Chapter 9

Interval Estimation

9.1 Denote $A = \{x\colon L(x) \le \theta\}$ and $B = \{x\colon U(x) \ge \theta\}$. Then $A \cap B = \{x\colon L(x) \le \theta \le U(x)\}$ and $1 \ge P\{A \cup B\} = P\{L(X) \le \theta \text{ or } \theta \le U(X)\} \ge P\{L(X) \le \theta \text{ or } \theta \le L(X)\} = 1$, since $L(x) \le U(x)$. Therefore, $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1 - \alpha_1 + 1 - \alpha_2 - 1 = 1 - \alpha_1 - \alpha_2$.

9.3 a. The MLE of β is $X_{(n)} = \max X_i$. Since β is a scale parameter, $X_{(n)}/\beta$ is a pivot, and

$$.05 = P_\beta(X_{(n)}/\beta \le c) = P_\beta(\text{all } X_i \le c\beta) = \left(\frac{c\beta}{\beta}\right)^{\alpha_0 n} = c^{\alpha_0 n}$$

implies $c = .05^{1/\alpha_0 n}$. Thus, $.95 = P_\beta(X_{(n)}/\beta > c) = P_\beta(X_{(n)}/c > \beta)$, and $\{\beta : \beta < X_{(n)}/(.05^{1/\alpha_0 n})\}$ is a 95% upper confidence limit for β.

b. From 7.10, $\hat\alpha = 12.59$ and $X_{(n)} = 25$. So the confidence interval is $(0, 25/[.05^{1/(12.59\cdot 14)}]) = (0, 25.43)$.

9.4 a.

$$\lambda(x,y) = \frac{\sup_{\lambda=\lambda_0} L\left(\sigma_X^2, \sigma_Y^2 \,\middle|\, x, y\right)}{\sup_{\lambda\in(0,+\infty)} L\left(\sigma_X^2, \sigma_Y^2 \,\middle|\, x, y\right)}$$

The unrestricted MLEs of σ_X^2 and σ_Y^2 are $\hat\sigma_X^2 = \frac{\Sigma X_i^2}{n}$ and $\hat\sigma_Y^2 = \frac{\Sigma Y_i^2}{m}$, as usual. Under the restriction, $\lambda = \lambda_0$, $\sigma_Y^2 = \lambda_0\sigma_X^2$, and

$$\begin{aligned} L\left(\sigma_X^2, \lambda_0\sigma_X^2 \,\middle|\, x, y\right) &= \left(2\pi\sigma_X^2\right)^{-n/2}\left(2\pi\lambda_0\sigma_X^2\right)^{-m/2} e^{-\Sigma x_i^2/(2\sigma_X^2)} \cdot e^{-\Sigma y_i^2/(2\lambda_0\sigma_X^2)} \\ &= \left(2\pi\sigma_X^2\right)^{-(m+n)/2} \lambda_0^{-m/2} e^{-(\lambda_0\Sigma x_i^2 + \Sigma y_i^2)/(2\lambda_0\sigma_X^2)} \end{aligned}$$

Differentiating the log likelihood gives

$$\begin{aligned} \frac{d \log L}{d\left(\sigma_X^2\right)^2} &= \frac{d}{d\sigma_X^2}\left[-\frac{m+n}{2}\log\sigma_X^2 - \frac{m+n}{2}\log(2\pi) - \frac{m}{2}\log\lambda_0 - \frac{\lambda_0\Sigma x_i^2 + \Sigma y_i^2}{2\lambda_0\sigma_X^2}\right] \\ &= -\frac{m+n}{2}\left(\sigma_X^2\right)^{-1} + \frac{\lambda_0\Sigma x_i^2 + \Sigma y_i^2}{2\lambda_0}\left(\sigma_X^2\right)^{-2} \quad \stackrel{\text{set}}{=} \quad 0 \end{aligned}$$

which implies

$$\hat\sigma_0^2 = \frac{\lambda_0\Sigma x_i^2 + \Sigma y_i^2}{\lambda_0(m+n)}.$$

To see this is a maximum, check the second derivative:

$$\begin{aligned} \frac{d^2 \log L}{d\left(\sigma_X^2\right)^2} &= \frac{m+n}{2}\left(\sigma_X^2\right)^{-2} - \frac{1}{\lambda_0}\left(\lambda_0\Sigma x_i^2 + \Sigma y_i^2\right)\left(\sigma_X^2\right)^{-3}\Big|_{\sigma_X^2 = \hat\sigma_0^2} \\ &= -\frac{m+n}{2}(\hat\sigma_0^2)^{-2} \quad < \quad 0, \end{aligned}$$

therefore $\hat\sigma_0^2$ is the MLE. The LRT statistic is

$$\frac{\left(\hat\sigma_X^2\right)^{n/2}\left(\hat\sigma_Y^2\right)^{m/2}}{\lambda_0^{m/2}\left(\hat\sigma_0^2\right)^{(m+n)/2}},$$

and the test is: Reject H_0 if $\lambda(x,y)<k$, where k is chosen to give the test size α.

b. Under H_0, $\sum Y_i^2/(\lambda_0\sigma_X^2)\sim\chi_m^2$ and $\sum X_i^2/\sigma_X^2\sim\chi_n^2$, independent. Also, we can write

$$\begin{aligned}
\lambda(X,Y) &= \left(\frac{1}{\frac{n}{m+n}+\frac{(\Sigma Y_i^2/\lambda_0\sigma_X^2)/m}{(\Sigma X_i^2/\sigma_X^2)/n}\cdot\frac{m}{m+n}}\right)^{n/2}\left(\frac{1}{\frac{m}{m+n}+\frac{(\Sigma X_i^2/\sigma_X^2)/n}{(\Sigma Y_i^2/\lambda_0\sigma_X^2)/m}\cdot\frac{n}{m+n}}\right)^{m/2}\\
&= \left[\frac{1}{\frac{n}{n+m}+\frac{m}{m+n}F}\right]^{n/2}\left[\frac{1}{\frac{m}{m+n}+\frac{n}{m+n}F^{-1}}\right]^{m/2}
\end{aligned}$$

where $F=\frac{\Sigma Y_i^2/\lambda_0 m}{\Sigma X_i^2/n}\sim F_{m,n}$ under H_0. The rejection region is

$$\left\{(x,y)\colon\frac{1}{\left[\frac{n}{n+m}+\frac{m}{m+n}F\right]^{n/2}}\cdot\frac{1}{\left[\frac{m}{m+n}+\frac{n}{m+n}F^{-1}\right]^{m/2}}<c_\alpha\right\}$$

where c_α is chosen to satisfy

$$P\left\{\left[\frac{n}{n+m}+\frac{m}{m+n}F\right]^{-n/2}\left[\frac{m}{n+m}+\frac{n}{m+n}F^{-1}\right]^{-m/2}<c_\alpha\right\}=\alpha.$$

c. To ease notation, let $a=m/(n+m)$ and $b=a\sum y_i^2/\sum x_i^2$. From the duality of hypothesis tests and confidence sets, the set

$$c(\lambda)=\left\{\lambda\colon\left(\frac{1}{a+b/\lambda}\right)^{n/2}\left(\frac{1}{(1-a)+\frac{a(1-a)}{b}\lambda}\right)^{m/2}\geq c_\alpha\right\}$$

is a $1-\alpha$ confidence set for λ. We now must establish that this set is indeed an interval. To do this, we establish that the function on the left hand side of the inequality has only an interior maximum. That is, it looks like an upside-down bowl. Furthermore, it is straightforward to establish that the function is zero at both $\lambda=0$ and $\lambda=\infty$. These facts imply that the set of λ values for which the function is greater than or equal to c_α must be an interval. We make some further simplifications. If we multiply both sides of the inequality by $[(1-a)/b]^{m/2}$, we need be concerned with only the behavior of the function

$$h(\lambda)=\left(\frac{1}{a+b/\lambda}\right)^{n/2}\left(\frac{1}{b+a\lambda}\right)^{m/2}.$$

Moreover, since we are most interested in the sign of the derivative of h, this is the same as the sign of the derivative of $\log h$, which is much easier to work with. We have

$$\begin{aligned}
\frac{d}{d\lambda}\log h(\lambda) &= \frac{d}{d\lambda}\left[-\frac{n}{2}\log(a+b/\lambda)-\frac{m}{2}\log(b+a\lambda)\right]\\
&= \frac{n}{2}\frac{b/\lambda^2}{a+b/\lambda}-\frac{m}{2}\frac{a}{b+a\lambda}\\
&= \frac{1}{2\lambda^2(a+b/\lambda)(b+a\lambda)}\left[-a^2m\lambda^2+ab(n-m)\lambda+nb^2\right].
\end{aligned}$$

The sign of the derivative is given by the expression in square brackets, a parabola. It is easy to see that for $\lambda \geq 0$, the parabola changes sign from positive to negative. Since this is the sign change of the derivative, the function must increase then decrease. Hence, the function is an upside-down bowl, and the set is an interval.

