

Bivariate Transformations

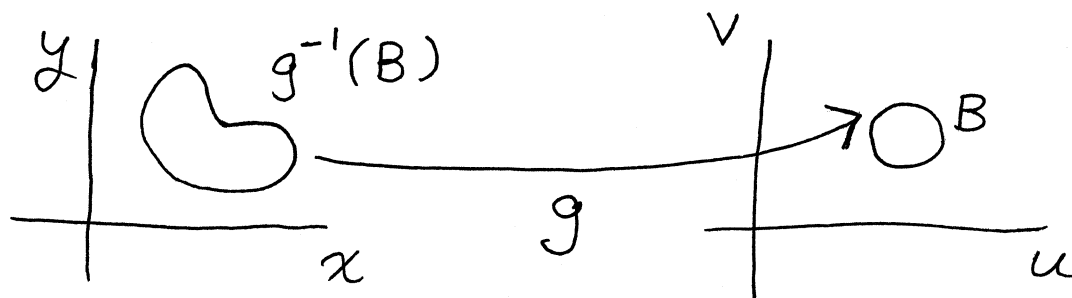
(U, V) is a function of (X, Y) .

$$(U, V) = g(X, Y) \text{ where } g: \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

Given the distn. of (X, Y) ,

find the distn. of (U, V) .

First principles



Define $g^{-1}(B) = \{(x, y) : g(x, y) \in B\}$
for any set $B \subset \mathbb{R}^2$.

$$\text{Then } P((U, V) \in B) = P((X, Y) \in g^{-1}(B)).$$

Formula for joint density (in a special case)

Suppose:

① (X, Y) has a joint pdf $f_{X, Y}(x, y)$.

② g is 1-1 and "smooth"

from $\mathcal{A} = \{(x, y) : f_{X, Y}(x, y) > 0\}$

to $\mathcal{B}$ = image of $\mathcal{A}$ under the map g

$$= \{(u, v) : (u, v) = g(x, y) \text{ for some } (x, y) \in \mathcal{A}\}.$$

Then g has an inverse $h = g^{-1}$ and h is also smooth.

Notation: Let $g = (g_1, g_2)$ with
 $u = g_1(x, y)$ and $v = g_2(x, y)$

and $h = (h_1, h_2)$ with

$$x = h_1(u, v) \text{ and } y = h_2(u, v).$$

Result: If ① and ② are true, then
 (U, V) has a joint pdf given by

$$f_{U, V}(u, v) = f_{X, Y}(h_1(u, v), h_2(u, v)) |J| \quad \text{for } (u, v) \in \mathcal{B}$$

where J is the Jacobian of the
inverse transformation h .

The Jacobian

$$J = J(u, v) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

determinant matrix of partial derivatives

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

where $x = h_1(u, v)$,
 $y = h_2(u, v)$.

Comments:

$$\textcircled{1} \quad f_{u,v}(u,v) = f_{x,y}(h_1(u,v), h_2(u,v)) |J|$$

for $(u,v) \in B$

is the "vector" version of

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

for $y \in \mathcal{Y}$.

$\textcircled{2}$

$f_{x,y}(h_1(u,v), h_2(u,v))$ is simply $f_{x,y}(x,y)$ re-expressed in terms of u and v .

Just replace x and y by expressions in terms of u and v .

Heuristic derivation of the density of $(U, V) = g(X, Y)$

Let (u, v) be a particular point in the (U, V) plane.

Suppose the function g maps (x, y) to the point (u, v) (so that $(x, y) = h(u, v)$ where $h = g^{-1}$).

Let A be a small (infinitesimal) region containing (x, y) .

Suppose g maps A into the small region B containing (u, v) .

Then $P((X, Y) \in A) = P((U, V) \in B)$.

We also know:

$$P((X, Y) \in A) \approx f_{X,Y}(x, y) \text{Area}(A),$$

$$P((U, V) \in B) \approx f_{U,V}(u, v) \text{Area}(B)$$

and the approximations can be made as accurate as we please by making the areas sufficiently small.

