathbf{x},\mathbf{y})}{\sup_{\Theta} L(\theta|\mathbf{x},\mathbf{y})} = \frac{\sup_\theta \prod_{i=1}^n \frac{1}{\theta}e^{-x_i/\theta} \prod_{j=1}^m \frac{1}{\theta}e^{-y_j/\theta}}{\sup_{\theta,\mu} \prod_{i=1}^n \frac{1}{\theta}e^{-x_i/\theta} \prod_{j=1}^m \frac{1}{\mu}e^{-y_j/\mu}} \\ &= \frac{\sup_\theta \frac{1}{\theta^{m+n}}\exp\left\{-\left(\sum_{i=1}^n x_i + \sum_{j=1}^m y_j\right)\Big/\theta\right\}}{\sup_{\theta,\mu}\frac{1}{\theta^n}\exp\left\{-\sum_{i=1}^n x_i/\theta\right\}\frac{1}{\mu^m}\exp\left\{-\sum_{j=1}^m y_j/\mu\right\}}.\end{aligned}$$

Differentiation will show that in the numerator $\hat{\theta}_0 = (\sum_i x_i + \sum_j y_j)/(n+m)$, while in the denominator $\hat{\theta} = \bar{x}$ and $\hat{\mu} = \bar{y}$. Therefore,

$$\begin{aligned}\lambda(\mathbf{x},\mathbf{y}) &= \frac{\left(\frac{n+m}{\sum_i x_i+\sum_j y_j}\right)^{n+m} \exp\left\{-\left(\frac{n+m}{\sum_i x_i+\sum_j y_j}\right)\left(\sum_i x_i+\sum_j y_j\right)\right\}}{\left(\frac{n}{\sum_i x_i}\right)^{n} \exp\left\{-\left(\frac{n}{\sum_i x_i}\right)\sum_i x_i\right\}\left(\frac{m}{\sum_j y_j}\right)^{m} \exp\left\{-\left(\frac{m}{\sum_j y_j}\right)\sum_j y_j\right\}} \\ &= \frac{(n+m)^{n+m}}{n^n m^m}\frac{\left(\sum_i x_i\right)^n\left(\sum_j y_j\right)^m}{\left(\sum_i x_i+\sum_j y_j\right)^{n+m}}.\end{aligned}$$

And the LRT is to reject H_0 if $\lambda(\mathbf{x},\mathbf{y}) \le c$.

b.

$$\lambda = \frac{(n+m)^{n+m}}{n^n m^m}\left(\frac{\sum_i x_i}{\sum_i x_i+\sum_j y_j}\right)^n\left(\frac{\sum_j y_j}{\sum_i x_i+\sum_j y_j}\right)^m = \frac{(n+m)^{n+m}}{n^n m^m}T^n(1-T)^m.$$

Therefore λ is a function of T. λ is a unimodal function of T which is maximized when $T = \frac{n}{m+n}$. Rejection for $\lambda \le c$ is equivalent to rejection for $T \le a$ or $T \ge b$, where a and b are constants that satisfy $a^n(1-a)^m = b^n(1-b)^m$.

c. When H_0 is true, $\sum_i X_i \sim \text{gamma}(n,\theta)$ and $\sum_j Y_j \sim \text{gamma}(m,\theta)$ and they are independent. So by an extension of Exercise 4.19b, $T \sim \text{beta}(n,m)$.

8.7 a.

$$L(\theta,\lambda|\mathbf{x}) = \prod_{i=1}^n \frac{1}{\lambda}e^{-(x_i-\theta)/\lambda}I_{[\theta,\infty)}(x_i) = \left(\frac{1}{\lambda}\right)^n e^{-(\Sigma_i x_i - n\theta)/\lambda}I_{[\theta,\infty)}(x_{(1)}),$$

which is increasing in θ if $x_{(1)} \ge \theta$ (regardless of λ). So the MLE of θ is $\hat{\theta} = x_{(1)}$. Then

$$\frac{\partial \log L}{\partial\lambda} = -\frac{n}{\lambda} + \frac{\sum_i x_i - n\hat{\theta}}{\lambda^2} \overset{\text{set}}{=} 0 \quad\Rightarrow\quad n\hat{\lambda} = \sum_i x_i - n\hat{\theta} \quad\Rightarrow\quad \hat{\lambda} = \bar{x} - x_{(1)}.$$

Because

$$\frac{\partial^2 \log L}{\partial\lambda^2} = \frac{n}{\lambda^2} - 2\left.\frac{\sum_i x_i - n\hat{\theta}}{\lambda^3}\right|_{\bar{x}-x_{(1)}} = \frac{n}{(\bar{x}-x_{(1)})^2} - \frac{2n(\bar{x}-x_{(1)})}{(\bar{x}-x_{(1)})^3} = \frac{-n}{(\bar{x}-x_{(1)})^2} < 0,$$

we have $\hat{\theta} = x_{(1)}$ and $\hat{\lambda} = \bar{x} - x_{(1)}$ as the unrestricted MLEs of θ and λ. Under the restriction $\theta \le 0$, the MLE of θ (regardless of λ) is

$$\hat{\theta}_0 = \begin{cases} 0 & \text{if } x_{(1)} > 0 \\ x_{(1)} & \text{if } x_{(1)} \le 0. \end{cases}$$

For $x_{(1)} > 0$, substituting $\hat{\theta}_0 = 0$ and maximizing with respect to λ, as above, yields $\hat{\lambda}_0 = \bar{x}$. Therefore,

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta,\lambda \mid \mathbf{x})}{\sup_{\Theta} L(\theta,\lambda \mid \mathbf{x})} = \frac{\sup_{\{(\lambda,\theta):\theta\le 0\}} L(\lambda,\theta \mid \mathbf{x})}{L(\hat{\theta},\hat{\lambda} \mid \mathbf{x})} = \begin{cases} 1 & \text{if } x_{(1)} \le 0 \\ \frac{L(\bar{x},0|\mathbf{x})}{L(\hat{\lambda},\hat{\theta}|\mathbf{x})} & \text{if } x_{(1)} > 0, \end{cases}$$

where

$$\frac{L(\bar{x},0 \mid \mathbf{x})}{L(\hat{\lambda},\hat{\theta} \mid \mathbf{x})} = \frac{(1/\bar{x})^n e^{-n\bar{x}/\bar{x}}}{\left(1/\hat{\lambda}\right)^n e^{-n(\bar{x}-x_{(1)})/(\bar{x}-x_{(1)})}} = \left(\frac{\hat{\lambda}}{\bar{x}}\right)^n = \left(\frac{\bar{x}-x_{(1)}}{\bar{x}}\right)^n = \left(1-\frac{x_{(1)}}{\bar{x}}\right)^n.$$

So rejecting if $\lambda(\mathbf{x}) \le c$ is equivalent to rejecting if $x_{(1)}/\bar{x} \ge c^*$, where c^* is some constant.

b. The LRT statistic is

$$\lambda(\mathbf{x}) = \frac{\sup_\beta (1/\beta^n) e^{-\Sigma_i x_i/\beta}}{\sup_{\beta,\gamma} (\gamma^n/\beta^n)(\prod_i x_i)^{\gamma-1} e^{-\Sigma_i x_i^\gamma/\beta}}.$$

The numerator is maximized at $\hat{\beta}_0 = \bar{x}$. For fixed γ, the denominator is maximized at $\hat{\beta}_\gamma = \sum_i x_i^\gamma/n$. Thus

$$\lambda(\mathbf{x}) = \frac{\bar{x}^{-n} e^{-n}}{\sup_\gamma (\gamma^n/\hat{\beta}_\gamma^n)(\prod_i x_i)^{\gamma-1} e^{-\Sigma_i x_i^\gamma/\hat{\beta}_\gamma}} = \frac{\bar{x}^{-n}}{\sup_\gamma (\gamma^n/\hat{\beta}_\gamma^n)(\prod_i x_i)^{\gamma-1}}.$$

The denominator cannot be maximized in closed form. Numeric maximization could be used to compute the statistic for observed data $\mathbf{x}$.

8.8 a. We will first find the MLEs of a and θ. We have

$$\begin{aligned} L(a,\theta \mid \mathbf{x}) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi a\theta}} e^{-(x_i-\theta)^2/(2a\theta)}, \\ \log L(a,\theta \mid \mathbf{x}) &= \sum_{i=1}^n -\frac{1}{2}\log(2\pi a\theta) - \frac{1}{2a\theta}(x_i-\theta)^2. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial \log L}{\partial a} &= \sum_{i=1}^n \left(-\frac{1}{2a} + \frac{1}{2\theta a^2}(x_i-\theta)^2\right) = -\frac{n}{2a} + \frac{1}{2\theta a^2}\sum_{i=1}^n (x_i-\theta)^2 \stackrel{\text{set}}{=} 0 \\ \frac{\partial \log L}{\partial \theta} &= \sum_{i=1}^n \left[-\frac{1}{2\theta} + \frac{1}{2a\theta^2}(x_i-\theta)^2 + \frac{1}{a\theta}(x_i-\theta)\right] \\ &= -\frac{n}{2\theta} + \frac{1}{2a\theta^2}\sum_{i=1}^n (x_i-\theta)^2 + \frac{n\bar{x}-n\theta}{a\theta} \stackrel{\text{set}}{=} 0. \end{aligned}$$

We have to solve these two equations simultaneously to get MLEs of a and θ, say $\hat{a}$ and $\hat{\theta}$. Solve the first equation for a in terms of θ to get

$$a = \frac{1}{n\theta}\sum_{i=1}^n (x_i-\theta)^2.$$

Substitute this into the second equation to get

$$-\frac{n}{2\theta} + \frac{n}{2\theta} + \frac{n(\bar{x}-\theta)}{a\theta} = 0.$$

So we get $\hat{\theta} = \bar{x}$, and

$$\hat{a} = \frac{1}{n\bar{x}}\sum_{i=1}^n (x_i-\bar{x})^2 = \frac{\hat{\sigma}^2}{\bar{x}},$$

the ratio of the usual MLEs of the mean and variance. (Verification that this is a maximum is lengthy. We omit it.) For $a = 1$, we just solve the second equation, which gives a quadratic in θ that leads to the restricted MLE

$$\hat{\theta}_R = \frac{-1+\sqrt{1+4(\hat{\sigma}^2+\bar{x}^2)}}{2}.$$

Noting that $\hat{a}\hat{\theta} = \hat{\sigma}^2$, we obtain

$$
\begin{aligned}
\lambda(\mathbf{x}) &= \frac{L(\hat{\theta}_R \mid \mathbf{x})}{L(\hat{a},\hat{\theta} \mid \mathbf{x})} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\hat{\theta}_R}} e^{-(x_i-\hat{\theta}_R)^2/(2\hat{\theta}_R)}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\hat{a}\hat{\theta}}} e^{-(x_i-\hat{\theta})^2/(2\hat{a}\hat{\theta})}} \\
&= \frac{\left(1/(2\pi\hat{\theta}_R)\right)^{n/2} e^{-\Sigma_i (x_i-\hat{\theta}_R)^2/(2\hat{\theta}_R)}}{(1/(2\pi\hat{\sigma}^2))^{n/2}\, e^{-\Sigma_i (x_i-\bar{x})^2/(2\hat{\sigma}^2)}} \\
&= \left(\hat{\sigma}^2/\hat{\theta}_R\right)^{n/2} e^{(n/2)-\Sigma_i (x_i-\hat{\theta}_R)^2/(2\hat{\theta}_R)}.
\end{aligned}
$$

b. In this case we have

$$
\log L(a,\theta \mid \mathbf{x}) = \sum_{i=1}^{n} \left[-\frac{1}{2}\log(2\pi a\theta^2) - \frac{1}{2a\theta^2}(x_i-\theta)^2 \right].
$$

