Joint density of Order Statistics

Suppose $X_1, X_2, \ldots, X_n$ are iid with pdf $f(x)$.

Let $(U_1, U_2, \ldots, U_n) = (X_{(1)}, X_{(2)}, \ldots, X_{(n)})$.

Then

$$\begin{aligned} & f_{U_1,U_2,\ldots,U_n}(u_1, u_2, \ldots, u_n) \\ & \quad = \quad n!\, f(u_1)f(u_2)\cdots f(u_n)I(u_1 < u_2 < \cdots < u_n)\,. \end{aligned}$$

The order statistics are dependent (**not** independent).

The support of the joint density is the set

$$\{(u_1, u_2, \ldots, u_n) : u_1 < u_2 < \cdot < u_n\}\,.$$

This is **not** a Cartesian product set.

We know that $U_i < U_j$ for $i < j$. Thus, knowing U_i tells us something about U_j. So (intuitively) they must be dependent rv's.

Manipulating Joint Distributions

Obtaining Marginal Density from Joint Density (continuous case)

$$f_W(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{W,X,Y,Z}(w,x,y,z)\,dx\,dy\,dz$$

$$f_{W,Y}(w,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{W,X,Y,Z}(w,x,y,z)\,dx\,dz$$

$$f_{X,Y,Z}(x,y,z) = \int_{-\infty}^{\infty} f_{W,X,Y,Z}(w,x,y,z)\,dw$$

Conditional Densities

$$f_{W|X,Y,Z}(w|x,y,z) = \frac{f_{W,X,Y,Z}(w,x,y,z)}{f_{X,Y,Z}(x,y,z)}$$

$$f_{X,Z|W,Y}(x,z|w,y) = \frac{f_{W,X,Y,Z}(w,x,y,z)}{f_{W,Y}(w,y)}$$

$$f_{X,Y,Z|W}(x,y,z|w) = \frac{f_{W,X,Y,Z}(w,x,y,z)}{f_W(w)}$$

Joint density as Product

$$\begin{aligned} & f_{W,X,Y,Z}(w,x,y,z) \\ &= f_W(w)f_{X|W}(x|w)f_{Y|W,X}(y|w,x)f_{Z|W,X,Y}(z|w,x,y) \end{aligned}$$

Example:

X_1, X_2, X_3, X_4 iid with pdf f.

$(U_1, \ldots, U_4) = (X_{(1)}, \ldots, X_{(4)})$.

Find pdf of U_3.

$$f_{U_3}(u_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{f_{U_1,\ldots,U_4}(u_1,\ldots,u_4)}_{4!\, f(u_1) f(u_2) f(u_3) f(u_4)\, I(u_1<u_2<u_3<u_4)} du_1\, du_2\, du_4$$

$$= \int_{-\infty}^{u_3} du_1 \int_{u_1}^{u_3} du_2 \int_{u_3}^{\infty} du_4 \; 4! \prod_{i=1}^{4} f(u_i)$$

$$= 4!\, f(u_3) \int_{-\infty}^{u_3} du_1\, f(u_1) \underbrace{\int_{u_1}^{u_3} du_2\, f(u_2)}_{F(u_3)-F(u_1)} \underbrace{\int_{u_3}^{\infty} du_4\, f(u_4)}_{\substack{1-F(u_3) \\ \text{(constant, factor out)}}}$$

$$= 4!\, f(u_3)(1-F(u_3)) \underbrace{\int_{-\infty}^{u_3} du_1\, (F(u_3)-F(u_1))\, f(u_1)}_{-\frac{(F(u_3)-F(u_1))^2}{2}\Big|_{-\infty}^{u_3}}$$

$$= 4!\, f(u_3)(1-F(u_3)) \frac{(F(u_3))^2}{2}$$

$$= 12 f(u_3) (F(u_3))^2 (1-F(u_3)) \quad \text{(Change } u_3 \text{ to } u.\text{)}$$

Joint pdf of (U_1, U_3)?

$$f_{U_1,U_3}(u_1,u_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{f_{U_1,\ldots,U_4}(u_1,\ldots,u_4)}_{4!\prod_{i=1}^{4} f(u_i)\, I(u_1<u_2<u_3<u_4)} du_2\, du_4$$

$$= \int_{u_1}^{u_3} du_2 \int_{u_3}^{\infty} du_4 \; 4! \prod_{i=1}^{4} f(u_i)$$

$$= 4!\, f(u_1) f(u_3) \underbrace{\int_{u_1}^{u_3} du_2\, f(u_2)}_{F(u_3)-F(u_1)} \underbrace{\int_{u_3}^{\infty} du_4\, f(u_4)}_{1-F(u_3)}$$

$$= 4!\, f(u_1) f(u_3) (F(u_3)-F(u_1))(1-F(u_3))$$

(Can change (u_1, u_3) to (u, v).)