Equating the two expressions leads to

$$f_{X,Y}(x, y) \text{Area}(A) = f_{U,V}(u, v) \text{Area}(B)$$

so that

$$f_{U,V}(u, v) = f_{X,Y}(x, y) \cdot \frac{\text{Area}(A)}{\text{Area}(B)}.$$

This ratio of areas is (in the limit of areas going to zero) just the absolute value of the Jacobian $|J(u, v)|$ of the inverse transformation h , which tells us the amount of contraction or expansion produced by the mapping h at the point (u, v) .

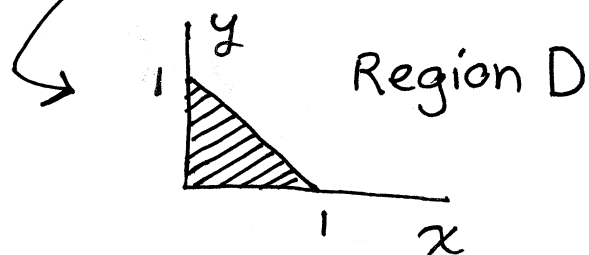
Thus we have:

$$f_{U,V}(u, v) = f_{X,Y}(h(u, v)) |J(u, v)|$$

as desired.

Example : (X, Y) with joint pdf

$$f_{X,Y}(x,y) = 6(1-x-y)I_D(x,y).$$



Define $(U, V) = g(X, Y)$ by

$$U = -\log(1-X) = g_1(X, Y),$$

$$V = \frac{Y}{1-X} = g_2(X, Y).$$

Find joint pdf $f_{U,V}(u,v)$.

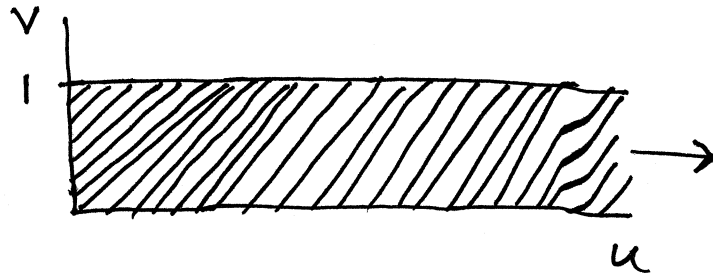
① What are the supports of (X, Y) and (U, V) ?

As X ranges from 0 to 1, $U = -\log(1-X)$ ranges from 0 to ∞ .

For fixed X between 0 and 1, $V = \frac{Y}{1-X}$ ranges from 0 to 1 as Y ranges from 0 to $1-X$.

Thus the supports are

$\mathcal{A} = \{(x, y) : x > 0, y > 0, x + y < 1\} = D,$
 $\mathcal{B} = \{(u, v) : 0 < u < \infty, 0 < v < 1\}$
 $= (0, \infty) \times (0, 1)$ which is the infinite strip pictured below.



② Find the inverse transformation.

Solve for X, Y in $U = -\log(1-X), \quad (1)$

$$V = \frac{Y}{1-X}. \quad (2)$$

(1) gives $X = 1 - e^{-U} = h_1(U, V).$

Then (2) gives

$$Y = V(1-X) = V e^{-U} = h_2(U, V).$$

The solution is unique. Thus $g = (g_1, g_2)$ is a 1-1 and "smooth" transformation from $\mathcal{A}$ onto $\mathcal{B}$.

③ Find the Jacobian of the inverse transformation $h = (h_1, h_2)$.