9.5 a. Analogous to Example 9.2.5, the test here will reject H_0 if $T < k(p_0)$. Thus the confidence set is $C = \{p\colon T \geq k(p)\}$. Since $k(p)$ is nondecreasing, this gives an upper bound on p.

b. $k(p)$ is the integer that simultaneously satisfies

$$\sum_{y=k(p)}^{n} \binom{n}{y} p^y (1-p)^{n-y} \geq 1-\alpha \quad \text{and} \quad \sum_{y=k(p)+1}^{n} \binom{n}{y} p^y (1-p)^{n-y} < 1-\alpha.$$

9.6 a. For $Y = \sum X_i \sim \text{binomial}(n,p)$, the LRT statistic is

$$\lambda(y) = \frac{\binom{n}{y} p_0^y (1-p_0)^{n-y}}{\binom{n}{y} \hat{p}^y (1-\hat{p})^{n-y}} = \left(\frac{p_0(1-\hat{p})}{\hat{p}(1-p_0)}\right)^y \left(\frac{1-p_0}{1-\hat{p}}\right)^n$$

where $\hat{p} = y/n$ is the MLE of p. The acceptance region is

$$A(p_0) = \left\{ y\colon \left(\frac{p_0}{\hat{p}}\right)^y \left(\frac{1-p_0}{1-\hat{p}}\right)^{n-y} \geq k^* \right\}$$

where k^* is chosen to satisfy $P_{p_0}(Y \in A(p_0)) = 1-\alpha$. Inverting the acceptance region to a confidence set, we have

$$C(y) = \left\{ p\colon \left(\frac{p}{\hat{p}}\right)^y \left(\frac{(1-p)}{1-\hat{p}}\right)^{n-y} \geq k^* \right\}.$$

b. For given n and observed y, write

$$C(y) = \left\{ p\colon (n/y)^y \left(n/(n-y)\right)^{n-y} p^y (1-p)^{n-y} \geq k^* \right\}.$$

This is clearly a highest density region. The endpoints of $C(y)$ are roots of the n^{th} degree polynomial (in p), $(n/y)^y \left(n/(n-y)\right)^{n-y} p^y (1-p)^{n-y} - k^*$. The interval of (10.4.4) is

$$\left\{ p\colon \left| \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \right| \leq z_{\alpha/2} \right\}.$$

The endpoints of this interval are the roots of the second degree polynomial (in p), $(\hat{p}-p)^2 - z_{\alpha/2}^2 p(1-p)/n$. Typically, the second degree and n^{th} degree polynomials will not have the same roots. Therefore, the two intervals are different. (Note that when $n \to \infty$ and $y \to \infty$, the density becomes symmetric (CLT). Then the two intervals are the same.)

9.7 These densities have already appeared in Exercise 8.8, where LRT statistics were calculated for testing $H_0\colon a = 1$.

a. Using the result of Exercise 8.8(a), the restricted MLE of θ (when $a = a_0$) is

$$\hat{\theta}_0 = \frac{-a_0 + \sqrt{a_0^2 + 4\sum x_i^2/n}}{2},$$

and the unrestricted MLEs are

$$\hat{\theta} = \bar{x} \quad \text{and} \quad \hat{a} = \frac{\sum (x_i - \bar{x})^2}{n\bar{x}}.$$

The LRT statistic is

$$\lambda(x)=\frac{\left(\frac{\hat{a}\hat{\theta}}{a_0\hat{\theta}_0}\right)^{n/2}e^{-\frac{1}{2a_0\hat{\theta}_0}\Sigma(x_i-\hat{\theta}_0)^2}}{e^{-\frac{1}{2\hat{a}\hat{\theta}}\Sigma(x_i-\hat{\theta})^2}}=\left(\frac{1}{2\pi a_0\hat{\theta}_0}\right)^{n/2}e^{n/2}e^{-\frac{1}{2a_0\hat{\theta}_0}\Sigma(x_i-\hat{\theta}_0)^2}$$

The rejection region of a size α test is $\{x\colon \lambda(x)\le c_\alpha\}$, and a $1-\alpha$ confidence set is $\{a_0\colon \lambda(x)\ge c_\alpha\}$.

b. Using the results of Exercise 8.8b, the restricted MLE (for $a=a_0$) is found by solving

$$-a_0\theta^2+[\hat{\sigma}^2+(\bar{x}-\theta)^2]+\theta(\bar{x}-\theta)=0,$$

yielding the MLE

$$\hat{\theta}_R=\bar{x}+\sqrt{\bar{x}+4a_0(\hat{\sigma}^2+\bar{x}^2)}/2a_0.$$

The unrestricted MLEs are

$$\hat{\theta}=\bar{x}\quad\text{and}\quad\hat{a}=\frac{1}{n\bar{x}^2}\sum_{i=1}^n(x_i-\bar{x})^2=\frac{\hat{\sigma}^2}{\bar{x}^2},$$

yielding the LRT statistic

$$\lambda(x)=\left(\hat{\sigma}/\hat{\theta}_R\right)^n e^{(n/2)-\Sigma(x_i-\hat{\theta}_R)^2/(2\hat{\theta}_R)}.$$

The rejection region of a size α test is $\{x\colon \lambda(x)\le c_\alpha\}$, and a $1-\alpha$ confidence set is $\{a_0\colon \lambda(x)\ge c_\alpha\}$.

9.9 Let $Z_1,\dots,Z_n$ be iid with pdf $f(z)$.

a. For $X_i\sim f(x-\mu)$, $(X_1,\dots,X_n)\sim(Z_1+\mu,\dots,Z_n+\mu)$, and $\bar{X}-\mu\sim\overline{Z+\mu}-\mu=\bar{Z}$. The distribution of $\bar{Z}$ does not depend on μ.

b. For $X_i\sim f(x/\sigma)/\sigma$, $(X_1,\dots,X_n)\sim(\sigma Z_1,\dots,\sigma Z_n)$, and $\bar{X}/\sigma\sim\overline{\sigma Z}/\sigma=\bar{Z}$. The distribution of $\bar{Z}$ does not depend on σ.

c. For $X_i\sim f((x-\mu)/\sigma)/\sigma$, $(X_1,\dots,X_n)\sim(\sigma Z_1+\mu,\dots,\sigma Z_n+\mu)$, and $(\bar{X}-\mu)/S_X\sim(\overline{\sigma Z+\mu}-\mu)/S_{\sigma Z+\mu}=\sigma\bar{Z}/(\sigma S_Z)=\bar{Z}/S_Z$. The distribution of $\bar{Z}/S_Z$ does not depend on μ or σ.

9.11 Recall that if θ is the true parameter, then $F_T(T|\theta)\sim\text{uniform}(0,1)$. Thus,

$$P_{\theta_0}(\{T\colon \alpha_1\le F_T(T|\theta_0)\le 1-\alpha_2\})=P(\alpha_1\le U\le 1-\alpha_2)=1-\alpha_2-\alpha_1,$$

where $U\sim\text{uniform}(0,1)$. Since

$$t\in\{t\colon \alpha_1\le F_T(t|\theta)\le 1-\alpha_2\}\quad\Leftrightarrow\quad\theta\in\{\theta:\alpha_1\le F_T(t|\theta)\le 1-\alpha_2\}$$

the same calculation shows that the interval has confidence $1-\alpha_2-\alpha_1$.

9.12 If $X_1,\dots,X_n\sim$ iid $\text{n}(\theta,\theta)$, then $\sqrt{n}(\bar{X}-\theta)/\sqrt{\theta}\sim\text{n}(0,1)$ and a $1-\alpha$ confidence interval is $\{\theta\colon |\sqrt{n}(\bar{x}-\theta)/\sqrt{\theta}|\le z_{\alpha/2}\}$. Solving for θ, we get

$$\left\{\theta\colon n\theta^2-\theta\left(2n\bar{x}+z_{\alpha/2}^2\right)+n\bar{x}^2\le 0\right\}=\left\{\theta\colon\theta\in\left(2n\bar{x}+z_{\alpha/2}^2\pm\sqrt{4n\bar{x}z_{\alpha/2}^2+z_{\alpha/2}^4}\right)/2n\right\}.$$

Simpler answers can be obtained using the t pivot, $(\bar{X}-\theta)/(S/\sqrt{n})$, or the χ^2 pivot, $(n-1)S^2/\theta^2$. (Tom Werhley of Texas A&M university notes the following: The largest probability of getting a negative discriminant (hence empty confidence interval) occurs when $\sqrt{n\theta}=\frac{1}{2}z_{\alpha/2}$, and the probability is equal to $\alpha/2$. The behavior of the intervals for negative values of $\bar{x}$ is also interesting. When $\bar{x}=0$ the lefthand endpoint is also equal to 0, but when $\bar{x}<0$, the lefthand endpoint is positive. Thus, the interval based on $\bar{x}=0$ contains smaller values of θ than that based on $\bar{x}<0$. The intervals get smaller as $\bar{x}$ decreases, finally becoming empty.)