Thus

$$
\begin{aligned}
\frac{\partial \mathrm{log} L}{\partial a} &= \sum_{i=1}^{n} \left(-\frac{1}{2a} + \frac{1}{2a^2\theta^2}(x_i-\theta)^2 \right) = -\frac{n}{2a} + \frac{1}{2a^2\theta^2}\sum_{i=1}^{n}(x_i-\theta)^2 \overset{\text{set}}{=} 0. \\
\frac{\partial \mathrm{log} L}{\partial \theta} &= \sum_{i=1}^{n} \left[-\frac{1}{\theta} + \frac{1}{a\theta^3}(x_i-\theta)^2 + \frac{1}{a\theta^2}(x_i-\theta) \right] \\
&= -\frac{n}{\theta} + \frac{1}{a\theta^3}\sum_{i=1}^{n}(x_i-\theta)^2 + \frac{1}{a\theta^2}\sum_{i=1}^{n}(x_i-\theta) \overset{\text{set}}{=} 0.
\end{aligned}
$$

Solving the first equation for a in terms of θ yields

$$
a = \frac{1}{n\theta^2}\sum_{i=1}^{n}(x_i-\theta)^2.
$$

Substituting this into the second equation, we get

$$
-\frac{n}{\theta} + \frac{n}{\theta} + n\frac{\sum_i (x_i-\theta)}{\sum_i (x_i-\theta)^2} = 0.
$$

So again, $\hat{\theta} = \bar{x}$ and

$$
\hat{a} = \frac{1}{n\bar{x}^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{\hat{\sigma}^2}{\bar{x}^2}
$$

in the unrestricted case. In the restricted case, set $a = 1$ in the second equation to obtain

$$
\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n}(x_i-\theta)^2 + \frac{1}{\theta^2}\sum_{i=1}^{n}(x_i-\theta) \overset{\text{set}}{=} 0.
$$

Multiply through by θ^3/n to get

$$
-\theta^2 + \frac{1}{n}\sum_{i=1}^{n}(x_i-\theta)^2 - \frac{\theta}{n}\sum_{i=1}^{n}(x_i-\theta) = 0.
$$

Add $\pm\bar{x}$ inside the square and complete all sums to get the equation

$$
-\theta^2 + \hat{\sigma}^2 + (\bar{x}-\theta)^2 + \theta(\bar{x}-\theta) = 0.
$$

This is a quadratic in θ with solution for the MLE

$$\hat{\theta}_R = \bar{x} + \sqrt{\bar{x}+4(\hat{\sigma}^2+\bar{x}^2)} \Big/ 2.$$

which yields the LRT statistic

$$\lambda(\mathbf{x}) = \frac{L(\hat{\theta}_R \mid \mathbf{x})}{L(\hat{a}, \hat{\theta} \mid \mathbf{x})} = \frac{\prod_{i=1}^n \frac{1}{\sqrt{2\pi\hat{\theta}_R^2}} e^{-(x_i-\hat{\theta}_R)^2/(2\hat{\theta}_R^2)}}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi\hat{a}\hat{\theta}^2}} e^{-(x_i-\hat{\theta})^2/(2\hat{a}\hat{\theta}^2)}} = \left(\frac{\hat{\sigma}}{\hat{\theta}_R}\right)^n e^{(n/2)-\Sigma_i(x_i-\hat{\theta}_R)^2/(2\hat{\theta}_R)}.$$

8.9 a. The MLE of λ under H_0 is $\hat{\lambda}_0 = \left(\bar{Y}\right)^{-1}$, and the MLE of λ_i under H_1 is $\hat{\lambda}_i = Y_i^{-1}$. The LRT statistic is bounded above by 1 and is given by

$$1 \geq \frac{\left(\bar{Y}\right)^{-n} e^{-n}}{\left(\prod_i Y_i\right)^{-1} e^{-n}}.$$

Rearrangement of this inequality yields $\bar{Y} \geq \left(\prod_i Y_i\right)^{1/n}$, the arithmetic-geometric mean inequality.

b. The pdf of X_i is $f(x_i|\lambda_i) = (\lambda_i/x_i^2)e^{-\lambda_i/x_i}$, $x_i > 0$. The MLE of λ under H_0 is $\hat{\lambda}_0 = n/\left[\sum_i(1/X_i)\right]$, and the MLE of λ_i under H_1 is $\hat{\lambda}_i = X_i$. Now, the argument proceeds as in part (a).

8.10 Let $Y = \sum_i X_i$. The posterior distribution of $\lambda|y$ is gamma $(y+\alpha, \beta/(\beta+1))$.

a.

$$P(\lambda \leq \lambda_0|y) = \frac{(\beta+1)^{y+\alpha}}{\Gamma(y+\alpha)\beta^{y+\alpha}} \int_0^{\lambda_0} t^{y+\alpha-1} e^{-t(\beta+1)/\beta}\, dt.$$

$P(\lambda > \lambda_0|y) = 1 - P(\lambda \leq \lambda_0|y)$.

b. Because $\beta/(\beta+1)$ is a scale parameter in the posterior distribution, $(2(\beta+1)\lambda/\beta)|y$ has a gamma$(y+\alpha, 2)$ distribution. If 2α is an integer, this is a $\chi^2_{2y+2\alpha}$ distribution. So, for $\alpha = 5/2$ and $\beta = 2$,

$$P(\lambda \leq \lambda_0|y) = P\left(\frac{2(\beta+1)\lambda}{\beta} \leq \frac{2(\beta+1)\lambda_0}{\beta} \,\middle|\, y\right) = P(\chi^2_{2y+5} \leq 3\lambda_0).$$

8.11 a. From Exercise 7.23, the posterior distribution of σ^2 given S^2 is IG(γ, δ), where $\gamma = \alpha + (n-1)/2$ and $\delta = [(n-1)S^2/2 + 1/\beta]^{-1}$. Let $Y = 2/(\sigma^2\delta)$. Then $Y|S^2 \sim$ gamma$(\gamma, 2)$. (Note: If 2α is an integer, this is a $\chi^2_{2\gamma}$ distribution.) Let M denote the median of a gamma$(\gamma, 2)$ distribution. Note that M depends on only α and n, not on S^2 or β. Then we have $P(Y \geq 2/\delta|S^2) = P(\sigma^2 \leq 1|S^2) > 1/2$ if and only if

$$M > \frac{2}{\delta} = (n-1)S^2 + \frac{2}{\beta}, \quad \text{that is,} \quad S^2 < \frac{M - 2/\beta}{n-1}.$$

b. From Example 7.2.11, the unrestricted MLEs are $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = (n-1)S^2/n$. Under H_0, $\hat{\mu}$ is still $\bar{X}$, because this was the maximizing value of μ, regardless of σ^2. Then because $L(\bar{x}, \sigma^2|\mathbf{x})$ is a unimodal function of σ^2, the restricted MLE of σ^2 is $\hat{\sigma}^2$, if $\hat{\sigma}^2 \leq 1$, and is 1, if $\hat{\sigma}^2 > 1$. So the LRT statistic is

$$\lambda(\mathbf{x}) = \begin{cases} 1 & \text{if } \hat{\sigma}^2 \leq 1 \\ (\hat{\sigma}^2)^{n/2} e^{-n(\hat{\sigma}^2-1)/2} & \text{if } \hat{\sigma}^2 > 1. \end{cases}$$

We have that, for $\hat{\sigma}^2 > 1$,

$$\frac{\partial}{\partial(\hat{\sigma}^2)} \log \lambda(\mathbf{x}) = \frac{n}{2}\left(\frac{1}{\hat{\sigma}^2} - 1\right) < 0.$$

So $\lambda(\mathbf{x})$ is decreasing in $\hat{\sigma}^2$, and rejecting H_0 for small values of $\lambda(\mathbf{x})$ is equivalent to rejecting for large values of $\hat{\sigma}^2$, that is, large values of S^2. The LRT accepts H_0 if and only if $S^2 < k$, where k is a constant. We can pick the prior parameters so that the acceptance regions match in this way. First, pick α large enough that $M/(n-1) > k$. Then, as β varies between 0 and ∞, $(M-2/\beta)/(n-1)$ varies between $-\infty$ and $M/(n-1)$. So, for some choice of β, $(M-2/\beta)/(n-1) = k$ and the acceptance regions match.

8.12 a. For $H_0\colon \mu \le 0$ vs. $H_1\colon \mu > 0$ the LRT is to reject H$_0$ if $\bar{x} > c\sigma/\sqrt{n}$ (Example 8.3.3). For $\alpha = .05$ take $c = 1.645$. The power function is

$$\beta(\mu) = P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > 1.645 - \frac{\mu}{\sigma/\sqrt{n}}\right) = P\left(Z > 1.645 - \frac{\sqrt{n}\mu}{\sigma}\right).$$

Note that the power will equal .5 when $\mu = 1.645\sigma/\sqrt{n}$.

b. For $H_0\colon \mu = 0$ vs. $H_A\colon \mu \ne 0$ the LRT is to reject H_0 if $|\bar{x}| > c\sigma/\sqrt{n}$ (Example 8.2.2). For $\alpha = .05$ take $c = 1.96$. The power function is

$$\beta(\mu) = P\left(-1.96 - \sqrt{n}\mu/\sigma \le Z \le 1.96 + \sqrt{n}\mu/\sigma\right).$$

In this case, $\mu = \pm 1.96\sigma/\sqrt{n}$ gives power of approximately .5.