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{with} \quad \begin{aligned} x &= 1 - e^{-u} \\ y &= ve^{-u} \end{aligned}$$

$$= \begin{vmatrix} e^{-u} & -ve^{-u} \\ 0 & e^{-u} \end{vmatrix} = (e^{-u})(e^{-u}) - (0)(-ve^{-u})$$

$$= e^{-2u} - 0 = e^{-2u} = J.$$

④ The joint pdf is now given by

$$f_{U, V}(u, v) = f_{X, Y}(h_1(u, v), h_2(u, v)) |J|$$

$$= 6(1-x-y) \bigg|_{\substack{x=1-e^{-u} \\ y=ve^{-u}}} \cdot |e^{-2u}| \quad \text{for } (u, v) \in \mathcal{B}$$

$$= 6(1 - (1 - e^{-u}) - ve^{-u}) e^{-2u}$$

$$= 6(e^{-u} - ve^{-u}) e^{-2u}$$

$$= 6(1-v) e^{-3u} \quad \text{for } 0 < u < \infty, 0 < v < 1.$$

(*)

Comments:

The marginals for U and V are :

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v) dv$$

$$= \int_0^1 6(1-v)e^{-3u} dv$$

$$= 6e^{-3u} \int_0^1 (1-v) dv = 6e^{-3u} \cdot \frac{1}{2}$$

$$= 3e^{-3u} \text{ for } 0 < u < \infty$$

which is an exponential pdf with $\beta = \frac{1}{3}$,

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v) du$$

$$= 6(1-v) \int_0^{\infty} e^{-3u} du = 6(1-v) \cdot \frac{1}{3}$$

$$= 2(1-v) \text{ for } 0 < v < 1$$

which is a Beta pdf with $\alpha = 1, \beta = 2$.

Note that $f_{U,V}(u,v) = f_U(u)f_V(v)$ for all

u, v (that is, for $-\infty < u < \infty$ and $-\infty < v < \infty$):

$$6(1-v)e^{-3u} I_{(0,\infty) \times (0,1)}(u,v)$$

$$= (3e^{-3u} I_{(0,\infty)}(u)) (2(1-v) I_{(0,1)}(v)).$$

Thus U and V are independent rv's.

We can see this by inspection (using the following lemma) directly from (*).

Lemma: If $f_{X,Y}(x,y) = c p(x) q(y)$

for $-\infty < x < \infty$ and $-\infty < y < \infty$,

then X and Y are independent rv's with marginal densities found by "normalizing" $p(x)$ and $q(y)$.

Going back to (*):

$$\begin{aligned} f_{U,V}(u,v) &= 6(1-v)e^{-3u} \text{ for } 0 < u < \infty, 0 < v < 1 \\ &= c p(u) q(v) \text{ for all } u, v \end{aligned}$$

where $c = 6$, $p(u) = e^{-3u} I_{(0,\infty)}(u)$,

and $q(v) = (1-v) I_{(0,1)}(v)$.

Thus U and V are independent.

If you don't like indicator functions, modify the lemma as follows.

Lemma: If the support of $f_{X,Y}(x,y)$ is a Cartesian product set, and on this support

$$f_{X,Y}(x,y) = c p(x) q(y),$$

then X and Y are independent.

A more precise and detailed wording:

Lemma: If

$$f_{X,Y}(x,y) = \begin{cases} c p(x) q(y) & \text{for } (x,y) \in A \times B \\ 0 & \text{otherwise} \end{cases}$$

then X and Y are independent

and $f_X(x) = a p(x)$ for $x \in A$,
(0 o.w.)

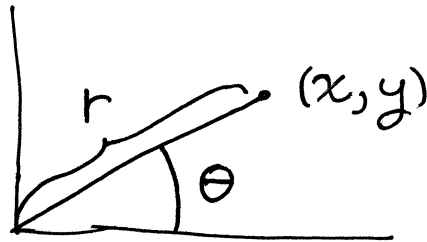
and $f_Y(y) = b q(y)$ for $y \in B$,

where a, b are normalizing constants.

Example :

Suppose X, Y iid $N(0,1)$.

Define $(R, \Theta) =$ polar coordinates of (X, Y) .



$$\text{Then } R = \sqrt{X^2 + Y^2}$$

$$\Theta = \sin^{-1}\left(\frac{Y}{\sqrt{X^2 + Y^2}}\right) \text{ or } = \tan^{-1}\left(\frac{Y}{X}\right).$$

Define $S = R^2$.