9.13 a. For $Y = -(\log X)^{-1}$, the pdf of Y is $f_Y(y) = \frac{\theta}{y^2}e^{-\theta/y}$, $0 < y < \infty$, and

$$P(Y/2 \le \theta \le Y) = \int_\theta^{2\theta} \frac{\theta}{y^2} e^{-\theta/y} dy = e^{-\theta/y}\Big|_\theta^{2\theta} = e^{-1/2} - e^{-1} = .239.$$

b. Since $f_X(x) = \theta x^{\theta-1}$, $0 < x < 1$, $T = X^\theta$ is a good guess at a pivot, and it is since $f_T(t) = 1$, $0 < t < 1$. Thus a pivotal interval is formed from $P(a < X^\theta < b) = b - a$ and is

$$\left\{\theta \colon \frac{\log b}{\log x} \le \theta \le \frac{\log a}{\log x}\right\}.$$

Since $X^\theta \sim \text{uniform}(0,1)$, the interval will have confidence .239 as long as $b - a = .239$.

c. The interval in part a) is a special case of the one in part b). To find the best interval, we minimize $\log b - \log a$ subject to $b - a = 1 - \alpha$, or $b = 1 - \alpha + a$. Thus we want to minimize $\log(1 - \alpha + a) - \log a = \log\left(1 + \frac{1-\alpha}{a}\right)$, which is minimized by taking a as big as possible. Thus, take $b = 1$ and $a = \alpha$, and the best $1 - \alpha$ pivotal interval is $\left\{\theta \colon 0 \le \theta \le \frac{\log \alpha}{\log x}\right\}$. Thus the interval in part a) is nonoptimal. A shorter interval with confidence coefficient .239 is $\{\theta \colon 0 \le \theta \le \log(1 - .239)/\log(x)\}$.

9.14 a. Recall the Bonferroni Inequality (1.2.9), $P(A_1 \cap A_2) \ge P(A_1) + P(A_2) - 1$. Let $A_1 = P(\text{interval covers } \mu)$ and $A_2 = P(\text{interval covers } \sigma^2)$. Use the interval (9.2.14), with $t_{n-1,\alpha/4}$ to get a $1 - \alpha/2$ confidence interval for μ. Use the interval after (9.2.14) with $b = \chi^2_{n-1,\alpha/4}$ and $a = \chi^2_{n-1,1-\alpha/4}$ to get a $1-\alpha/2$ confidence interval for σ. Then the natural simultaneous set is

$$\begin{aligned} C_a(x) &= \Bigg\{(\mu, \sigma^2) \colon \bar{x} - t_{n-1,\alpha/4}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{n-1,\alpha/4}\frac{s}{\sqrt{n}} \\ &\qquad \text{and } \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/4}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/4}}\Bigg\} \end{aligned}$$

and $P\left(C_a(X) \text{ covers } (\mu, \sigma^2)\right) = P(A_1 \cap A_2) \ge P(A_1) + P(A_2) - 1 = 2(1 - \alpha/2) - 1 = 1 - \alpha$.

b. If we replace the μ interval in a) by $\left\{\mu \colon \bar{x} - \frac{k\sigma}{\sqrt{n}} \le \mu \le \bar{x} + \frac{k\sigma}{\sqrt{n}}\right\}$ then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \text{n}(0,1)$, so we use $z_{\alpha/4}$ and

$$C_b(x) = \left\{(\mu, \sigma^2) \colon \bar{x} - z_{\alpha/4}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/4}\frac{\sigma}{\sqrt{n}} \text{ and } \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/4}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/4}}\right\}$$

and $P\left(C_b(X) \text{ covers } (\mu, \sigma^2)\right) \ge 2(1 - \alpha/2) - 1 = 1 - \alpha$.

c. The sets can be compared graphically in the (μ, σ) plane: C_a is a rectangle, since μ and σ^2 are treated independently, while C_b is a trapezoid, with larger σ^2 giving a longer interval. Their areas can also be calculated

$$\begin{aligned} \text{Area of } C_a &= \left[2t_{n-1,\alpha/4}\frac{s}{\sqrt{n}}\right]\left\{\sqrt{(n-1)s^2}\left(\frac{1}{\chi^2_{n-1,1-\alpha/4}} - \frac{1}{\chi^2_{n-1,\alpha/4}}\right)\right\} \\ \text{Area of } C_b &= \left[z_{\alpha/4}\frac{s}{\sqrt{n}}\left(\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/4}}} + \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/4}}}\right)\right] \\ &\qquad \times \left\{\sqrt{(n-1)s^2}\left(\frac{1}{\chi^2_{n-1,1-\alpha/4}} - \frac{1}{\chi^2_{n-1,\alpha/4}}\right)\right\} \end{aligned}$$

and compared numerically.

9.15 Fieller's Theorem says that a $1-\alpha$ confidence set for $\theta = \mu_Y/\mu_X$ is

$$\left\{\theta\colon \left(\bar{x}^2 - \frac{t^2_{n-1,\alpha/2}}{n-1}s_X^2\right)\theta^2 - 2\left(\bar{x}\bar{y} - \frac{t^2_{n-1,\alpha/2}}{n-1}s_{YX}\right)\theta + \left(\bar{y}^2 - \frac{t^2_{n-1,\alpha/2}}{n-1}s_Y^2\right) \le 0\right\}.$$

a. Define $a = \bar{x}^2 - ts_X^2$, $b = \bar{x}\bar{y} - ts_{YX}$, $c = \bar{y}^2 - ts_Y^2$, where $t = \frac{t^2_{n-1,\alpha/2}}{n-1}$. Then the parabola opens upward if $a > 0$. Furthermore, if $a > 0$, then there always exists at least one real root. This follows from the fact that at $\theta = \bar{y}/\bar{x}$, the value of the function is negative. For $\bar{\theta} = \bar{y}/\bar{x}$ we have

$$\begin{aligned}
&\left(\bar{x}^2 - ts_X^2\right)\left(\frac{\bar{y}}{\bar{x}}\right)^2 - 2\left(\bar{x}\bar{y} - ts_{XY}\right)\left(\frac{\bar{y}}{\bar{x}}\right) + \left(\bar{y}^2 - as_Y^2\right) \\
&= -t\left[\frac{\bar{y}^2}{\bar{x}^2}s_X^2 - 2\frac{\bar{y}}{\bar{x}}s_{XY} + s_Y^2\right] \\
&= -t\left[\sum_{i=1}^n \left(\frac{\bar{y}^2}{\bar{x}^2}(x_i-\bar{x})^2 - 2\frac{\bar{y}}{\bar{x}}(x_i-\bar{x})(y_i-\bar{y}) + (y_i-\bar{y})^2\right)\right] \\
&= -t\left[\sum_{i=1}^n \left(\frac{\bar{y}}{\bar{x}}(x_i-\bar{x}) - (y_i-\bar{y})\right)^2\right]
\end{aligned}$$

which is negative.

b. The parabola opens downward if $a < 0$, that is, if $\bar{x}^2 < ts_X^2$. This will happen if the test of $H_0\colon \mu_X = 0$ accepts H_0 at level α.

c. The parabola has no real roots if $b^2 < ac$. This can only occur if $a < 0$.

9.16 a. The LRT (see Example 8.2.1) has rejection region $\{x : |\bar{x} - \theta_0| > z_{\alpha/2}\sigma/\sqrt{n}\}$, acceptance region $A(\theta_0) = \{x\colon -z_{\alpha/2}\sigma/\sqrt{n} \le \bar{x} - \theta_0 \le z_{\alpha/2}\sigma/\sqrt{n}\}$, and $1-\alpha$ confidence interval $C(\theta) = \{\theta\colon \bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \le \theta \le \bar{x} + z_{\alpha/2}\sigma/\sqrt{n}\}$.

b. We have a UMP test with rejection region $\{x : \bar{x} - \theta_0 < -z_\alpha\sigma/\sqrt{n}\}$, acceptance region $A(\theta_0) = \{x\colon \bar{x} - \theta_0 \ge -z_\alpha\sigma/\sqrt{n}\}$, and $1-\alpha$ confidence interval $C(\theta) = \{\theta\colon \bar{x} + z_\alpha\sigma/\sqrt{n} \ge \theta\}$.

c. Similar to b), the UMP test has rejection region $\{x\colon \bar{x} - \theta_0 > z_\alpha\sigma/\sqrt{n}\}$, acceptance region $A(\theta_0) = \{x\colon \bar{x} - \theta_0 \le z_\alpha\sigma/\sqrt{n}\}$, and $1-\alpha$ confidence interval $C(\theta) = \{\theta\colon \bar{x} - z_\alpha\sigma/\sqrt{n} \le \theta\}$.