8.13 a. The size of ϕ_1 is $\alpha_1 = P(X_1 > .95|\theta = 0) = .05$. The size of ϕ_2 is $\alpha_2 = P(X_1+X_2 > C|\theta = 0)$. If $1 \le C \le 2$, this is

$$\alpha_2 = P(X_1 + X_2 > C|\theta = 0) = \int_{1-C}^{1}\int_{C-x_1}^{1} 1\, dx_2\, dx_1 = \frac{(2-C)^2}{2}.$$

Setting this equal to α and solving for C gives $C = 2 - \sqrt{2\alpha}$, and for $\alpha = .05$, we get $C = 2 - \sqrt{.1} \approx 1.68$.

b. For the first test we have the power function

$$\beta_1(\theta) = P_\theta(X_1 > .95) = \begin{cases} 0 & \text{if } \theta \le -.05 \\ \theta + .05 & \text{if } -.05 < \theta \le .95 \\ 1 & \text{if } .95 < \theta. \end{cases}$$

Using the distribution of $Y = X_1 + X_2$, given by

$$f_Y(y|\theta) = \begin{cases} y - 2\theta & \text{if } 2\theta \le y < 2\theta + 1 \\ 2\theta + 2 - y & \text{if } 2\theta{+}1 \le y < 2\theta + 2 \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the power function for the second test as

$$\beta_2(\theta) = P_\theta(Y > C) = \begin{cases} 0 & \text{if } \theta \le (C/2) - 1 \\ (2\theta + 2 - C)^2/2 & \text{if } (C/2) - 1 < \theta \le (C-1)/2 \\ 1 - (C - 2\theta)^2/2 & \text{if } (C-1)/2 < \theta \le C/2 \\ 1 & \text{if } C/2 < \theta. \end{cases}$$

c. From the graph it is clear that ϕ_1 is more powerful for θ near 0, but ϕ_2 is more powerful for larger θs. ϕ_2 is not uniformly more powerful than ϕ_1.

d. If either $X_1 \geq 1$ or $X_2 \geq 1$, we should reject H_0, because if $\theta = 0$, $P(X_i < 1) = 1$. Thus, consider the rejection region given by

$$\{(x_1, x_2)\colon x_1 + x_2 > C\} \bigcup \{(x_1, x_2)\colon x_1 > 1\} \bigcup \{(x_1, x_2)\colon x_2 > 1\}.$$

The first set is the rejection region for ϕ_2. The test with this rejection region has the same size as ϕ_2 because the last two sets both have probability 0 if $\theta = 0$. But for $0 < \theta < C - 1$, The power function of this test is strictly larger than $\beta_2(\theta)$. If $C - 1 \leq \theta$, this test and ϕ_2 have the same power.

8.14 The CLT tells us that $Z = (\sum_i X_i - np)/\sqrt{np(1-p)}$ is approximately $\mathrm{n}(0,1)$. For a test that rejects H_0 when $\sum_i X_i > c$, we need to find c and n to satisfy

$$P\left(Z > \frac{c - n(.49)}{\sqrt{n(.49)(.51)}}\right) = .01 \quad \text{and} \quad P\left(Z > \frac{c - n(.51)}{\sqrt{n(.51)(.49)}}\right) = .99.$$

We thus want

$$\frac{c - n(.49)}{\sqrt{n(.49)(.51)}} = 2.33 \quad \text{and} \quad \frac{c - n(.51)}{\sqrt{n(.51)(.49)}} = -2.33.$$

Solving these equations gives $n = 13{,}567$ and $c = 6{,}783.5$.

8.15 From the Neyman-Pearson lemma the UMP test rejects H_0 if

$$\frac{f(x \mid \sigma_1)}{f(x \mid \sigma_0)} = \frac{(2\pi\sigma_1^2)^{-n/2} e^{-\Sigma_i x_i^2/(2\sigma_1^2)}}{(2\pi\sigma_0^2)^{-n/2} e^{-\Sigma_i x_i^2/(2\sigma_0^2)}} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{\frac{1}{2}\sum_i x_i^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right\} > k$$

for some $k \geq 0$. After some algebra, this is equivalent to rejecting if

$$\sum_i x_i^2 > \frac{2\log\left(k\left(\sigma_1/\sigma_0\right)^n\right)}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)} = c \qquad \left(\text{because } \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0\right).$$

This is the UMP test of size α, where $\alpha = P_{\sigma_0}(\sum_i X_i^2 > c)$. To determine c to obtain a specified α, use the fact that $\sum_i X_i^2/\sigma_0^2 \sim \chi_n^2$. Thus

$$\alpha = P_{\sigma_0}\left(\sum_i X_i^2/\sigma_0^2 > c/\sigma_0^2\right) = P\left(\chi_n^2 > c/\sigma_0^2\right),$$

so we must have $c/\sigma_0^2 = \chi^2_{n,\alpha}$, which means $c = \sigma_0^2 \chi^2_{n,\alpha}$.

8.16 a.

$$\begin{aligned} \text{Size} &= P(\text{reject } H_0 \mid H_0 \text{ is true}) = 1 \Rightarrow \text{Type I error} = 1. \\ \text{Power} &= P(\text{reject } H_0 \mid H_A \text{ is true}) = 1 \Rightarrow \text{Type II error} = 0. \end{aligned}$$

b.

$$\begin{aligned} \text{Size} &= P(\text{reject } H_0 \mid H_0 \text{ is true}) = 0 \Rightarrow \text{Type I error} = 0. \\ \text{Power} &= P(\text{reject } H_0 \mid H_A \text{ is true}) = 0 \Rightarrow \text{Type II error} = 1. \end{aligned}$$

8.17 a. The likelihood function is

$$L(\mu, \theta|\mathbf{x}, \mathbf{y}) = \mu^n \left(\prod_i x_i\right)^{\mu-1} \theta^n \left(\prod_j y_j\right)^{\theta-1}.$$

Maximizing, by differentiating the log-likelihood, yields the MLEs

$$\hat{\mu} = -\frac{n}{\sum_i \log x_i} \quad \text{and} \quad \hat{\theta} = -\frac{m}{\sum_j \log y_j}.$$

Under H_0, the likelihood is

$$L(\theta|\mathbf{x}, \mathbf{y}) = \theta^{n+m} \left(\prod_i x_i \prod_j y_j \right)^{\theta-1},$$

and maximizing as above yields the restricted MLE,

$$\hat{\theta}_0 = -\frac{n+m}{\sum_i \log x_i + \sum_j \log y_j}.$$

The LRT statistic is

$$\lambda(\mathbf{x}, \mathbf{y}) = \frac{\hat{\theta}_0^{m+n}}{\hat{\mu}^n \hat{\theta}^m} \left(\prod_i x_i \right)^{\hat{\theta}_0 - \hat{\mu}} \left(\prod_j y_j \right)^{\hat{\theta}_0 - \hat{\theta}}.$$

b. Substituting in the formulas for $\hat{\theta}$, $\hat{\mu}$ and $\hat{\theta}_0$ yields $(\prod_i x_i)^{\hat{\theta}_0 - \hat{\mu}} \left(\prod_j y_j\right)^{\hat{\theta}_0 - \hat{\theta}} = 1$ and

$$\lambda(\mathbf{x}, \mathbf{y}) = \frac{\hat{\theta}_0^{m+n}}{\hat{\mu}^n \hat{\theta}^m} = \frac{\hat{\theta}_0^n}{\hat{\mu}^n} \frac{\hat{\theta}_0^m}{\hat{\theta}^m} = \left(\frac{m+n}{m} \right)^m \left(\frac{m+n}{n} \right)^n (1-T)^m T^n.$$

This is a unimodal function of T. So rejecting if $\lambda(\mathbf{x}, \mathbf{y}) \le c$ is equivalent to rejecting if $T \le c_1$ or $T \ge c_2$, where c_1 and c_2 are appropriately chosen constants.

c. Simple transformations yield $-\log X_i \sim$ exponential$(1/\mu)$ and $-\log Y_i \sim$ exponential$(1/\theta)$. Therefore, $T = W/(W+V)$ where W and V are independent, $W \sim$ gamma$(n, 1/\mu)$ and $V \sim$ gamma$(m, 1/\theta)$. Under H_0, the scale parameters of W and V are equal. Then, a simple generalization of Exercise 4.19b yields $T \sim$ beta(n, m). The constants c_1 and c_2 are determined by the two equations

$$P(T \le c_1) + P(T \ge c_2) = \alpha \quad \text{and} \quad (1-c_1)^m c_1^n = (1-c_2)^m c_2^n.$$

8.18 a.

$$\begin{aligned}
\beta(\theta) &= P_\theta\left(\frac{|\bar{X} - \theta_0|}{\sigma/\sqrt{n}} > c \right) = 1 - P_\theta\left(\frac{|\bar{X} - \theta_0|}{\sigma/\sqrt{n}} \le c \right) \\
&= 1 - P_\theta\left(-\frac{c\sigma}{\sqrt{n}} \le \bar{X} - \theta_0 \le \frac{c\sigma}{\sqrt{n}} \right) \\
&= 1 - P_\theta\left(\frac{-c\sigma/\sqrt{n} + \theta_0 - \theta}{\sigma/\sqrt{n}} \le \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \le \frac{c\sigma/\sqrt{n} + \theta_0 - \theta}{\sigma/\sqrt{n}} \right) \\
&= 1 - P\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \le Z \le c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right) \\
&= 1 + \Phi\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right) - \Phi\left(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right),
\end{aligned}$$

where $Z \sim$ n$(0, 1)$ and Φ is the standard normal cdf.

b. The size is $.05 = \beta(\theta_0) = 1 + \Phi(-c) - \Phi(c)$ which implies $c = 1.96$. The power $(1-$ type II error) is

$$.75 \leq \beta(\theta_0 + \sigma) = 1 + \Phi(-c - \sqrt{n}) - \Phi(c - \sqrt{n}) = 1 + \underbrace{\Phi(-1.96 - \sqrt{n})}_{\approx 0} - \Phi(1.96 - \sqrt{n}).$$

$\Phi(-.675) \approx .25$ implies $1.96 - \sqrt{n} = -.675$ implies $n = 6.943 \approx 7$.