Conditional density of (U_2, U_4) given (U_1, U_3).

$$
\begin{aligned}
& f_{U_2,U_4|U_1,U_3}(u_2, u_4|u_1, u_3) \\
&= \frac{f_{U_1,U_2,U_3,U_4}(u_1, u_2, u_3, u_4)}{f_{U_1,U_3}(u_1, u_3)} \\
&= \frac{4!\, f(u_1)f(u_2)f(u_3)f(u_4)I(u_1 < u_2 < u_3 < u_4)}{4!\, f(u_1)f(u_3)(F(u_3) - F(u_1))(1 - F(u_3))I(u_1 < u_3)} \\
&= \frac{f(u_2)f(u_4)I(u_1 < u_2 < u_3 < u_4)}{(F(u_3) - F(u_1))(1 - F(u_3))} \quad \text{if } u_1 < u_3 \\
&= \frac{f(u_2)I(u_1 < u_2 < u_3)}{F(u_3) - F(u_1)} \cdot \frac{f(u_4)I(u_3 < u_4)}{1 - F(u_3)}
\end{aligned}
$$

This last factorization says that U_2 and U_4 are **conditionally** independent given U_1 and U_3.

Example 4.6.13 on page 185 (with notational changes)

Suppose X_1, X_2, X_3, X_4 are iid with pdf $f(x) = e^{-x}$ for $x > 0$.

Define $(Y_1, Y_2, Y_3, Y_4) = (X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})$
(the order statistics).

Find the joint density of (Z_1, Z_2, Z_3, Z_4) defined by

$$\begin{aligned} Z_1 &= Y_1 \\ Z_2 &= Y_2 - Y_1 \\ Z_3 &= Y_3 - Y_2 \\ Z_4 &= Y_4 - Y_3 \,. \end{aligned}$$

Solution: The inverse transformation is

$$\begin{aligned} Y_1 &= Z_1 \\ Y_2 &= Z_1 + Z_2 \\ Y_3 &= Z_1 + Z_2 + Z_3 \\ Y_4 &= Z_1 + Z_2 + Z_3 + Z_4 \end{aligned}$$

which has Jacobian equal to 1.

$$\begin{aligned} & f_{Y_1,Y_2,Y_3,Y_4}(y_1, y_2, y_3, y_4) \\ = \;& 4!\, e^{-y_1} e^{-y_2} e^{-y_3} e^{-y_4} I(0 < y_1 < y_2 < y_3 < y_4) \end{aligned}$$

The transformation maps

$$\mathcal{A} = \{(y_1, y_2, y_3, y_4) : 0 < y_1 < y_2 < y_3 < y_4\}$$

to $\quad \mathcal{B} = (0, \infty)^4 \,.$

$$
\begin{aligned}
& f_{Z_1,Z_2,Z_3,Z_4}(z_1, z_2, z_3, z_4) \\
&\quad = f_{Y_1,Y_2,Y_3,Y_4}(z_1, z_1 + z_2, z_1 + z_2 + z_3, z_1 + \cdots + z_4) \cdot 1 \\
&\quad = 4!\, e^{-z_1-(z_1+z_2)-(z_1+z_2+z_3)-(z_1+\cdots+z_4)} \\
&\quad = 4e^{-4z_1} \cdot 3e^{-3z_2} \cdot 2e^{-2z_3} \cdot e^{-z_4} \quad \text{for } z_1, z_2, z_3, z_4 > 0
\end{aligned}
$$

so that Z_1, Z_2, Z_3, Z_4 are independent exponential rv's with means $1/4, 1/3, 1/2, 1$.

We obtained this earlier in the course by using the memoryless property.