9.17 a. Since $X - \theta \sim$ uniform$(-1/2, 1/2)$, $P(a \le X - \theta \le b) = b - a$. Any a and b satisfying $b = a + 1 - \alpha$ will do. One choice is $a = -\frac{1}{2} + \frac{\alpha}{2}$, $b = \frac{1}{2} - \frac{\alpha}{2}$.

b. Since $T = X/\theta$ has pdf $f(t) = 2t$, $0 \le t \le 1$,

$$P(a \le X/\theta \le b) = \int_a^b 2t\,dt = b^2 - a^2.$$

Any a and b satisfying $b^2 = a^2 + 1 - \alpha$ will do. One choice is $a = \sqrt{\alpha/2}$, $b = \sqrt{1-\alpha/2}$.

9.18 a. $P_p(X=1) = \binom{3}{1}p^1(1-p)^{3-1} = 3p(1-p)^2$, maximum at $p = 1/3$.
$P_p(X=2) = \binom{3}{2}p^2(1-p)^{3-2} = 3p^2(1-p)$, maximum at $p = 2/3$.

b. $P(X=0) = \binom{3}{0}p^0(1-p)^{3-0} = (1-p)^3$, and this is greater than $P(X=2)$ if $(1-p)^2 > 3p^2$, or $2p^2 + 2p - 1 < 0$. At $p = 1/3$, $2p^2 + 2p - 1 = -1/9$.

c. To show that this is a $1-\alpha = .442$ interval, compare with the interval in Example 9.2.11. There are only two discrepancies. For example,

$$P(p \in \text{interval} \mid .362 < p < .634) = P(X = 1 \text{ or } X = 2) > .442$$

by comparison with Sterne's procedure, which is given by

x	interval
0	[.000,.305)
1	[.305,.634)
2	[.362,.762)
3	[.695,1].

9.19 For $F_T(t|\theta)$ increasing in θ, there are unique values $\theta_U(t)$ and $\theta_L(t)$ such that $F_T(t|\theta) < 1-\frac{\alpha}{2}$ if and only if $\theta < \theta_U(t)$ and $F_T(t|\theta) > \frac{\alpha}{2}$ if and only if $\theta > \theta_L(t)$. Hence,

$$\begin{aligned} P\left(\theta_L(T) \le \theta \le \theta_U(T)\right) &= P\left(\theta \le \theta_U(T)\right) - P\left(\theta \le \theta_L(T)\right) \\ &= P\left(F_T(T) \le 1-\frac{\alpha}{2}\right) - P\left(F_T(T) \le \frac{\alpha}{2}\right) \\ &= 1-\alpha. \end{aligned}$$

9.21 To construct a $1-\alpha$ confidence interval for p of the form $\{p\colon \ell \le p \le u\}$ with $P(\ell \le p \le u) = 1-\alpha$, we use the method of Theorem 9.2.12. We must solve for ℓ and u in the equations

$$(1)\quad \frac{\alpha}{2} = \sum_{k=0}^{x} \binom{n}{k} u^k (1-u)^{n-k} \quad \text{and} \quad (2)\quad \frac{\alpha}{2} = \sum_{k=x}^{n} \binom{n}{k} \ell^k (1-\ell)^{n-k}.$$

In equation (1) $\alpha/2 = P(K \le x) = P(Y \le 1-u)$, where $Y \sim \text{beta}(n-x, x+1)$ and $K \sim \text{binomial}(n,u)$. This is Exercise 2.40. Let $Z \sim F_{2(n-x),2(x+1)}$ and $c = (n-x)/(x+1)$. By Theorem 5.3.8c, $cZ/(1+cZ) \sim \text{beta}(n-x, x+1) \sim Y$. So we want

$$\alpha/2 = P\left(\frac{cZ}{(1+cZ)} \le 1-u\right) = P\left(\frac{1}{Z} \ge \frac{cu}{1-u}\right).$$

From Theorem 5.3.8a, $1/Z \sim F_{2(x+1),2(n-x)}$. So we need $cu/(1-u) = F_{2(x+1),2(n-x),\alpha/2}$. Solving for u yields

$$u = \frac{\frac{x+1}{n-x} F_{2(x+1),2(n-x),\alpha/2}}{1 + \frac{x+1}{n-x} F_{2(x+1),2(n-x),\alpha/2}}.$$

A similar manipulation on equation (2) yields the value for ℓ.

9.23 a. The LRT statistic for $H_0\colon \lambda = \lambda_0$ versus $H_1\colon \lambda \ne \lambda_0$ is

$$g(y) = e^{-n\lambda_0}(n\lambda_0)^y / e^{-n\hat{\lambda}}(n\hat{\lambda})^y,$$

where $Y = \sum X_i \sim \text{Poisson}(n\lambda)$ and $\hat{\lambda} = y/n$. The acceptance region for this test is $A(\lambda_0) = \{y\colon g(y) > c(\lambda_0)\}$ where $c(\lambda_0)$ is chosen so that $P(Y \in A(\lambda_0)) \ge 1-\alpha$. $g(y)$ is a unimodal function of y so $A(\lambda_0)$ is an interval of y values. Consider constructing $A(\lambda_0)$ for each $\lambda_0 > 0$. Then, for a fixed y, there will be a smallest λ_0, call it $a(y)$, and a largest λ_0, call it $b(y)$, such that $y \in A(\lambda_0)$. The confidence interval for λ is then $C(y) = (a(y), b(y))$. The values $a(y)$ and $b(y)$ are not expressible in closed form. They can be determined by a numerical search, constructing $A(\lambda_0)$ for different values of λ_0 and determining those values for which $y \in A(\lambda_0)$. (Jay Beder of the University of Wisconsin, Milwaukee, reminds us that since c is a function of λ, the resulting confidence set need not be a highest density region of a likelihood function. This is an example of the effect of the imposition of one type of inference (frequentist) on another theory (likelihood).)

b. The procedure in part a) was carried out for $y = 558$ and the confidence interval was found to be $(57.78, 66.45)$. For the confidence interval in Example 9.2.15, we need the values $\chi^2_{1116,.95} = 1039.444$ and $\chi^2_{1118,.05} = 1196.899$. This confidence interval is $(1039.444/18, 1196.899/18) = (57.75, 66.49)$. The two confidence intervals are virtually the same.

9.25 The confidence interval derived by the method of Section 9.2.3 is

$$C(y) = \left\{\mu\colon y + \frac{1}{n}\log\left(\frac{\alpha}{2}\right) \le \mu \le y + \frac{1}{n}\log\left(1 - \frac{\alpha}{2}\right)\right\}$$

where $y = \min_i x_i$. The LRT method derives its interval from the test of $H_0: \mu = \mu_0$ versus $H_1\colon \mu \ne \mu_0$. Since Y is sufficient for μ, we can use $f_Y(y \mid \mu)$. We have

$$\begin{aligned}\lambda(y) &= \frac{\sup_{\mu=\mu_0} L(\mu|y)}{\sup_{\mu\in(-\infty,\infty)} L(\mu|y)} = \frac{ne^{-n}(y-\mu_0)I_{[\mu_0,\infty)(y)}}{ne^{-(y-y)}I_{[\mu,\infty)(y)}}\\ &= e^{-n(y-\mu_0)}I_{[\mu_0,\infty)}(y) = \begin{cases}0 & \text{if } y < \mu_0\\ e^{-n(y-\mu_0)} & \text{if } y \ge \mu_0.\end{cases}\end{aligned}$$

We reject H_0 if $\lambda(y) = e^{-n(y-\mu_0)} < c_\alpha$, where $0 \le c_\alpha \le 1$ is chosen to give the test level α. To determine c_α, set

$$\begin{aligned}\alpha &= P\{\text{reject } H_0|\, \mu = \mu_0\} = P\left\{Y > \mu_0 - \frac{\log c_\alpha}{n} \text{ or } Y < \mu_0 \,\middle|\, \mu = \mu_0\right\}\\ &= P\left\{Y > \mu_0 - \frac{\log c_\alpha}{n} \,\middle|\, \mu = \mu_0\right\} = \int_{\mu_0 - \frac{\log c_\alpha}{n}}^{\infty} ne^{-n(y-\mu_0)}\,dy\\ &= -e^{-n(y-\mu_0)}\Big|_{\mu_0-\frac{\log c_\alpha}{n}}^{\infty} = e^{\log c_\alpha} = c_\alpha.\end{aligned}$$