8.19 The pdf of Y is

$$f(y|\theta) = \frac{1}{\theta} y^{(1/\theta)-1} e^{-y^{1/\theta}}, \quad y > 0.$$

By the Neyman-Pearson Lemma, the UMP test will reject if

$$\frac{1}{2} y^{-1/2} e^{y - y^{1/2}} = \frac{f(y|2)}{f(y|1)} > k.$$

To see the form of this rejection region, we compute

$$\frac{d}{dy}\left(\frac{1}{2} y^{-1/2} e^{y-y^{1/2}}\right) = \frac{1}{2} y^{-3/2} e^{y-y^{1/2}} \left(y - \frac{y^{1/2}}{2} - \frac{1}{2}\right)$$

which is negative for $y < 1$ and positive for $y > 1$. Thus $f(y|2)/f(y|1)$ is decreasing for $y \leq 1$ and increasing for $y \geq 1$. Hence, rejecting for $f(y|2)/f(y|1) > k$ is equivalent to rejecting for $y \leq c_0$ or $y \geq c_1$. To obtain a size α test, the constants c_0 and c_1 must satisfy

$$\alpha = P(Y \leq c_0|\theta = 1) + P(Y \geq c_1|\theta = 1) = 1 - e^{-c_0} + e^{-c_1} \quad \text{and} \quad \frac{f(c_0|2)}{f(c_0|1)} = \frac{f(c_1|2)}{f(c_1|1)}.$$

Solving these two equations numerically, for $\alpha = .10$, yields $c_0 = .076546$ and $c_1 = 3.637798$. The Type II error probability is

$$P(c_0 < Y < c_1|\theta = 2) = \int_{c_0}^{c_1} \frac{1}{2} y^{-1/2} e^{-y^{1/2}}\, dy = -e^{-y^{1/2}}\Big|_{c_0}^{c_1} = .609824.$$

8.20 By the Neyman-Pearson Lemma, the UMP test rejects for large values of $f(x|H_1)/f(x|H_0)$. Computing this ratio we obtain

x	1	2	3	4	5	6	7
$\frac{f(x\|H_1)}{f(x\|H_0)}$	6	5	4	3	2	1	.84

The ratio is decreasing in x. So rejecting for large values of $f(x|H_1)/f(x|H_0)$ corresponds to rejecting for small values of x. To get a size α test, we need to choose c so that $P(X \leq c|H_0) = \alpha$. The value $c = 4$ gives the UMP size $\alpha = .04$ test. The Type II error probability is $P(X = 5, 6, 7|H_1) = .82$.

8.21 The proof is the same with integrals replaced by sums.

8.22 a. From Corollary 8.3.13 we can base the test on $\sum_i X_i$, the sufficient statistic. Let $Y = \sum_i X_i \sim \text{binomial}(10, p)$ and let $f(y|p)$ denote the pmf of Y. By Corollary 8.3.13, a test that rejects if $f(y|1/4)/f(y|1/2) > k$ is UMP of its size. By Exercise 8.25c, the ratio $f(y|1/2)/f(y|1/4)$ is increasing in y. So the ratio $f(y|1/4)/f(y|1/2)$ is decreasing in y, and rejecting for large value of the ratio is equivalent to rejecting for small values of y. To get $\alpha = .0547$, we must find c such that $P(Y \leq c|p = 1/2) = .0547$. Trying values $c = 0, 1, \ldots$, we find that for $c = 2$, $P(Y \leq 2|p = 1/2) = .0547$. So the test that rejects if $Y \leq 2$ is the UMP size $\alpha = .0547$ test. The power of the test is $P(Y \leq 2|p = 1/4) \approx .526$.

b. The size of the test is $P(Y \geq 6|p = 1/2) = \sum_{k=6}^{10} \binom{10}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{10-k} \approx .377$. The power function is $\beta(\theta) = \sum_{k=6}^{10} \binom{10}{k} \theta^k (1-\theta)^{10-k}$

c. There is a nonrandomized UMP test for all α levels corresponding to the probabilities $P(Y \leq i|p = 1/2)$, where i is an integer. For $n = 10$, α can have any of the values 0, $\frac{1}{1024}$, $\frac{11}{1024}$, $\frac{56}{1024}$, $\frac{176}{1024}$, $\frac{386}{1024}$, $\frac{638}{1024}$, $\frac{848}{1024}$, $\frac{968}{1024}$, $\frac{1013}{1024}$, $\frac{1023}{1024}$, and 1.

8.23 a. The test is Reject H_0 if $X > 1/2$. So the power function is

$$\beta(\theta) = P_\theta(X > 1/2) = \int_{1/2}^{1} \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1}(1-x)^{1-1}\, dx = \theta \left. \frac{1}{\theta} x^\theta \right|_{1/2}^{1} = 1 - \frac{1}{2^\theta}.$$

The size is $\sup_{\theta \in H_0} \beta(\theta) = \sup_{\theta \leq 1}(1 - 1/2^\theta) = 1 - 1/2 = 1/2$.

b. By the Neyman-Pearson Lemma, the most powerful test of $H_0\colon \theta = 1$ vs. $H_1\colon \theta = 2$ is given by Reject H_0 if $f(x \mid 2)/f(x \mid 1) > k$ for some $k \geq 0$. Substituting the beta pdf gives

$$\frac{f(x|2)}{f(x|1)} = \frac{\frac{1}{\beta(2,1)} x^{2-1}(1-x)^{1-1}}{\frac{1}{\beta(1,1)} x^{1-1}(1-x)^{1-1}} = \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} x = 2x.$$

Thus, the MP test is Reject H_0 if $X > k/2$. We now use the α level to determine k. We have

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(1) = \int_{k/2}^{1} f_X(x|1)\, dx = \int_{k/2}^{1} \frac{1}{\beta(1,1)} x^{1-1}(1-x)^{1-1}\, dx = 1 - \frac{k}{2}.$$

Thus $1 - k/2 = \alpha$, so the most powerful α level test is reject H_0 if $X > 1 - \alpha$.

c. For $\theta_2 > \theta_1$, $f(x|\theta_2)/f(x|\theta_1) = (\theta_2/\theta_1)x^{\theta_2 - \theta_1}$, an increasing function of x because $\theta_2 > \theta_1$. So this family has MLR. By the Karlin-Rubin Theorem, the test that rejects H_0 if $X > t$ is the UMP test of its size. By the argument in part (b), use $t = 1 - \alpha$ to get size α.

8.24 For $H_0\colon \theta = \theta_0$ vs. $H_1\colon \theta = \theta_1$, the LRT statistic is

$$\lambda(\mathbf{x}) = \frac{L(\theta_0|\mathbf{x})}{\max\{L(\theta_0|\mathbf{x}), L(\theta_1|\mathbf{x})\}} = \begin{cases} 1 & \text{if } L(\theta_0|\mathbf{x}) \geq L(\theta_1|\mathbf{x}) \\ L(\theta_0|\mathbf{x})/L(\theta_1|\mathbf{x}) & \text{if } L(\theta_0|\mathbf{x}) < L(\theta_1|\mathbf{x}). \end{cases}$$

The LRT rejects H_0 if $\lambda(\mathbf{x}) < c$. The Neyman-Pearson test rejects H_0 if $f(\mathbf{x}|\theta_1)/f(\mathbf{x}|\theta_0) = L(\theta_1|\mathbf{x})/L(\theta_0|\mathbf{x}) > k$. If $k = 1/c > 1$, this is equivalent to $L(\theta_0|\mathbf{x})/L(\theta_1|\mathbf{x}) < c$, the LRT. But if $c \geq 1$ or $k \leq 1$, the tests will not be the same. Because c is usually chosen to be small (k large) to get a small size α, in practice the two tests are often the same.

8.25 a. For $\theta_2 > \theta_1$,

$$\frac{g(x \mid \theta_2)}{g(x \mid \theta_1)} = \frac{e^{-(x-\theta_2)^2/2\sigma^2}}{e^{-(x-\theta_1)^2/2\sigma^2}} = e^{x(\theta_2-\theta_1)/\sigma^2} e^{(\theta_1^2-\theta_2^2)/2\sigma^2}.$$

Because $\theta_2 - \theta_1 > 0$, the ratio is increasing in x. So the families of $\mathrm{n}(\theta, \sigma^2)$ have MLR.

b. For $\theta_2 > \theta_1$,

$$\frac{g(x \mid \theta_2)}{g(x \mid \theta_1)} = \frac{e^{-\theta_2}\theta_2^x/x!}{e^{-\theta_1}\theta_1^x/x!} = \left(\frac{\theta_2}{\theta_1}\right)^x e^{\theta_1-\theta_2},$$

which is increasing in x because $\theta_2/\theta_1 > 1$. Thus the Poisson(θ) family has an MLR.

c. For $\theta_2 > \theta_1$,

$$\frac{g(x \mid \theta_2)}{g(x \mid \theta_1)} = \frac{\binom{n}{x}\theta_2^x(1-\theta_2)^{n-x}}{\binom{n}{x}\theta_1^x(1-\theta_1)^{n-x}} = \left(\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}\right)^x \left(\frac{1-\theta_2}{1-\theta_1}\right)^n.$$

Both $\theta_2/\theta_1 > 1$ and $(1-\theta_1)/(1-\theta_2) > 1$. Thus the ratio is increasing in x, and the family has MLR.

(Note: You can also use the fact that an exponential family $h(x)c(\theta)\exp(w(\theta)x)$ has MLR if $w(\theta)$ is increasing in θ (Exercise 8.27). For example, the Poisson(θ) pmf is $e^{-\theta}\exp(x \log \theta)/x!$, and the family has MLR because $\log \theta$ is increasing in θ.)

8.26 a. We will prove the result for continuous distributions. But it is also true for discrete MLR families. For $\theta_1 > \theta_2$, we must show $F(x|\theta_1) \le F(x|\theta_2)$. Now

$$\frac{d}{dx}\left[F(x|\theta_1) - F(x|\theta_2)\right] = f(x|\theta_1) - f(x|\theta_2) = f(x|\theta_2)\left(\frac{f(x|\theta_1)}{f(x|\theta_2)} - 1\right).$$

Because f has MLR, the ratio on the right-hand side is increasing, so the derivative can only change sign from negative to positive showing that any interior extremum is a minimum. Thus the function in square brackets is maximized by its value at ∞ or $-\infty$, which is zero.

b. From Exercise 3.42, location families are stochastically increasing in their location parameter, so the location Cauchy family with pdf $f(x|\theta) = (\pi[1+(x-\theta)^2])^{-1}$ is stochastically increasing. The family does not have MLR.

8.27 For $\theta_2 > \theta_1$,

$$\frac{g(t|\theta_2)}{g(t|\theta_1)} = \frac{c(\theta_2)}{c(\theta_1)} e^{[w(\theta_2)-w(\theta_1)]t}$$

which is increasing in t because $w(\theta_2) - w(\theta_1) > 0$. Examples include $\mathrm{n}(\theta, 1)$, $\mathrm{beta}(\theta, 1)$, and Bernoulli(θ).

8.28 a. For $\theta_2 > \theta_1$, the likelihood ratio is

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = e^{\theta_1-\theta_2}\left[\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right]^2.$$

The derivative of the quantity in brackets is

$$\frac{d}{dx}\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}} = \frac{e^{x-\theta_1} - e^{x-\theta_2}}{(1+e^{x-\theta_2})^2}.$$

Because $\theta_2 > \theta_1$, $e^{x-\theta_1} > e^{x-\theta_2}$, and, hence, the ratio is increasing. This family has MLR.

b. The best test is to reject H_0 if $f(x|1)/f(x|0) > k$. From part (a), this ratio is increasing in x. Thus this inequality is equivalent to rejecting if $x > k'$. The cdf of this logistic is $F(x|\theta) = e^{x-\theta}\Big/(1 + e^{x-\theta})$. Thus

$$\alpha = 1 - F(k'|0) = \frac{1}{1+e^{k'}} \quad \text{and} \quad \beta = F(k'|1) = \frac{e^{k'-1}}{1+e^{k'-1}}.$$

For a specified α, $k' = \log(1-\alpha)/\alpha$. So for $\alpha = .2$, $k' \approx 1.386$ and $\beta \approx .595$.

c. The Karlin-Rubin Theorem is satisfied, so the test is UMP of its size.