Find the joint pdf of (S, Θ) .

X, Y iid $N(0,1)$ implies

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

(by independence)

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2} \quad \text{for } -\infty < x < \infty, \\ -\infty < y < \infty.$$

We know that (x, y) may be obtained from the polar coordinates (r, θ) by

$$x = r \cos \theta$$

$$y = r \sin \theta.$$

Thus we have

$$S = x^2 + y^2 = g_1(x, y)$$

$$\Theta = \tan^{-1}\left(\frac{y}{x}\right) = g_2(x, y)$$

and the inverse transformation

$$x = \sqrt{S} \cos \Theta = h_1(S, \Theta)$$

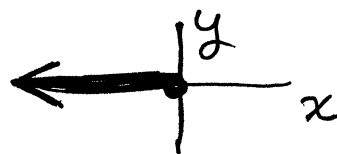
$$y = \sqrt{S} \sin \Theta = h_2(S, \Theta).$$

The transformation $g = (g_1, g_2)$ is 1-1 and smooth from $\mathcal{A}$ to $\mathcal{B}$ where

$$\mathcal{A} = \mathbb{R}^2 - \{(x, y) : x \leq 0, y = 0\}, \text{ and}$$

$$\mathcal{B} = \{(s, \theta) : s > 0, -\pi < \theta < \pi\}.$$

Note : g is discontinuous on the ray $\{(x, y) : x \leq 0, y = 0\}$ consisting of the points where $\theta = \pm \pi$.



This does not cause any problems because this ray has probability zero:

$$P\{(X,Y) \in \text{ray}\} = 0.$$

The Jacobian of the inverse transformation

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \quad \text{with } \begin{aligned} x &= \sqrt{s} \cos \theta, \\ y &= \sqrt{s} \sin \theta \end{aligned}$$

$$= \begin{vmatrix} \frac{\cos \theta}{2\sqrt{s}} & \frac{\sin \theta}{2\sqrt{s}} \\ -\sqrt{s} \sin \theta & \sqrt{s} \cos \theta \end{vmatrix} = \frac{1}{2} (\cos^2 \theta + \sin^2 \theta)$$

$$= \frac{1}{2}$$

The Joint pdf

$$f_{S,\Theta}(s,\theta) = f_{X,Y}(h_1(s,\theta), h_2(s,\theta)) |J|$$

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2} \cdot \frac{1}{2}$$

$\begin{aligned} x &= \sqrt{s} \cos \theta \\ y &= \sqrt{s} \sin \theta \end{aligned}$

$$= \frac{1}{4\pi} e^{-s/2} \text{ for } \underbrace{s > 0, -\pi < \theta < \pi}_{(s, \theta) \in B}$$

$$= \underbrace{\left(\frac{1}{2} e^{-s/2} I_{(0, \infty)}(s) \right)}_{\text{pdf of exponential with } \beta=2.} \cdot \underbrace{\left(\frac{1}{2\pi} I_{(-\pi, \pi)}(\theta) \right)}_{\text{pdf of Uniform}(-\pi, \pi).}$$

Thus S and Θ are independent with the distributions given above.

Note: We have discovered that

$S = X^2 + Y^2 \sim \text{exponential } (\beta=2)$ which is the same as χ^2_2 .

The technique we used is generally useful: To find the distrn. of $U = \text{function of } (X, Y)$, introduce another convenient random variable $V = \text{function of } (X, Y)$, then find the joint pdf of (U, V) and integrate over V .

Distributions of Sums and Ratios

If X and Y have joint pdf $f_{X,Y}(x,y)$, then

① $Z = X + Y$ has pdf

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) dx,$$

② $Z = Y/X$ has pdf

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f_{X,Y}(x, zx) dx.$$

When X, Y are independent

$$f_{X,Y}(x, z-x) = f_X(x) f_Y(z-x),$$

$$f_{X,Y}(x, zx) = f_X(x) f_Y(zx).$$

In this case ① is known as a "convolution integral" or as the "convolution of two densities".