Therefore, $c_\alpha = \alpha$ and the $1-\alpha$ confidence interval is

$$C(y) = \left\{\mu\colon \mu \le y \le \mu - \frac{\log\alpha}{n}\right\} = \left\{\mu\colon y + \frac{1}{n}\ \log\alpha \le \mu \le y\right\}.$$

To use the pivotal method, note that since μ is a location parameter, a natural pivotal quantity is $Z = Y - \mu$. Then, $f_Z(z) = ne^{-nz}I_{(0,\infty)}(z)$. Let $P\{a \le Z \le b\} = 1-\alpha$, where a and b satisfy

$$\begin{aligned}\frac{\alpha}{2} = \int_0^a ne^{-nz}\,dz = -e^{-nz}\Big|_0^a = 1 - e^{-na} \quad &\Rightarrow\quad e^{-na} = 1 - \frac{\alpha}{2}\\ &\Rightarrow\quad a = \frac{-\log\left(1-\frac{\alpha}{2}\right)}{n}\\ \frac{\alpha}{2} = \int_b^\infty ne^{-nz}\,dz = -e^{-nz}\Big|_b^\infty = e^{-nb} \quad &\Rightarrow\quad -nb = \log\frac{\alpha}{2}\\ &\Rightarrow\quad b = -\frac{1}{n}\log\left(\frac{\alpha}{2}\right)\end{aligned}$$

Thus, the pivotal interval is $Y + \log(\alpha/2)/n \le \mu \le Y + \log(1-\alpha/2)$, the same interval as from Example 9.2.13. To compare the intervals we compare their lengths. We have

$$\begin{aligned}\text{Length of LRT interval} &= y - (y + \frac{1}{n}\log\alpha) = -\frac{1}{n}\log\alpha\\ \text{Length of Pivotal interval} &= \left(y + \frac{1}{n}\log(1-\alpha/2)\right) - (y + \frac{1}{n}\log\alpha/2) = \frac{1}{n}\log\frac{1-\alpha/2}{\alpha/2}\end{aligned}$$

Thus, the LRT interval is shorter if $-\log\alpha < \log[(1-\alpha/2)/(\alpha/2)]$, but this is always satisfied.

9.27 a. $Y = \sum X_i \sim \text{gamma}(n,\lambda)$, and the posterior distribution of λ is

$$\pi(\lambda|y) = \frac{(y+\frac{1}{b})^{n+a}}{\Gamma(n+a)}\frac{1}{\lambda^{n+a+1}}e^{-\frac{1}{\lambda}(y+\frac{1}{b})},$$

an IG $\left(n+a,(y+\frac{1}{b})^{-1}\right)$. The Bayes HPD region is of the form $\{\lambda\colon \pi(\lambda|y)\geq k\}$, which is an interval since $\pi(\lambda|y)$ is unimodal. It thus has the form $\{\lambda\colon a_1(y)\leq\lambda\leq a_2(y)\}$, where a_1 and a_2 satisfy

$$\frac{1}{a_1{}^{n+a+1}}e^{-\frac{1}{a_1}(y+\frac{1}{b})}=\frac{1}{a_2{}^{n+a+1}}e^{-\frac{1}{a_2}(y+\frac{1}{b})}.$$

b. The posterior distribution is IG$(((n-1)/2)+a,(((n-1)s^2/2)+1/b)^{-1})$. So the Bayes HPD region is as in part a) with these parameters replacing $n+a$ and $y+1/b$.

c. As $a\to 0$ and $b\to\infty$, the condition on a_1 and a_2 becomes

$$\frac{1}{a_1{}^{((n-1)/2)+1}}e^{-\frac{1}{a_1}\frac{(n-1)s^2}{2}}=\frac{1}{a_2{}^{((n-1)/2)+1}}e^{-\frac{1}{a_2}\frac{(n-1)s^2}{2}}.$$

9.29 a. We know from Example 7.2.14 that if $\pi(p)\sim\text{beta}(a,b)$, the posterior is $\pi(p|y)\sim\text{beta}(y+a,n-y+b)$ for $y=\sum x_i$. So a $1-\alpha$ credible set for p is:

$$\{p\colon \beta_{y+a,n-y+b,1-\alpha/2}\leq p\leq\beta_{y+a,n-y+b,\alpha/2}\}.$$

b. Converting to an F distribution, $\beta_{c,d}=\frac{(c/d)F_{2c,2d}}{1+(c/d)F_{2c,2d}}$, the interval is

$$\frac{\frac{y+a}{n-y+b}F_{2(y+a),2(n-y+b),1-\alpha/2}}{1+\frac{y+a}{n-y+b}F_{2(y+a),2(n-y+b),1-\alpha/2}}\leq p\leq\frac{\frac{y+a}{n-y+b}F_{2(y+a),2(n-y+b),\alpha/2}}{1+\frac{y+a}{n-y+b}F_{2(y+a),2(n-y+b),\alpha/2}}$$

or, using the fact that $F_{m,n}=F^{-1}_{n,m}$,

$$\frac{1}{1+\frac{n-y+b}{y+a}F_{2(n-y+b),2(y+a),\alpha/2}}\leq p\leq\frac{\frac{y+a}{n-y+b}F_{2(y+a),2(n+b),\alpha/2}}{1+\frac{y+a}{n-y+b}F_{2(y+a),2(n-y+b),\alpha/2}}.$$

For this to match the interval of Exercise 9.21, we need $x=y$ and

$$\begin{aligned}\text{Lower limit: } n-y+b=n-x+1 &\quad\Rightarrow\quad b=1\\ y+a=x &\quad\Rightarrow\quad a=0\\ \text{Upper limit: } y+a=x+1 &\quad\Rightarrow\quad a=1\\ n-y+b=n-x &\quad\Rightarrow\quad b=0.\end{aligned}$$

So no values of a and b will make the intervals match.

9.31 a. We continually use the fact that given $Y=y$, χ^2_{2y} is a central χ^2 random variable with $2y$ degrees of freedom. Hence

$$\begin{aligned}\mathrm{E}\chi^2_{2Y} &= \mathrm{E}[\mathrm{E}(\chi^2_{2Y}|Y)] = \mathrm{E}2Y = 2\lambda\\ \mathrm{Var}\chi^2_{2Y} &= \mathrm{E}[\mathrm{Var}(\chi^2_{2Y}|Y)]+\mathrm{Var}[\mathrm{E}(\chi^2_{2Y}|Y)]\\ &= \mathrm{E}[4Y]+\mathrm{Var}[2Y] = 4\lambda+4\lambda = 8\lambda\\ \text{mgf} &= \mathrm{E}e^{t\chi^2_{2Y}} = \mathrm{E}[\mathrm{E}(e^{t\chi^2_{2Y}}|Y)] = \mathrm{E}\left(\frac{1}{1-2t}\right)^Y\\ &= \sum_{y=0}^{\infty}\frac{e^{-\lambda}\left(\frac{\lambda}{1-2t}\right)^y}{y!} = e^{-\lambda+\frac{\lambda}{1-2t}}.\end{aligned}$$

From Theorem 2.3.15, the mgf of $(\chi^2_{2Y}-2\lambda)/\sqrt{8\lambda}$ is

$$e^{-t\sqrt{\lambda/2}}\left[e^{-\lambda+\frac{\lambda}{1-t/\sqrt{2\lambda}}}\right].$$

The log of this is

$$-\sqrt{\lambda/2}t-\lambda+\frac{\lambda}{1-t/\sqrt{2\lambda}}=\frac{t^2\sqrt{\lambda}}{-t\sqrt{2}+2\sqrt{\lambda}}=\frac{t^2}{-(t\sqrt{2}/\sqrt{\lambda})+2}\to t^2/2 \text{ as } \lambda\to\infty,$$

so the mgf converges to $e^{t^2/2}$, the mgf of a standard normal.

b. Since $P(\chi^2_{2Y}\le\chi^2_{2Y,\alpha})=\alpha$ for all λ,

$$\frac{\chi^2_{2Y,\alpha}-2\lambda}{\sqrt{8\lambda}}\to z_\alpha \text{ as } \lambda\to\infty.$$

In standardizing (9.2.22), the upper bound is

$$\frac{\frac{nb}{nb+1}\chi^2_{2(Y+a),\alpha/2}-2\lambda}{\sqrt{8\lambda}}=\sqrt{\frac{8(\lambda+a)}{8\lambda}}\left[\frac{\frac{nb}{nb+1}[\chi^2_{2(Y+a),\alpha/2}-2(\lambda+a)]}{\sqrt{8(\lambda+a)}}+\frac{\frac{nb}{nb+1}2(\lambda+a)-2\lambda}{\sqrt{8(\lambda+a)}}\right].$$

While the first quantity in square brackets $\to z_{\alpha/2}$, the second one has limit

$$\lim_{\lambda\to\infty}\frac{-2\frac{1}{nb+1}\lambda+a\frac{nb}{nb+1}}{\sqrt{8(\lambda+a)}}\to-\infty,$$

so the coverage probability goes to zero.