8.29 a. Let $\theta_2 > \theta_1$. Then

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2} = \frac{1 + (1+\theta_1)^2/x^2 - 2\theta_1/x}{1 + (1+\theta_2)^2/x^2 - 2\theta_2/x}.$$

The limit of this ratio as $x \to \infty$ or as $x \to -\infty$ is 1. So the ratio cannot be monotone increasing (or decreasing) between $-\infty$ and ∞. Thus, the family does not have MLR.

b. By the Neyman-Pearson Lemma, a test will be UMP if it rejects when $f(x|1)/f(x|0) > k$, for some constant k. Examination of the derivative shows that $f(x|1)/f(x|0)$ is decreasing for $x \le (1-\sqrt{5})/2 = -.618$, is increasing for $(1-\sqrt{5})/2 \le x \le (1+\sqrt{5})/2 = 1.618$, and is decreasing for $(1+\sqrt{5})/2 \le x$. Furthermore, $f(1|1)/f(1|0) = f(3|1)/f(3|0) = 2$. So rejecting if $f(x|1)/f(x|0) > 2$ is equivalent to rejecting if $1 < x < 3$. Thus, the given test is UMP of its size. The size of the test is

$$P(1 < X < 3|\theta = 0) = \int_1^3 \frac{1}{\pi}\frac{1}{1+x^2}\,dx = \frac{1}{\pi}\arctan x\Big|_1^3 \approx .1476.$$

The Type II error probability is

$$1-P(1<X<3|\theta=1)=1-\int_1^3 \frac{1}{\pi}\frac{1}{1+(x-1)^2}\,dx = 1-\frac{1}{\pi}\arctan(x-1)\Big|_1^3 \approx .6476.$$

c. We will not have $f(1|\theta)/f(1|0) = f(3|\theta)/f(3|0)$ for any other value of $\theta \neq 1$. Try $\theta = 2$, for example. So the rejection region $1 < x < 3$ will not be most powerful at any other value of θ. The test is not UMP for testing $H_0\colon \theta \le 0$ versus $H_1\colon \theta > 0$.

8.30 a. For $\theta_2 > \theta_1 > 0$, the likelihood ratio and its derivative are

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2}{\theta_1}\frac{\theta_1^2+x^2}{\theta_2^2+x^2} \quad\text{and}\quad \frac{d}{dx}\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\theta_2}{\theta_1}\frac{\theta_2^2-\theta_1^2}{(\theta_2^2+x^2)^2}x.$$

The sign of the derivative is the same as the sign of x (recall, $\theta_2^2-\theta_1^2>0$), which changes sign. Hence the ratio is not monotone.

b. Because $f(x|\theta) = (\theta/\pi)(\theta^2+|x|^2)^{-1}$, $Y=|X|$ is sufficient. Its pdf is

$$f(y|\theta) = \frac{2\theta}{\pi}\frac{1}{\theta^2+y^2}, \quad y>0.$$

Differentiating as above, the sign of the derivative is the same as the sign of y, which is positive. Hence the family has MLR.

8.31 a. By the Karlin-Rubin Theorem, the UMP test is to reject H_0 if $\sum_i X_i > k$, because $\sum_i X_i$ is sufficient and $\sum_i X_i \sim \text{Poisson}(n\lambda)$ which has MLR. Choose the constant k to satisfy $P\left(\sum_i X_i > k|\lambda=\lambda_0\right) = \alpha$.

b.

$$\begin{aligned} P\left(\sum_i X_i > k \,\middle|\, \lambda=1\right) &\approx P\left(Z>(k-n)/\sqrt{n}\right) \overset{\text{set}}{=} .05,\\ P\left(\sum_i X_i > k \,\middle|\, \lambda=2\right) &\approx P\left(Z>(k-2n)/\sqrt{2n}\right) \overset{\text{set}}{=} .90.\end{aligned}$$

Thus, solve for k and n in

$$\frac{k-n}{\sqrt{n}} = 1.645 \quad\text{and}\quad \frac{k-2n}{\sqrt{2n}} = -1.28,$$

yielding $n=12$ and $k=17.70$.

8.32 a. This is Example 8.3.15.

b. This is Example 8.3.19.

8.33 a. From Theorems 5.4.4 and 5.4.6, the marginal pdf of Y_1 and the joint pdf of (Y_1,Y_n) are

$$\begin{aligned} f(y_1|\theta) &= n(1-(y_1-\theta))^{n-1}, \quad \theta<y_1<\theta+1,\\ f(y_1,y_n|\theta) &= n(n-1)(y_n-y_1)^{n-2}, \quad \theta<y_1<y_n<\theta+1.\end{aligned}$$

Under H_0, $P(Y_n\ge 1)=0$. So

$$\alpha = P(Y_1\ge k|0) = \int_k^1 n(1-y_1)^{n-1}\,dy_1 = (1-k)^n.$$

Thus, use $k=1-\alpha^{1/n}$ to have a size α test.

b. For $\theta \le k-1$, $\beta(\theta) = 0$. For $k-1 < \theta \le 0$,

$$\beta(\theta) = \int_k^{\theta+1} n(1-(y_1-\theta))^{n-1}\,dy_1 = (1-k+\theta)^n.$$

For $0 < \theta \le k$,

$$\begin{aligned}\beta(\theta) &= \int_k^{\theta+1} n(1-(y_1-\theta))^{n-1}\,dy_1 + \int_\theta^k \int_1^{\theta+1} n(n-1)(y_n-y_1)^{n-2}\,dy_n\,dy_1 \\ &= \alpha + 1 - (1-\theta)^n.\end{aligned}$$

And for $k < \theta$, $\beta(\theta) = 1$.

c. (Y_1, Y_n) are sufficient statistics. So we can attempt to find a UMP test using Corollary 8.3.13 and the joint pdf $f(y_1, y_n|\theta)$ in part (a). For $0 < \theta < 1$, the ratio of pdfs is

$$\frac{f(y_1,y_n|\theta)}{f(y_1,y_n|0)} = \begin{cases} 0 & \text{if } 0 < y_1 \le \theta,\ y_1 < y_n < 1 \\ 1 & \text{if } \theta < y_1 < y_n < 1 \\ \infty & \text{if } 1 \le y_n < \theta+1, \theta < y_1 < y_n. \end{cases}$$

For $1 \le \theta$, the ratio of pdfs is

$$\frac{f(y_1,y_n|\theta)}{f(y_1,y_n|0)} = \begin{cases} 0 & \text{if } y_1 < y_n < 1 \\ \infty & \text{if } \theta < y_1 < y_n < \theta+1. \end{cases}$$

For $0 < \theta < k$, use $k' = 1$. The given test always rejects if $f(y_1,y_n|\theta)/f(y_1,y_n|0) > 1$ and always accepts if $f(y_1,y_n|\theta)/f(y_1,y_n|0) < 1$. For $\theta \ge k$, use $k' = 0$. The given test always rejects if $f(y_1,y_n|\theta)/f(y_1,y_n|0) > 0$ and always accepts if $f(y_1,y_n|\theta)/f(y_1,y_n|0) < 0$. Thus the given test is UMP by Corollary 8.3.13.

d. According to the power function in part (b), $\beta(\theta) = 1$ for all $\theta \ge k = 1-\alpha^{1/n}$. So these conditions are satisfied for any n.

8.34 a. This is Exercise 3.42a.

b. This is Exercise 8.26a.

8.35 a. We will use the equality in Exercise 3.17 which remains true so long as $\nu > -\alpha$. Recall that $Y \sim \chi^2_\nu = \text{gamma}(\nu/2, 2)$. Thus, using the independence of X and Y we have

$$\mathrm{E}\,T' = \mathrm{E}\frac{X}{\sqrt{Y/\nu}} = (\mathrm{E}\,X)\sqrt{\nu}\mathrm{E}\,Y^{-1/2} = \mu\sqrt{\nu}\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)\sqrt{2}}$$

if $\nu > 1$. To calculate the variance, compute

$$\mathrm{E}(T')^2 = \mathrm{E}\frac{X^2}{Y/\nu} = (\mathrm{E}\,X^2)\nu\mathrm{E}\,Y^{-1} = (\mu^2+1)\nu\frac{\Gamma((\nu-2)/2)}{\Gamma(\nu/2)2} = \frac{(\mu^2+1)\nu}{\nu-2}$$

if $\nu > 2$. Thus, if $\nu > 2$,

$$\mathrm{Var}\,T' = \frac{(\mu^2+1)\nu}{\nu-2} - \left(\mu\sqrt{\nu}\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)\sqrt{2}}\right)^2$$

b. If $\delta = 0$, all the terms in the sum for $k = 1, 2, \ldots$ are zero because of the δ^k term. The expression with just the $k = 0$ term and $\delta = 0$ simplifies to the central t pdf.

c. The argument that the noncentral t has an MLR is fairly involved. It may be found in Lehmann (1986, p. 295).