Caution:

when applying

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) dx,$$

you must think carefully about where the integrand is positive and where it is zero.

Example Suppose X, Y independent with

$$f_X(x) = \alpha e^{-\alpha x} \text{ for } x > 0$$

$$f_Y(y) = \beta e^{-\beta y} \text{ for } y > 0$$

Then $Z = X + Y$ has density

$$f_Z(z) = \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{\text{zero for } x < 0} \underbrace{f_Y(z-x)}_{\text{zero when } z-x < 0 \text{ or } x > z} dx$$

$$= \int_0^z \alpha e^{-\alpha x} \beta e^{-\beta(z-x)} dx$$

for $z > 0$ ($f_Z(z) = 0$ for $z < 0$.)

Example : If X, Y iid with common density $f(x) = 6x(1-x)$, $0 < x < 1$, find the density of $Z = X + Y$.

$$f_Z(z) = \int_{-\infty}^{\infty} \underbrace{f(x)}_{\substack{\text{positive} \\ \text{for } 0 < x < 1}} \underbrace{f(z-x)}_{\substack{\text{positive for} \\ 0 < z-x < 1 \\ \text{which means} \\ x < z \text{ and} \\ x > z-1}} dx$$

$$= \int_{\max(0, z-1)}^{\min(1, z)} 6x(1-x) 6(z-x)(1-z+x) dx$$

Proof:

"First principles" or "Formula" approach possible.

"Formula" approach: $X+Y$

(X, Y) (U, V)

$$U = X$$

$$V = X+Y$$

} many other choices

Get joint density $f_{U,V}(u,v)$.

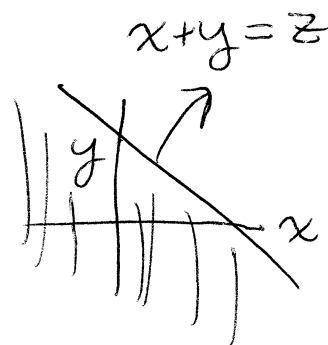
Integrate out u to get $f_V(v)$

"First principles"

Find $F_Z(z)$ where $Z = X+Y$.

$$f_Z(z) = \frac{d}{dz} F_Z(z).$$

$$\{Z \leq z\} = \{X+Y \leq z\}$$



$$= \iint_A \underbrace{f_{X,Y}(x,y)}_{f_X(x)f_Y(y)} dx dy$$

"Formula" approach for Y/X

Define the transformation

$$\begin{aligned} U &= X \\ V &= Y/X \end{aligned} \quad \left(\begin{aligned} &= g_1(X, Y) \\ &= g_2(X, Y) \end{aligned} \right)$$

with inverse transformation

$$\begin{aligned} X &= U \\ Y &= UV \end{aligned} \quad \left(\begin{aligned} &= h_1(U, V) \\ &= h_2(U, V) \end{aligned} \right).$$

The mapping $(X, Y) \mapsto (U, V)$ is 1-1

from $\mathcal{A} = \{(x, y) : x \neq 0\}$

onto $\mathcal{B} = \{(u, v) : u \neq 0\}$.

The Jacobian of the inverse transformation is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & v \\ 0 & u \end{vmatrix} = u$$

so that

$$f_{U,V}(u,v) = f_{X,Y}(u, uv) |u| \text{ for } u \neq 0 \text{ and}$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v) du = \int_{-\infty}^{\infty} |u| f_{X,Y}(u, uv) du.$$

First principles $Z = Y/X$ (Assume $f_X(x) = 0$ for $x \leq 0$)

$$\textcircled{P} \quad F_Z(z) = P\{Z \leq z\}$$

$$= P\left\{ \begin{array}{c} Y/X \leq z \\ Y \leq zX \end{array} \right\} = \iint_A f_X(x) f_Y(y) dx dy$$