9.33 a. Since $0\in C_a(x)$ for every x, $P(0\in C_a(X)|\,\mu=0)=1$. If $\mu>0$,

$$\begin{aligned}P(\mu\in C_a(X)) &= P(\mu\le\max\{0,X+a\}) = P(\mu\le X+a) && \text{(since } \mu>0)\\ &= P(Z\ge -a) && (Z\sim \mathrm{n}(0,1))\\ &= .95 && (a=1.645.)\end{aligned}$$

A similar calculation holds for $\mu<0$.

b. The credible probability is

$$\begin{aligned}\int_{\min(0,x-a)}^{\max(0,x+a)}\frac{1}{\sqrt{2\pi}}e^{-\frac12(\mu-x)^2}\,d\mu &= \int_{\min(-x,-a)}^{\max(-x,a)}\frac{1}{\sqrt{2\pi}}e^{-\frac12 t^2}\,dt\\ &= P\left(\min(-x,-a)\le Z\le\max(-x,a)\right).\end{aligned}$$

To evaluate this probability we have two cases:

(i) $|x|\le a \;\Rightarrow\;$ credible probability $=P(|Z|\le a)$
(ii) $|x|>a \;\Rightarrow\;$ credible probability $=P(-a\le Z\le|x|)$

Thus we see that for $a=1.645$, the credible probability is equal to .90 if $|x|\le 1.645$ and increases to .95 as $|x|\to\infty$.

9.34 a. A $1-\alpha$ confidence interval for μ is $\{\mu:\bar x-1.96\sigma/\sqrt n\le\mu\le\bar x+1.96\sigma/\sqrt n\}$. We need $2(1.96)\sigma/\sqrt n\le\sigma/4$ or $\sqrt n\ge 4(2)(1.96)$. Thus we need $n\ge 64(1.96)^2\approx 245.9$. So $n=246$ suffices.

b. The length of a 95% confidence interval is $2t_{n-1,.025}S/\sqrt n$. Thus we need

$$\begin{aligned}P\left(2t_{n-1,.025}\frac{S}{\sqrt n}\le\frac{\sigma}{4}\right)\ge .9 \;&\Rightarrow\; P\left(4t^2_{n-1,.025}\frac{S^2}{n}\le\frac{\sigma^2}{16}\right)\ge .9\\ &\Rightarrow\; P\left(\underbrace{\frac{(n-1)S^2}{\sigma^2}}_{\sim\chi^2_{n-1}}\le\frac{(n-1)n}{t^2_{n-1,.025}\cdot 64}\right)\ge .9.\end{aligned}$$

We need to solve this numerically for the smallest n that satisfies the inequality

$$\frac{(n-1)n}{t^2_{n-1,.025}\cdot 64} \geq \chi^2_{n-1,.1}.$$

Trying different values of n we find that the smallest such n is $n = 276$ for which

$$\frac{(n-1)n}{t^2_{n-1,.025}\cdot 64} = 306.0 \geq 305.5 = \chi^2_{n-1,.1}.$$

As to be expected, this is somewhat larger than the value found in a).

9.35 length $= 2z_{\alpha/2}\sigma/\sqrt{n}$, and if it is unknown, E(length) $= 2t_{\alpha/2,n-1}c\sigma/\sqrt{n}$, where

$$c = \frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(n/2)}$$

and E$cS = \sigma$ (Exercise 7.50). Thus the difference in lengths is $(2\sigma/\sqrt{n})(z_{\alpha/2} - ct_{\alpha/2})$. A little work will show that, as $n \to \infty$, $c \to$ constant. (This can be done using Stirling's formula along with Lemma 2.3.14. In fact, some careful algebra will show that $c \to 1$ as $n \to \infty$.) Also, we know that, as $n \to \infty$, $t_{\alpha/2,n-1} \to z_{\alpha/2}$. Thus, the difference in lengths $(2\sigma/\sqrt{n})(z_{\alpha/2} - ct_{\alpha/2}) \to 0$ as $n \to \infty$.

9.36 The sample pdf is

$$f(x_1,\ldots,x_n|\theta) = \prod_{i=1}^n e^{i\theta - x_i} I_{(i\theta,\infty)}(x_i) = e^{\Sigma(i\theta - x_i)} I_{(\theta,\infty)}[\min(x_i/i)].$$

Thus $T = \min(X_i/i)$ is sufficient by the Factorization Theorem, and

$$P(T > t) = \prod_{i=1}^n P(X_i > it) = \prod_{i=1}^n \int_{it}^\infty e^{i\theta - x}\,dx = \prod_{i=1}^n e^{i(\theta - t)} = e^{-\frac{n(n+1)}{2}(t-\theta)},$$

and

$$f_T(t) = \frac{n(n+1)}{2} e^{-\frac{n(n+1)}{2}(t-\theta)}, \quad t \geq \theta.$$

Clearly, θ is a location parameter and $Y = T - \theta$ is a pivot. To find the shortest confidence interval of the form $[T + a, T + b]$, we must minimize $b - a$ subject to the constraint $P(-b \leq Y \leq -a) = 1 - \alpha$. Now the pdf of Y is strictly decreasing, so the interval length is shortest if $-b = 0$ and a satisfies

$$P(0 \leq Y \leq -a) = e^{-\frac{n(n+1)}{2}a} = 1 - \alpha.$$

So $a = 2\log(1-\alpha)/(n(n+1))$.

9.37 a. The density of $Y = X_{(n)}$ is $f_Y(y) = ny^{n-1}/\theta^n$, $0 < y < \theta$. So θ is a scale parameter, and $T = Y/\theta$ is a pivotal quantity. The pdf of T is $f_T(t) = nt^{n-1}$, $0 \leq t \leq 1$.

b. A pivotal interval is formed from the set

$$\{\theta\colon a \leq t \leq b\} = \left\{\theta\colon a \leq \frac{y}{\theta} \leq b\right\} = \left\{\theta\colon \frac{y}{b} \leq \theta \leq \frac{y}{a}\right\},$$

and has length $Y(1/a - 1/b) = Y(b-a)/ab$. Since $f_T(t)$ is increasing, $b - a$ is minimized and ab is maximized if $b = 1$. Thus shortest interval will have $b = 1$ and a satisfying $\alpha = \int_0^a nt^{n-1}dt = a^n \;\Rightarrow\; a = \alpha^{1/n}$. So the shortest $1-\alpha$ confidence interval is $\{\theta\colon y \leq \theta \leq y/\alpha^{1/n}\}$.

9.39 Let a be such that $\int_{-\infty}^{a} f(x)\,dx = \alpha/2$. This value is unique for a unimodal pdf if $\alpha > 0$. Let μ be the point of symmetry and let $b = 2\mu - a$. Then $f(b) = f(a)$ and $\int_b^\infty f(x)\,dx = \alpha/2$. $a \le \mu$ since $\int_{-\infty}^{a} f(x)\,dx = \alpha/2 \le 1/2 = \int_{-\infty}^{\mu} f(x)\,dx$. Similarly, $b \ge \mu$. And, $f(b) = f(a) > 0$ since $f(a) \ge f(x)$ for all $x \le a$ and $\int_{-\infty}^{a} f(x)\,dx = \alpha/2 > 0 \Rightarrow f(x) > 0$ for some $x < a \Rightarrow f(a) > 0$. So the conditions of Theorem 9.3.2 are satisfied.