8.37 a. $P(\bar{X} > \theta_0 + z_\alpha \sigma/\sqrt{n}|\theta_0) = P\left((\bar{X}-\theta_0)/(\sigma/\sqrt{n}) > z_\alpha|\theta_0\right) = P(Z > z_\alpha) = \alpha$, where $Z \sim$ n$(0,1)$. Because $\bar{x}$ is the unrestricted MLE, and the restricted MLE is θ_0 if $\bar{x} > \theta_0$, the LRT statistic is, for $\bar{x} \geq \theta_0$

$$\lambda(\mathbf{x}) = \frac{(2\pi\sigma^2)^{-n/2} e^{-\Sigma_i(x_i-\theta_0)^2/2\sigma^2}}{(2\pi\sigma^2)^{-n/2} e^{-\Sigma_i(x_i-\bar{x})^2/2\sigma^2}} = \frac{e^{-\left[n(\bar{x}-\theta_0)^2+(n-1)s^2\right]\Big/2\sigma^2}}{e^{-(n-1)s^2/2\sigma^2}} = e^{-n(\bar{x}-\theta_0)^2/2\sigma^2}.$$

and the LRT statistic is 1 for $\bar{x} < \theta_0$. Thus, rejecting if $\lambda < c$ is equivalent to rejecting if $(\bar{x}-\theta_0)/(\sigma/\sqrt{n}) > c'$ (as long as $c < 1$ – see Exercise 8.24).

b. The test is UMP by the Karlin-Rubin Theorem.

c. $P(\bar{X} > \theta_0 + t_{n-1,\alpha}S/\sqrt{n}|\theta = \theta_0) = P(T_{n-1} > t_{n-1,\alpha}) = \alpha$, when T_{n-1} is a Student's t random variable with $n-1$ degrees of freedom. If we define $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$ and $\hat{\sigma}_0^2 = \frac{1}{n}\sum(x_i-\theta_0)^2$, then for $\bar{x} \geq \theta_0$ the LRT statistic is $\lambda = (\hat{\sigma}^2/\hat{\sigma}_0^2)^{n/2}$, and for $\bar{x} < \theta_0$ the LRT statistic is $\lambda = 1$. Writing $\hat{\sigma}^2 = \frac{n-1}{n}s^2$ and $\hat{\sigma}_0^2 = (\bar{x}-\theta_0)^2 + \frac{n-1}{n}s^2$, it is clear that the LRT is equivalent to the t-test because $\lambda < c$ when

$$\frac{\frac{n-1}{n}s^2}{(\bar{x}-\theta_0)^2 + \frac{n-1}{n}s^2} = \frac{(n-1)/n}{(\bar{x}-\theta_0)^2/s^2 + (n-1)/n} < c' \quad \text{and} \quad \bar{x} \geq \theta_0,$$

which is the same as rejecting when $(\bar{x}-\theta_0)/(s/\sqrt{n})$ is large.

d. The proof that the one-sided t test is UMP unbiased is rather involved, using the bounded completeness of the normal distribution and other facts. See Chapter 5 of Lehmann (1986) for a complete treatment.

8.38 a.

$$\begin{aligned}
\text{Size} &= P_{\theta_0}\left\{|\ \bar{X}-\theta_0\ | > t_{n-1,\alpha/2}\sqrt{S^2/n}\right\} \\
&= 1 - P_{\theta_0}\left\{-t_{n-1,\alpha/2}\sqrt{S^2/n} \leq \bar{X}-\theta_0 \leq t_{n-1,\alpha/2}\sqrt{S^2/n}\right\} \\
&= 1 - P_{\theta_0}\left\{-t_{n-1,\alpha/2} \leq \frac{\bar{X}-\theta_0}{\sqrt{S^2/n}} \leq t_{n-1,\alpha/2}\right\} \qquad \left(\frac{\bar{X}-\theta_0}{\sqrt{S^2/n}} \sim t_{n-1} \text{ under } H_0\right) \\
&= 1 - (1-\alpha) \quad = \quad \alpha.
\end{aligned}$$

b. The unrestricted MLEs are $\hat{\theta} = \bar{X}$ and $\hat{\sigma}^2 = \sum_i(X_i - \bar{X})^2/n$. The restricted MLEs are $\hat{\theta}_0 = \theta_0$ and $\hat{\sigma}_0^2 = \sum_i(X_i-\theta_0)^2/n$. So the LRT statistic is

$$\begin{aligned}
\lambda(\mathbf{x}) &= \frac{(2\pi\hat{\sigma}_0)^{-n/2}\exp\{-n\hat{\sigma}_0^2/(2\hat{\sigma}_0^2)\}}{(2\pi\hat{\sigma})^{-n/2}\exp\{-n\hat{\sigma}^2/(2\hat{\sigma}^2)\}} \\
&= \left[\frac{\sum_i(x_i-\bar{x})^2}{\sum_i(x_i-\theta_0)^2}\right]^{n/2} = \left[\frac{\sum_i(x_i-\bar{x})^2}{\sum_i(x_i-\bar{x})^2 + n(\bar{x}-\theta_0)^2}\right]^{n/2}.
\end{aligned}$$

For a constant c, the LRT is

$$\text{reject } H_0 \text{ if } \left[\frac{\sum_i(x_i-\bar{x})^2}{\sum_i(x_i-\bar{x})^2 + n(\bar{x}-\theta_0)^2}\right] = \frac{1}{1 + n(\bar{x}-\theta_0)^2/\sum_i(x_i-\bar{x})^2} < c^{2/n}.$$

After some algebra we can write the test as

$$\text{reject } H_0 \text{ if } |\bar{x}-\theta_0| > \left[\left(c^{-2/n}-1\right)(n-1)\frac{s^2}{n}\right]^{1/2}.$$

We now choose the constant c to achieve size α, and we

$$\text{reject if } |\bar{x}-\theta_0| > t_{n-1,\alpha/2}\sqrt{s^2/n}.$$

c. Again, see Chapter 5 of Lehmann (1986).

8.39 a. From Exercise 4.45c, $W_i = X_i - Y_i \sim \mathrm{n}(\mu_W, \sigma_W^2)$, where $\mu_X - \mu_Y = \mu_W$ and $\sigma_X^2 + \sigma_Y^2 - \rho\sigma_X\sigma_Y = \sigma_W^2$. The W_is are independent because the pairs (X_i, Y_i) are.

b. The hypotheses are equivalent to $H_0: \mu_W = 0$ vs $H_1: \mu_W \neq 0$, and, from Exercise 8.38, if we reject H_0 when $|\bar{W}| > t_{n-1,\alpha/2}\sqrt{S_W^2/n}$, this is the LRT (based on $W_1, \ldots, W_n$) of size α. (Note that if $\rho > 0$, $\operatorname{Var} W_i$ can be small and the test will have good power.)

8.41 a.

$$\lambda(\mathbf{x},\mathbf{y}) = \frac{\sup_{H_0} L(\mu_X, \mu_Y, \sigma^2 \mid \mathbf{x},\mathbf{y})}{\sup L(\mu_X, \mu_Y, \sigma^2 \mid \mathbf{x},\mathbf{y})} = \frac{L(\hat{\mu}, \hat{\sigma}_0^2 \mid \mathbf{x},\mathbf{y})}{L(\hat{\mu}_X, \hat{\mu}_Y, \hat{\sigma}_1^2 \mid \mathbf{x},\mathbf{y})}.$$

Under H_0, the X_is and Y_is are one sample of size $m+n$ from a $\mathrm{n}(\mu, \sigma^2)$ population, where $\mu = \mu_X = \mu_Y$. So the restricted MLEs are

$$\hat{\mu} = \frac{\sum_i X_i + \sum_i Y_i}{n+m} = \frac{n\bar{x}+n\bar{y}}{n+m} \quad \text{and} \quad \hat{\sigma}_0^2 = \frac{\sum_i (X_i - \hat{\mu})^2 + \sum_i (Y_i - \hat{\mu})^2}{n+m}.$$

To obtain the unrestricted MLEs, $\hat{\mu}_x$, $\hat{\mu}_y$, $\hat{\sigma}^2$, use

$$L(\mu_X, \mu_Y, \sigma^2 | x, y) = (2\pi\sigma^2)^{-(n+m)/2} e^{-[\Sigma_i (x_i - \mu_X)^2 + \Sigma_i (y_i - \mu_Y)^2]/2\sigma^2}.$$

Firstly, note that $\hat{\mu}_X = \bar{x}$ and $\hat{\mu}_Y = \bar{y}$, because maximizing over μ_X does not involve μ_Y and vice versa. Then

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n+m}{2}\frac{1}{\sigma^2} + \frac{1}{2}\left[\sum_i (x_i - \hat{\mu}_X)^2 + \sum_i (y_i - \hat{\mu}_Y)^2\right]\frac{1}{(\sigma^2)^2} \overset{\text{set}}{=} 0$$

implies

$$\hat{\sigma}^2 = \left[\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^m (y_i - \bar{y})^2\right]\frac{1}{n+m}.$$

To check that this is a maximum,

$$\begin{aligned}
\left.\frac{\partial^2 \log L}{\partial (\sigma^2)^2}\right|_{\hat{\sigma}^2} &= \left.\frac{n+m}{2}\frac{1}{(\sigma^2)^2} - \left[\sum_i (x_i - \hat{\mu}_X)^2 + \sum_i (y_i - \hat{\mu}_Y)^2\right]\frac{1}{(\sigma^2)^3}\right|_{\hat{\sigma}^2} \\
&= \frac{n+m}{2}\frac{1}{(\hat{\sigma}^2)^2} - (n+m)\frac{1}{(\hat{\sigma}^2)^2} = -\frac{n+m}{2}\frac{1}{(\hat{\sigma}^2)^2} < 0.
\end{aligned}$$

Thus, it is a maximum. We then have

$$\lambda(\mathbf{x},\mathbf{y}) = \frac{(2\pi\hat{\sigma}_0^2)^{-\frac{n+m}{2}} \exp\left\{-\frac{1}{2\hat{\sigma}_0^2}\left[\sum_{i=1}^n (x_i - \hat{\mu})^2 + \sum_{i=1}^m (y_i - \hat{\mu})^2\right]\right\}}{(2\pi\hat{\sigma}^2)^{-\frac{n+m}{2}} \exp\left\{-\frac{1}{2\hat{\sigma}^2}\left[\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^m (y_i - \bar{y})^2\right]\right\}} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{-\frac{n+m}{2}}$$

and the LRT is rejects H_0 if $\hat{\sigma}_0^2/\hat{\sigma}^2 > k$. In the numerator, first substitute $\hat{\mu} = (n\bar{x} + m\bar{y})/(n+m)$ and write

$$\sum_{i=1}^n \left(x_i - \frac{n\bar{x}+m\bar{y}}{n+m}\right)^2 = \sum_{i=1}^n \left((x_i - \bar{x}) + \left(\bar{x} - \frac{n\bar{x}+m\bar{y}}{n+m}\right)\right)^2 = \sum_{i=1}^n (x_i - \bar{x})^2 + \frac{nm^2}{(n+m)^2}(\bar{x}-\bar{y})^2,$$

because the cross term is zero. Performing a similar operation on the Y sum yields

$$\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} = \frac{\sum (x_i - \bar{x})^2 + \sum (y_i - \bar{y})^2 + \frac{nm}{n+m}(\bar{x}-\bar{y})^2}{\hat{\sigma}^2} = n + m + \frac{nm}{n+m}\frac{(\bar{x}-\bar{y})^2}{\hat{\sigma}^2}.$$

Because $\hat{\sigma}^2 = \frac{n+m-2}{n+m}S_p^2$, large values of $\hat{\sigma}_0^2 \big/ \hat{\sigma}^2$ are equivalent to large values of $(\bar{x}-\bar{y})^2 \big/ S_p^2$ and large values of $|T|$. Hence, the LRT is the two-sample t-test.

b.

$$T = \frac{\bar{X}-\bar{Y}}{\sqrt{S_p^2(1/n+1/m)}} = \frac{(\bar{X}-\bar{Y}) \Big/ \sqrt{\sigma^2(1/n+1/m)}}{\sqrt{[(n+m-2)S_p^2/\sigma^2]/(n+m-2)}}.$$

Under H_0, $(\bar{X}-\bar{Y}) \sim \mathrm{n}(0, \sigma^2(1/n+1/m))$. Under the model, $(n-1)S_X^2/\sigma^2$ and $(m-1)S_Y^2/\sigma^2$ are independent χ^2 random variables with $(n-1)$ and $(m-1)$ degrees of freedom. Thus, $(n+m-2)S_p^2/\sigma^2 = (n-1)S_X^2/\sigma^2 + (m-1)S_Y^2/\sigma^2 \sim \chi^2_{n+m-2}$. Furthermore, $\bar{X}-\bar{Y}$ is independent of S_X^2 and S_Y^2, and, hence, S_p^2. So $T \sim t_{n+m-2}$.

c. The two-sample t test is UMP unbiased, but the proof is rather involved. See Chapter 5 of Lehmann (1986).

d. For these data we have $n = 14$, $\bar{X} = 1249.86$, $S_X^2 = 591.36$, $m = 9$, $\bar{Y} = 1261.33$, $S_Y^2 = 176.00$ and $S_p^2 = 433.13$. Therefore, $T = -1.29$ and comparing this to a t_{21} distribution gives a p-value of .21. So there is no evidence that the mean age differs between the core and periphery.

8.42 a. The Satterthwaite approximation states that if $Y_i \sim \chi^2_{r_i}$, where the Y_i's are independent, then

$$\sum_i a_i Y_i \overset{\text{approx}}{\sim} \frac{\chi^2_{\hat{\nu}}}{\hat{\nu}} \quad \text{where} \quad \hat{\nu} = \frac{\left(\sum_i a_i Y_i\right)^2}{\sum_i a_i^2 Y_i^2 / r_i}.$$

We have $Y_1 = (n-1)S_X^2/\sigma_X^2 \sim \chi^2_{n-1}$ and $Y_2 = (m-1)S_Y^2/\sigma_Y^2 \sim \chi^2_{m-1}$. Now define

$$a_1 = \frac{\sigma_X^2}{n(n-1)\left[(\sigma_X^2/n)+(\sigma_Y^2/m)\right]} \quad \text{and} \quad a_2 = \frac{\sigma_Y^2}{m(m-1)\left[(\sigma_X^2/n)+(\sigma_Y^2/m)\right]}.$$

Then,

$$\begin{aligned} \sum a_i Y_i &= \frac{\sigma_X^2}{n(n-1)\left[(\sigma_X^2/n)+(\sigma_Y^2/m)\right]}\frac{(n-1)S_X^2}{\sigma_X^2} \\ &\quad + \frac{\sigma_Y^2}{m(m-1)\left[(\sigma_X^2/n)+(\sigma_Y^2/m)\right]}\frac{(m-1)S_Y^2}{\sigma_Y^2} \\ &= \frac{S_X^2/n + S_Y^2/m}{\sigma_X^2/n+\sigma_Y^2/m} \sim \frac{\chi^2_{\hat{\nu}}}{\hat{\nu}} \end{aligned}$$

where

$$\hat{\nu} = \frac{\left(\frac{S_X^2/n+S_Y^2/m}{\sigma_X^2/n+\sigma_Y^2/m}\right)^2}{\frac{1}{(n-1)}\frac{S_X^4}{n^2\left(\sigma_X^2/n+\sigma_Y^2/m\right)^2} + \frac{1}{(m-1)}\frac{S_Y^4}{m^2\left(\sigma_X^2/n+\sigma_Y^2/m\right)^2}} = \frac{\left(S_X^2/n+S_Y^2/m\right)^2}{\frac{S_X^4}{n^2(n-1)}+\frac{S_Y^4}{m^2(m-1)}}.$$

b. Because $\bar{X}-\bar{Y} \sim \mathrm{n}\left(\mu_X-\mu_Y, \sigma_X^2/n+\sigma_Y^2/m\right)$ and $\frac{S_X^2/n+S_Y^2/m}{\sigma_X^2/n+\sigma_Y^2/m} \overset{\text{approx}}{\sim} \chi^2_{\hat{\nu}}/\hat{\nu}$, under $H_0: \mu_X - \mu_Y = 0$ we have

$$T' = \frac{\bar{X}-\bar{Y}}{\sqrt{S_X^2/n+S_Y^2/m}} = \frac{(\bar{X}-\bar{Y}) \Big/ \sqrt{\sigma_X^2/n+\sigma_Y^2/m}}{\sqrt{\frac{\left(S_X^2/n+S_Y^2/m\right)}{\left(\sigma_X^2/n+\sigma_Y^2/m\right)}}} \overset{\text{approx}}{\sim} t_{\hat{\nu}}.$$

c. Using the values in Exercise 8.41d, we obtain $T' = -1.46$ and $\hat{\nu} = 20.64$. So the p-value is .16. There is no evidence that the mean age differs between the core and periphery.

d. $F = S_X^2/S_Y^2 = 3.36$. Comparing this with an $F_{13,8}$ distribution yields a p-value of $2P(F \geq 3.36) = .09$. So there is some slight evidence that the variance differs between the core and periphery.

8.43 There were typos in early printings. The t statistic should be

$$\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{1}{n_1}+\frac{\rho^2}{n_2}}\sqrt{\frac{(n_1-1)s_X^2+(n_2-1)s_Y^2/\rho^2}{n_1+n_2-2}}},$$

and the F statistic should be $s_Y^2/(\rho^2 s_X^2)$. Multiply and divide the denominator of the t statistic by σ to express it as

$$\frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma^2}{n_1}+\frac{\rho^2\sigma^2}{n_2}}}$$

divided by

$$\sqrt{\frac{(n_1-1)s_X^2/\sigma^2+(n_2-1)s_Y^2/(\rho^2\sigma^2)}{n_1+n_2-2}}.$$

The numerator has a $\mathrm{n}(0,1)$ distribution. In the denominator, $(n_1-1)s_X^2/\sigma^2 \sim \chi^2_{n_1-1}$ and $(n_2-1)s_Y^2/(\rho^2\sigma^2) \sim \chi^2_{n_2-1}$ and they are independent, so their sum has a $\chi^2_{n_1+n_2-2}$ distribution. Thus, the statistic has the form of $\mathrm{n}(0,1)/\sqrt{\chi^2_\nu/\nu}$ where $\nu = n_1+n_2-2$, and the numerator and denominator are independent because of the independence of sample means and variances in normal sampling. Thus the statistic has a $t_{n_1+n_2-2}$ distribution. The F statistic can be written as

$$\frac{s_Y^2}{\rho^2 s_X^2} = \frac{s_Y^2/(\rho^2\sigma^2)}{s_X^2/\sigma^2} = \frac{[(n_2-1)s_Y^2/(\rho^2\sigma^2)]/(n_2-1)}{[(n_1-1)s_X^2/(\sigma^2)]/(n_1-1)}$$

which has the form $[\chi^2_{n_2-1}/(n_2-1)]/[\chi^2_{n_1-1}/(n_1-1)]$ which has an F_{n_2-1,n_1-1} distribution. (Note, early printings had a typo with the numerator and denominator degrees of freedom switched.)

8.44 Test 3 rejects $H_0\colon \theta = \theta_0$ in favor of $H_1\colon \theta \neq \theta_0$ if $\bar{X} > \theta_0 + z_{\alpha/2}\sigma/\sqrt{n}$ or $\bar{X} < \theta_0 - z_{\alpha/2}\sigma/\sqrt{n}$. Let Φ and ϕ denote the standard normal cdf and pdf, respectively. Because $\bar{X} \sim \mathrm{n}(\theta, \sigma^2/n)$, the power function of Test 3 is

$$\begin{aligned}\beta(\theta) &= P_\theta(\bar{X} < \theta_0 - z_{\alpha/2}\sigma/\sqrt{n}) + P_\theta(\bar{X} > \theta_0 + z_{\alpha/2}\sigma/\sqrt{n}) \\ &= \Phi\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) + 1 - \Phi\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}} + z_{\alpha/2}\right),\end{aligned}$$

and its derivative is

$$\frac{d\beta(\theta)}{d\theta} = -\frac{\sqrt{n}}{\sigma}\phi\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) + \frac{\sqrt{n}}{\sigma}\phi\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}} + z_{\alpha/2}\right).$$

Because ϕ is symmetric and unimodal about zero, this derivative will be zero only if

$$-\left(\frac{\theta_0-\theta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) = \frac{\theta_0-\theta}{\sigma/\sqrt{n}} + z_{\alpha/2},$$

that is, only if $\theta = \theta_0$. So, $\theta = \theta_0$ is the only possible local maximum or minimum of the power function. $\beta(\theta_0) = \alpha$ and $\lim_{\theta\to\pm\infty}\beta(\theta) = 1$. Thus, $\theta = \theta_0$ is the global minimum of $\beta(\theta)$, and, for any $\theta' \neq \theta_0$, $\beta(\theta') > \beta(\theta_0)$. That is, Test 3 is unbiased.

8.45 The verification of size α is the same computation as in Exercise 8.37a. Example 8.3.3 shows that the power function $\beta_m(\theta)$ for each of these tests is an increasing function. So for $\theta > \theta_0$, $\beta_m(\theta) > \beta_m(\theta_0) = \alpha$. Hence, the tests are all unbiased.

8.47 a. This is very similar to the argument for Exercise 8.41.

b. By an argument similar to part (a), this LRT rejects H_0^+ if

$$T^+ = \frac{\bar{X} - \bar{Y} - \delta}{\sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}} \le -t_{n+m-2,\alpha}.$$

c. Because H_0 is the union of H_0^+ and H_0^-, by the IUT method of Theorem 8.3.23 the test that rejects H_0 if the tests in parts (a) and (b) both reject is a level α test of H_0. That is, the test rejects H_0 if $T^+ \le -t_{n+m-2,\alpha}$ and $T^- \ge t_{n+m-2,\alpha}$.

d. Use Theorem 8.3.24. Consider parameter points with $\mu_X - \mu_Y = \delta$ and $\sigma \to 0$. For any σ, $P(T^+ \le -t_{n+m-2,\alpha}) = \alpha$. The power of the T^- test is computed from the noncentral t distribution with noncentrality parameter $|\mu_x - \mu_Y - (-\delta)|/[\sigma(1/n + 1/m)] = 2\delta/[\sigma(1/n + 1/m)]$ which converges to ∞ as $\sigma \to 0$. Thus, $P(T^- \ge t_{n+m-2,\alpha}) \to 1$ as $\sigma \to 0$. By Theorem 8.3.24, this IUT is a size α test of H_0.