9.41 a. We show that for any interval $[a, b]$ and $\epsilon > 0$, the probability content of $[a-\epsilon, b-\epsilon]$ is greater (as long as $b - \epsilon > a$). Write

$$\begin{aligned}
\int_b^a f(x)\,dx - \int_{a-\epsilon}^{b-\epsilon} f(x)\,dx &= \int_{b-\epsilon}^{b} f(x)\,dx - \int_{a-\epsilon}^{a} f(x)\,dx \\
&\le f(b-\epsilon)[b-(b-\epsilon)] - f(a)[a-(a-\epsilon)] \\
&\le \epsilon[f(b-\epsilon) - f(a)] \;\le\; 0,
\end{aligned}$$

where all of the inequalities follow because $f(x)$ is decreasing. So moving the interval toward zero increases the probability, and it is therefore maximized by moving a all the way to zero.

b. $T = Y - \mu$ is a pivot with decreasing pdf $f_T(t) = ne^{-nt} I_{[0,\infty]}(t)$. The shortest $1-\alpha$ interval on T is $[0, -\frac{1}{n}\log\alpha]$, since

$$\int_0^b ne^{-nt}\,dt = 1 - \alpha \;\Rightarrow\; b = -\frac{1}{n}\log\alpha.$$

Since $a \le T \le b$ implies $Y - b \le \mu \le Y - a$, the best $1-\alpha$ interval on μ is $Y + \frac{1}{n}\log\alpha \le \mu \le Y$.

9.43 a. Using Theorem 8.3.12, identify $g(t)$ with $f(x|\theta_1)$ and $f(t)$ with $f(x|\theta_0)$. Define $\phi(t) = 1$ if $t \in C$ and 0 otherwise, and let ϕ' be the indicator of any other set C' satisfying $\int_{C'} f(t)\,dt \ge 1 - \alpha$. Then $(\phi(t) - \phi'(t))(g(t) - \lambda f(t)) \le 0$ and

$$0 \ge \int (\phi - \phi')(g - \lambda f) = \int_C g - \int_{C'} g - \lambda\left[\int_C f - \int_{C'} f\right] \ge \int_C g - \int_{C'} g,$$

showing that C is the best set.

b. For Exercise 9.37, the pivot $T = Y/\theta$ has density nt^{n-1}, and the pivotal interval $a \le T \le b$ results in the θ interval $Y/b \le \theta \le Y/a$. The length is proportional to $1/a - 1/b$, and thus $g(t) = 1/t^2$. The best set is $\{t\colon 1/t^2 \le \lambda n t^{n-1}\}$, which is a set of the form $\{t\colon a \le t \le 1\}$. This has probability content $1 - \alpha$ if $a = \alpha^{1/n}$. For Exercise 9.24 (or Example 9.3.4), the g function is the same and the density of the pivot is f_k, the density of a gamma$(k, 1)$. The set $\{t\colon 1/t^2 \le \lambda f_k(t)\} = \{t\colon f_{k+2}(t) \ge \lambda'\}$, so the best a and b satisfy $\int_a^b f_k(t)\,dt = 1 - \alpha$ and $f_{k+2}(a) = f_{k+2}(b)$.

9.45 a. Since $Y = \sum X_i \sim$ gamma(n, λ) has MLR, the Karlin-Rubin Theorem (Theorem 8.3.2) shows that the UMP test is to reject H_0 if $Y < k(\lambda_0)$, where $P\left(Y < k(\lambda_0) | \lambda = \lambda_0\right) = \alpha$.

b. $T = 2Y/\lambda \sim \chi^2_{2n}$ so choose $k(\lambda_0) = \frac{1}{2}\lambda_0\chi^2_{2n,\alpha}$ and

$$\{\lambda\colon Y \ge k(\lambda)\} = \left\{\lambda\colon Y \ge \frac{1}{2}\lambda\chi^2_{2n,\alpha}\right\} = \left\{\lambda\colon 0 < \lambda \le 2Y/\chi^2_{2n,\alpha}\right\}$$

is the UMA confidence set.

c. The expected length is $\mathrm{E}\frac{2Y}{\chi^2_{2n,\alpha}} = \frac{2n\lambda}{\chi^2_{2n,\alpha}}$.

d. $X_{(1)} \sim$ exponential(λ/n), so $\mathrm{E}X_{(1)} = \lambda/n$. Thus

$$\begin{aligned}
\mathrm{E}(\text{length}(C^*)) &= \frac{2 \times 120}{251.046}\lambda = .956\lambda \\
\mathrm{E}(\text{length}(C^m)) &= \frac{-\lambda}{120 \times \log(.99)} = .829\lambda.
\end{aligned}$$

9.46 The proof is similar to that of Theorem 9.3.5:

$$P_\theta\left(\theta' \in C^*(X)\right) = P_\theta\left(X \in A^*(\theta')\right) \le P_\theta\left(X \in A(\theta')\right) = P_\theta\left(\theta' \in C(X)\right),$$

where A and C are any competitors. The inequality follows directly from Definition 8.3.11.

9.47 Referring to (9.3.2), we want to show that for the upper confidence bound, $P_\theta(\theta' \in C) \le 1-\alpha$ if $\theta' \ge \theta$. We have

$$P_\theta(\theta' \in C) = P_\theta(\theta' \le \bar{X} + z_\alpha \sigma/\sqrt{n}).$$

Subtract θ from both sides and rearrange to get

$$P_\theta(\theta' \in C) = P_\theta\left(\frac{\theta'-\theta}{\sigma/\sqrt{n}} \le \frac{\bar{X}-\theta}{\sigma/\sqrt{n}} + z_\alpha\right) = P\left(Z \ge \frac{\theta'-\theta}{\sigma/\sqrt{n}} - z_\alpha\right),$$

which is less than $1-\alpha$ as long as $\theta' \ge \theta$. The solution for the lower confidence interval is similar.

9.48 a. Start with the hypothesis test $H_0\colon \theta \ge \theta_0$ versus $H_1\colon \theta < \theta_0$. Arguing as in Example 8.2.4 and Exercise 8.47, we find that the LRT rejects H_0 if $(\bar{X}-\theta_0)/(S/\sqrt{n}) < -t_{n-1,\alpha}$. So the acceptance region is $\{x\colon (\bar{x}-\theta_0)/(s/\sqrt{n}) \ge -t_{n-1,\alpha}\}$ and the corresponding confidence set is $\{\theta\colon \bar{x} + t_{n-1,\alpha}s/\sqrt{n} \ge \theta\}$.

b. The test in part a) is the UMP unbiased test so the interval is the UMA unbiased interval.

9.49 a. Clearly, for each σ, the conditional probability $P_{\theta_0}(\bar{X} > \theta_0 + z_\alpha\sigma/\sqrt{n} \mid \sigma) = \alpha$, hence the test has unconditional size α. The confidence set is $\{(\theta,\sigma)\colon \theta \ge \bar{x} - z_\alpha\sigma/\sqrt{n}\}$, which has confidence coefficient $1-\alpha$ conditionally and, hence, unconditionally.

b. From the Karlin-Rubin Theorem, the UMP test is to reject H_0 if $X > c$. To make this size α,

$$\begin{aligned}
P_{\theta_0}(X > c) &= P_{\theta_0}\left(X > c \middle| \sigma = 10\right)P(\sigma = 10) + P\left(X > c \middle| \sigma = 1\right)P(\sigma = 1)\\
&= pP\left(\frac{X-\theta_0}{10} > \frac{c-\theta_0}{10}\right) + (1-p)P(X-\theta_0 > c-\theta_0)\\
&= pP\left(Z > \frac{c-\theta_0}{10}\right) + (1-p)P(Z > c-\theta_0),
\end{aligned}$$

where $Z \sim \mathrm{n}(0,1)$. Without loss of generality take $\theta_0 = 0$. For $c = z_{(\alpha-p)/(1-p)}$ we have for the proposed test

$$\begin{aligned}
P_{\theta_0}(\text{reject}) &= p + (1-p)P\left(Z > z_{(\alpha-p)/(1-p)}\right)\\
&= p + (1-p)\frac{(\alpha-p)}{(1-p)} \quad = \quad p + \alpha - p \quad = \quad \alpha.
\end{aligned}$$

This is not UMP, but more powerful than part a. To get UMP, solve for c in $pP(Z > c/10) + (1-p)P(Z > c) = \alpha$, and the UMP test is to reject if $X > c$. For $p = 1/2$, $\alpha = .05$, we get $c = 12.81$. If $\alpha = .1$ and $p = .05$, $c = 1.392$ and $z_{\frac{.1-.05}{.95}} =_{.0526}= 1.62$.

9.51

$$\begin{aligned}
P_\theta\left(\theta \in C(X_1,\ldots,X_n)\right) &= P_\theta\left(\bar{X} - k_1 \le \theta \le \bar{X} + k_2\right)\\
&= P_\theta\left(-k_2 \le \bar{X} - \theta \le k_1\right)\\
&= P_\theta\left(-k_2 \le \sum Z_i/n \le k_1\right),
\end{aligned}$$

where $Z_i = X_i - \theta$, $i = 1,\ldots,n$. Since this is a location family, for any θ, $Z_1,\ldots,Z_n$ are iid with pdf $f(z)$, i. e., the Z_is are pivots. So the last probability does not depend on θ.