8.49 a. The p-value is

$$\begin{aligned} & P\left\{\left(\begin{matrix}\text{7 or more successes}\\ \text{out of 10 Bernoulli trials}\end{matrix}\right)\middle|\,\theta=\frac{1}{2}\right\} \\ = & \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 + \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + \binom{10}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1 + \binom{10}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0 \\ = & .171875. \end{aligned}$$

b.

$$\begin{aligned} \text{P-value} &= P\{X \ge 3 \mid \lambda = 1\} = 1 - P(X < 3 \mid \lambda = 1) \\ &= 1 - \left[\frac{e^{-1}1^2}{2!} + \frac{e^{-1}1^1}{1!} + \frac{e^{-1}1^0}{0!}\right] \quad \approx \quad .0803. \end{aligned}$$

c.

$$\begin{aligned} \text{P-value} &= P\{\sum_i X_i \ge 9 \mid 3\lambda = 3\} \quad = \quad 1 - P(Y < 9 \mid 3\lambda = 3) \\ &= 1 - e^{-3}\left[\frac{3^8}{8!} + \frac{3^7}{7!} + \frac{3^6}{6!} + \frac{3^5}{5!} + \cdots + \frac{3^1}{1!} + \frac{3^0}{0!}\right] \quad \approx \quad .0038, \end{aligned}$$

where $Y = \sum_{i=1}^3 X_i \sim \text{Poisson}(3\lambda)$.

8.50 From Exercise 7.26,

$$\pi(\theta|\mathbf{x}) = \sqrt{\frac{n}{2\pi\sigma^2}} e^{-n(\theta - \delta_\pm(\mathbf{x}))^2/(2\sigma^2)},$$

where $\delta_\pm(\mathbf{x}) = \bar{x} \pm \frac{\sigma^2}{na}$ and we use the "+" if $\theta > 0$ and the "−" if $\theta < 0$.

a. For $K > 0$,

$$P(\theta > K|\mathbf{x}, a) = \sqrt{\frac{n}{2\pi\sigma^2}} \int_K^\infty e^{-n(\theta - \delta_+(\mathbf{x}))^2/(2\sigma^2)}\, d\theta = P\left(Z > \frac{\sqrt{n}}{\sigma}[K - \delta_+(\mathbf{x})]\right),$$

where $Z \sim \mathrm{n}(0, 1)$.

b. As $a \to \infty$, $\delta_+(\mathbf{x}) \to \bar{x}$ so $P(\theta > K) \to P\left(Z > \frac{\sqrt{n}}{\sigma}(K-\bar{x})\right)$.

c. For $K = 0$, the answer in part (b) is $1 -$ (p-value) for $H_0\colon \theta \le 0$.

8.51 If $\alpha < p(\mathbf{x})$,

$$\sup_{\theta\in\Theta_0} P(W(\mathbf{X}) \ge c_\alpha) = \alpha < p(\mathbf{x}) = \sup_{\theta\in\Theta_0} P(W(\mathbf{X}) \ge W(\mathbf{x})).$$

Thus $W(\mathbf{x}) < c_\alpha$ and we could not reject H_0 at level α having observed $\mathbf{x}$. On the other hand, if $\alpha \ge p(\mathbf{x})$,

$$\sup_{\theta\in\Theta_0} P(W(\mathbf{X}) \ge c_\alpha) = \alpha \ge p(\mathbf{x}) = \sup_{\theta\in\Theta_0} P(W(\mathbf{X}) \ge W(\mathbf{x})).$$

Either $W(\mathbf{x}) \ge c_\alpha$ in which case we could reject H_0 at level α having observed $\mathbf{x}$ or $W(\mathbf{x}) < c_\alpha$. But, in the latter case we could use $c'_\alpha = W(\mathbf{x})$ and have $\{\mathbf{x}' : W(\mathbf{x}') \ge c'_\alpha\}$ define a size α rejection region. Then we could reject H_0 at level α having observed $\mathbf{x}$.

8.53 a.

$$P(-\infty < \theta < \infty) = \frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{2\pi\tau^2}}\int_{-\infty}^{\infty} e^{-\theta^2/(2\tau^2)}\, d\theta = \frac{1}{2} + \frac{1}{2} = 1.$$

b. First calculate the posterior density. Because

$$f(\bar{x}|\theta) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-n(\bar{x}-\theta)^2/(2\sigma^2)},$$

we can calculate the marginal density as

$$\begin{aligned} m_\pi(\bar{x}) &= \frac{1}{2} f(\bar{x}|0) + \frac{1}{2}\int_{-\infty}^{\infty} f(\bar{x}|\theta)\frac{1}{\sqrt{2\pi}\tau} e^{-\theta^2/(2\tau^2)} d\theta \\ &= \frac{1}{2}\frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-n\bar{x}^2/(2\sigma^2)} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{(\sigma^2/n)+\tau^2}} e^{-\bar{x}^2/[2((\sigma^2/n)+\tau^2)]} \end{aligned}$$

(see Exercise 7.22). Then $P(\theta = 0|\bar{x}) = \frac{1}{2} f(\bar{x}|0)/m_\pi(\bar{x})$.

c.

$$\begin{aligned} P\left(|\bar{X}| > \bar{x}\middle|\, \theta = 0\right) &= 1 - P\left(|\bar{X}| \le \bar{x}\middle|\, \theta = 0\right) \\ &= 1 - P\left(-\bar{x} \le \bar{X} \le \bar{x}\middle|\, \theta = 0\right) = 2\left[1 - \Phi\left(\bar{x}/(\sigma/\sqrt{n})\right)\right], \end{aligned}$$

where Φ is the standard normal cdf.

d. For $\sigma^2 = \tau^2 = 1$ and $n = 9$ we have a p-value of $2(1 - \Phi(3\bar{x}))$ and

$$P(\theta = 0|\,\bar{x}) = \left(1 + \sqrt{\frac{1}{10}} e^{81\bar{x}^2/20}\right)^{-1}.$$

The p-value of $\bar{x}$ is usually smaller than the Bayes posterior probability except when $\bar{x}$ is very close to the θ value specified by H_0. The following table illustrates this.

Some p-values and posterior probabilities ($n = 9$)

	$\bar{x}$								
	0	$\pm.1$	$\pm.15$	$\pm.2$	$\pm.5$	$\pm.6533$	$\pm.7$	±1	±2
p-value of $\bar{x}$	1	.7642	.6528	.5486	.1336	.05	.0358	.0026	≈ 0
posterior $P(\theta = 0\vert\bar{x})$	.7597	.7523	.7427	.7290	.5347	.3595	.3030	.0522	≈ 0

8.54 a. From Exercise 7.22, the posterior distribution of $\theta|\mathbf{x}$ is normal with mean $[\tau^2/(\tau^2+\sigma^2/n)]\bar{x}$ and variance $\tau^2/(1+n\tau^2/\sigma^2)$. So

$$\begin{aligned} P(\theta \le 0|\mathbf{x}) &= P\left(Z \le \frac{0-[\tau^2/(\tau^2+\sigma^2/n)]\bar{x}}{\sqrt{\tau^2/(1+n\tau^2/\sigma^2)}}\right) \\ &= P\left(Z \le -\frac{\tau}{\sqrt{(\sigma^2/n)(\tau^2+\sigma^2/n)}}\bar{x}\right) \;=\; P\left(Z \ge \frac{\tau}{\sqrt{(\sigma^2/n)(\tau^2+\sigma^2/n)}}\bar{x}\right). \end{aligned}$$

b. Using the fact that if $\theta = 0$, $\bar{X} \sim \mathrm{n}(0, \sigma^2/n)$, the p-value is

$$P(\bar{X} \ge \bar{x}) = P\left(Z \ge \frac{\bar{x}-0}{\sigma/\sqrt{n}}\right) = P\left(Z \ge \frac{1}{\sigma/\sqrt{n}}\bar{x}\right)$$

c. For $\sigma^2 = \tau^2 = 1$,

$$P(\theta \le 0|x) = P\left(Z \ge \frac{1}{\sqrt{(1/n)(1+1/n)}}\bar{x}\right) \quad \text{and} \quad P(\bar{X} \ge \bar{x}) = P\left(Z \ge \frac{1}{\sqrt{1/n}}\bar{x}\right).$$

Because

$$\frac{1}{\sqrt{(1/n)(1+1/n)}} < \frac{1}{\sqrt{1/n}},$$

the Bayes probability is larger than the p-value if $\bar{x} \ge 0$. (Note: The inequality is in the opposite direction for $\bar{x} < 0$, but the primary interest would be in large values of $\bar{x}$.)

d. As $\tau^2 \to \infty$, the constant in the Bayes probability,

$$\frac{\tau}{\sqrt{(\sigma^2/n)(\tau^2+\sigma^2/n)}} = \frac{1}{\sqrt{(\sigma^2/n)(1+\sigma^2/(\tau^2 n))}} \to \frac{1}{\sigma/\sqrt{n}},$$

the constant in the p-value. So the indicated equality is true.

8.55 The formulas for the risk functions are obtained from (8.3.14) using the power function $\beta(\theta) = \Phi(-z_\alpha + \theta_0 - \theta)$, where Φ is the standard normal cdf.

8.57 For 0–1 loss by (8.3.12) the risk function for any test is the power function $\beta(\mu)$ for $\mu \le 0$ and $1-\beta(\mu)$ for $\mu > 0$. Let $\alpha = P(1 < Z < 2)$, the size of test δ. By the Karlin-Rubin Theorem, the test δ_{z_α} that rejects if $X > z_\alpha$ is also size α and is uniformly more powerful than δ, that is, $\beta_{\delta_{z_\alpha}}(\mu) > \beta_\delta(\mu)$ for all $\mu > 0$. Hence,

$$R(\mu, \delta_{z_\alpha}) = 1 - \beta_{\delta_{z_\alpha}}(\mu) < 1 - \beta_\delta(\mu) = R(\mu, \delta), \quad \text{for all } \mu > 0.$$

Now reverse the roles of H_0 and H_1 and consider testing $H_0^*\colon \mu > 0$ versus $H_1^*\colon \mu \le 0$. Consider the test δ^* that rejects H_0^* if $X \le 1$ or $X \ge 2$, and the test $\delta^*_{z_\alpha}$ that rejects H_0^* if $X \le z_\alpha$. It is easily verified that for 0–1 loss δ and δ^* have the same risk functions, and $\delta^*_{z_\alpha}$ and δ_{z_α} have the same risk functions. Furthermore, using the Karlin-Rubin Theorem as before, we can conclude that $\delta^*_{z_\alpha}$ is uniformly more powerful than δ^*. Thus we have

$$R(\mu, \delta) = R(\mu, \delta^*) \ge R(\mu, \delta^*_{z_\alpha}) = R(\mu, \delta_{z_\alpha}), \quad \text{for all } \mu \le 0,$$

with strict inequality if $\mu < 0$. Thus, δ_{z_α} is better than δ.