9.52 a. The LRT of $H_0\colon \sigma=\sigma_0$ versus $H_1\colon \sigma\neq\sigma_0$ is based on the statistic

$$\lambda(x)=\frac{\sup_{\mu,\sigma=\sigma_0} L\left(\mu,\,\sigma_0|\,x\right)}{\sup_{\mu,\sigma\in(0,\infty)} L(\mu,\,\sigma^2|\,x)}.$$

In the denominator, $\hat\sigma^2=\sum(x_i-\bar x)^2/n$ and $\hat\mu=\bar x$ are the MLEs, while in the numerator, σ_0^2 and $\hat\mu$ are the MLEs. Thus

$$\lambda(x)=\frac{\left(2\pi\sigma_0^2\right)^{-n/2}e^{-\frac{\Sigma(x_i-\bar x)^2}{2\sigma_0^2}}}{\left(2\pi\hat\sigma^2\right)^{-n/2}e^{-\frac{\Sigma(x_i-\bar x)^2}{2\sigma^2}}}=\left(\frac{\sigma_0^2}{\hat\sigma^2}\right)^{-n/2}\frac{e^{-\frac{\Sigma(x_i-\bar x)^2}{2\sigma_0^2}}}{e^{-n/2}},$$

and, writing $\hat\sigma^2=[(n-1)/n]s^2$, the LRT rejects H_0 if

$$\left(\frac{\sigma_0^2}{\frac{n-1}{n}s^2}\right)^{-n/2}e^{-\frac{(n-1)s^2}{2\sigma_0^2}}<k_\alpha,$$

where k_α is chosen to give a size α test. If we denote $t=\frac{(n-1)s^2}{\sigma_0^2}$, then $T\sim\chi^2_{n-1}$ under H_0, and the test can be written: reject H_0 if $t^{n/2}e^{-t/2}<k'_\alpha$. Thus, a $1-\alpha$ confidence set is

$$\left\{\sigma^2\colon t^{n/2}e^{-t/2}\geq k'_\alpha\right\}=\left\{\sigma^2\colon \left(\frac{(n-1)s^2}{\sigma^2}\right)^{n/2}e^{-\frac{(n-1)s^2}{\sigma^2}/2}\geq k'_\alpha\right\}.$$

Note that the function $t^{n/2}e^{-t/2}$ is unimodal (it is the kernel of a gamma density) so it follows that the confidence set is of the form

$$\begin{aligned}\left\{\sigma^2\colon t^{n/2}e^{-t/2}\geq k'_\alpha\right\} &= \left\{\sigma^2\colon a\leq t\leq b\right\} = \left\{\sigma^2\colon a\leq\frac{(n-1)s^2}{\sigma^2}\leq b\right\}\\ &= \left\{\sigma^2\colon \frac{(n-1)s^2}{b}\leq\sigma^2\leq\frac{(n-1)s^2}{b}\right\},\end{aligned}$$

where a and b satisfy $a^{n/2}e^{-a/2}=b^{n/2}e^{-b/2}$ (since they are points on the curve $t^{n/2}e^{-t/2}$). Since $\frac{n}{2}=\frac{n+2}{2}-1$, a and b also satisfy

$$\frac{1}{\Gamma\left(\frac{n+2}{2}\right)2^{(n+2)/2}}a^{((n+2)/2)-1}e^{-a/2}=\frac{1}{\Gamma\left(\frac{n+2}{2}\right)2^{(n+2)/2}}b^{((n+2)/2)-1}e^{-b/2},$$

or, $f_{n+2}(a)=f_{n+2}(b)$.

b. The constants a and b must satisfy $f_{n-1}(b)b^2=f_{n-1}(a)a^2$. But since $b^{((n-1)/2)-1}b^2=b^{((n+3)/2)-1}$, after adjusting constants, this is equivalent to $f_{n+3}(b)=f_{n+3}(a)$. Thus, the values of a and b that give the minimum length interval must satisfy this along with the probability constraint. The confidence interval, say $I(s^2)$ will be unbiased if (Definition 9.3.7)

c.

$$P_{\sigma^2}\left(\sigma'^2\in I(S^2)\right)\leq P_{\sigma^2}\left(\sigma^2\in I(S^2)\right)=1-\alpha.$$

Some algebra will establish

$$\begin{aligned}P_{\sigma^2}\left(\sigma'^2\in I(S^2)\right) &= P_{\sigma^2}\left(\frac{(n-1)S^2}{b\sigma^2}\leq\frac{\sigma'^2}{\sigma^2}\leq\frac{(n-1)S^2}{a\sigma^2}\right)\\ &= P_{\sigma^2}\left(\frac{\chi^2_{n-1}}{b}\leq\frac{\sigma'^2}{\sigma^2}\leq\frac{\chi^2_{n-1}}{a}\right) = \int_{ac}^{bc}f_{n-1}(t)\,dt,\end{aligned}$$

where $c = \sigma'^2/\sigma^2$. The derivative (with respect to c) of this last expression is $bf_{n-1}(bc) - af_{n-1}(ac)$, and hence is equal to zero if both $c = 1$ (so the interval is unbiased) and $bf_{n-1}(b) = af_{n-1}(a)$. From the form of the chi squared pdf, this latter condition is equivalent to $f_{n+1}(b) = f_{n+1}(a)$.

d. By construction, the interval will be $1-\alpha$ equal-tailed.

9.53 a. $\mathrm{E}\left[b\mathrm{length}(C) - I_C(\mu)\right] = 2c\sigma b - P(|Z| \le c)$, where $Z \sim \mathrm{n}(0,1)$.

b. $\frac{d}{dc}\left[2c\sigma b - P(|Z| \le c)\right] = 2\sigma b - 2\left(\frac{1}{\sqrt{2\pi}} e^{-c^2/2}\right)$.

c. If $b\sigma > 1/\sqrt{2\pi}$ the derivative is always positive since $e^{-c^2/2} < 1$.

9.55

$$\begin{aligned}
\mathrm{E}[L((\mu,\sigma),C)] &= \mathrm{E}\left[L((\mu,\sigma),C)|S<K\right]P(S<K) + \mathrm{E}\left[L((\mu,\sigma),C)|S>K\right]P(S>K) \\
&= \mathrm{E}\left[L((\mu,\sigma),C')|S<K\right]P(S<K) + \mathrm{E}\left[L((\mu,\sigma),C)|S>K\right]P(S>K) \\
&= R\left[L((\mu,\sigma),C')\right] + \mathrm{E}\left[L((\mu,\sigma),C)|S>K\right]P(S>K),
\end{aligned}$$

where the last equality follows because $C' = \emptyset$ if $S > K$. The conditional expectation in the second term is bounded by

$$\begin{aligned}
\mathrm{E}\left[L((\mu,\sigma),C)|S>K\right] &= \mathrm{E}\left[b\mathrm{length}(C) - I_C(\mu)|S>K\right] \\
&= \mathrm{E}\left[2bcS - I_C(\mu)|S>K\right] \\
&> \mathrm{E}\left[2bcK - 1|S>K\right] \qquad (\text{since } S>K \text{ and } I_C \le 1) \\
&= 2bcK - 1,
\end{aligned}$$

which is positive if $K > 1/2bc$. For those values of K, C' dominates C.

9.57 a. The distribution of $X_{n+1} - \bar{X}$ is $\mathrm{n}[0, \sigma^2(1+1/n)]$, so

$$P\left(X_{n+1} \in \bar{X} \pm z_{\alpha/2}\sigma\sqrt{1+1/n}\right) = P(|Z| \le z_{\alpha/2}) = 1-\alpha.$$

b. p percent of the normal population is in the interval $\mu \pm z_{p/2}\sigma$, so $\bar{x} \pm k\sigma$ is a $1-\alpha$ tolerance interval if

$$P(\mu \pm z_{p/2} \subseteq \sigma\bar{X} \pm k\sigma) = P(\bar{X} - k\sigma \le \mu - z_{p/2}\sigma \text{ and } \bar{X} + k\sigma \ge \mu + z_{p/2}\sigma) \ge 1-\alpha.$$

This can be attained by requiring

$$P(\bar{X} - k\sigma \ge \mu - z_{p/2}\sigma) = \alpha/2 \quad \text{and} \quad P(\bar{X} + k\sigma \le \mu + z_{p/2}\sigma) = \alpha/2,$$

which is attained for $k = z_{p/2} + z_{\alpha/2}/\sqrt{n}$.

c. From part (a), $(X_{n+1} - \bar{X})/(S\sqrt{1+1/n}) \sim t_{n-1}$, so a $1-\alpha$ prediction interval is $\bar{X} \pm t_{n-1,\alpha/2}S\sqrt{1+1/n}$.