Independence

Defn: B, C are indep. if

$$P(B \cap C) = P(B)\,P(C)$$

or equivalently $P(B|C) = P(B)$

" " $P(C|B) = P(C)$.

Defn:

A, B, C are mutually indep. if all of the following are true:

$$P(A \cap B) = P(A)\,P(B)$$
$$P(A \cap C) = P(A)\,P(C)$$
$$P(B \cap C) = P(B)\,P(C)$$
$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

(To ~~check~~ verify mut. indep, you must verify all of these!)

General Defn: A list of events is mut. ind. if the probs of all possible intersections are given by multiplying the individual event probs.

To verify mutual independence, you must check all possible intersections, not just pairs.

Example: To show that pairwise independence does not imply mutual independence.

Suppose:

Coin with sides labeled $+1, -1$.

$P(\text{toss is } +1) = P(\text{toss is } -1) = 1/2$.

Toss the coin twice. (Tosses are independent.)

Let X_1 = result of first toss,

X_2 = result of second toss.

For this experiment

$$\Omega = \{(x_1, x_2) : x_i = \pm 1\},$$

$$\#(\Omega) = 4, \quad P(\omega) = \frac{1}{4} \text{ for all } \omega.$$

Define $X_3 = X_1 X_2$ (so that $X_3 = \pm 1$).

Define the events $A_i = \{X_i = +1\}$.

Fact: A_1, A_2, A_3 are pairwise indep., but not mutually independent.

Proof:

ω	$P(\omega)$	X_1	X_2	X_3
ω_1	1/4	+1	+1	+1
ω_2	1/4	+1	−1	−1
ω_3	1/4	−1	+1	−1
ω_4	1/4	−1	−1	+1

$A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_1, \omega_3\}$, $A_3 = \{\omega_1, \omega_4\}$.

Thus $P(A_1) = P(A_2) = P(A_3) = 1/2$.

A_1 and A_3 are independent because $A_1 \cap A_3 = \{\omega_1\}$ so $P(A_1 \cap A_3) = 1/4$ which agrees with $P(A_1)P(A_3) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$.

Similarly A_1, A_2 are independent and A_2, A_3 are independent.

But $A_1 \cap A_2 \cap A_3 = \{\omega_1\}$ so that $P(A_1 \cap A_2 \cap A_3) = 1/4$ which <u>disagrees</u> with $P(A_1)P(A_2)P(A_3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

The events A_1, A_2, A_3 are <u>not</u> mutually independent.

Fact: If $A_1, \ldots, A_j, B_1, \ldots, B_K$ are mutually independent events, then any event defined in terms of $A_1, \ldots, A_j$ is independent of any event defined in terms of $B_1, \ldots, B_K$.

Example: If A, B, C, D are mut. indep., then $A \cup B$ is indep. of $C \cup D$. Also $A^c \cup B^c$ is indep. of $C \cap D$, etc.

How would you prove these facts? In the first, we must show

$$P((A \cup B) \cap (C \cup D)) = P(A \cup B)\, P(C \cup D).$$

This can be done in many ways. Here is one. By repeated use of the distributive property

$$(A \cup B) \cap (C \cup D) = AC \cup BC \cup AD \cup BD$$

where $AC \equiv (A \cap C)$, etc.

Now use Inclusion–Exclusion:

$$P(AC \cup BC \cup AD \cup BD)$$

$$= P(AC) + P(BC) + P(AD) + P(BD)$$

$$- P(ABC) - P(ACD) - P(ABCD)$$

$$- P(ABCD) - P(BCD) - P(ABD)$$

$$+ P(ABCD) + P(ABCD) + P(ABCD)$$

$$+ P(ABCD) - P(ABCD)$$

$$= P(AC) + P(BC) + P(AD) + P(BD)$$

$$- P(ABC) - P(ACD) - P(BCD) - P(ABD)$$

$$+ P(ABCD)$$

Now note that $P(A \cup B)\, P(C \cup D)$

$$= (P(A) + P(B) - P(AB))(P(C) + P(D) - P(CD))$$

Expand the product and compare with earlier expression. They are the same since independence implies

$$P(ABCD) = P(A)\, P(B)\, P(C)\, P(D)$$

$$= P(AB)\, P(CD), \text{ etc.}$$

Independence versus Conditional Independence.

Example: Hat with two coins.
Coin #1 has $P(\text{heads}) = 1/4 = \pi_1$
Coin #2 " " $= 3/4 = \pi_2$

Pick a coin at random.
Toss it repeatedly.
Let $A_i = \{ i^{th} \text{ toss is heads} \}$.

Are A_1 and A_2 independent?

Answer: No!

Discussion: Let $B_j = \{ \text{pick coin } j \}$.

B_1, B_2 is a partition of Ω.

Problem implicitly assumes

$$P(A_1 \cap A_2 \mid B_j) = P(A_1 \mid B_j)\, P(A_2 \mid B_j)$$
$$= \pi_j \cdot \pi_j = \pi_j^2 ,$$

$$P(A_1 \cap A_2 \cap A_3 \mid B_j)$$
$$= P(A_1 \mid B_j)\, P(A_2 \mid B_j)\, P(A_3 \mid B_j)$$
$$= \pi_j^3 ,$$

and more generally

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_K} \mid B_j)$$

$$= \prod_{l=1}^{K} P(A_{i_l} \mid B_j) = \pi_j^{K} .$$

We say that the events $A_1, A_2, A_3, \ldots$ are <u>conditionally</u> <u>independent</u> given B_j.

Back to the original question:

$$P(A_1) = P(B_1)P(A_1 \mid B_1) + P(B_2)P(A_1 \mid B_2)$$

$$= \frac{1}{2} \cdot \underbrace{\pi_1}_{1/4} + \frac{1}{2} \cdot \underbrace{\pi_2}_{3/4} = \frac{1}{2}$$

Similarly $P(A_2) = \frac{1}{2}$.

$$P(A_1 \cap A_2) = P(B_1)P(A_1 \cap A_2 \mid B_1) + P(B_2)P(A_1 \cap A_2 \mid B_2)$$

$$= \frac{1}{2} \cdot \pi_1^2 + \frac{1}{2} \cdot \pi_2^2 = \frac{5}{16}$$

Since $\frac{5}{16} \neq \frac{1}{2} \cdot \frac{1}{2} = P(A_1)P(A_2)$, we see A_1 and A_2 are <u>not</u> independent.

Note that

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{5/16}{1/2}$$

$$= \frac{5}{8} > \frac{1}{2} = P(A_2).$$

Intuitively, $P(A_2|A_1) > P(A_2)$ because getting a head on the first toss makes it more likely you have drawn coin #2 which is biased towards heads.

Random Variables (r.v.)

"Experiment" is (Ω, P)

sample space   prob. function

R.V. is a function.

$$X : \Omega \to \mathbb{R}$$

rv

$$\omega \mapsto X(\omega)$$

Example 1

<u>Experiment</u>: Toss pair of dice.
(Record results.)

$\Omega = \{(i,j) : 1 \le i \le 6, 1 \le j \le 6\}$

$$P(A) = \frac{\#(A)}{\#(\Omega)}.$$

Some rv's

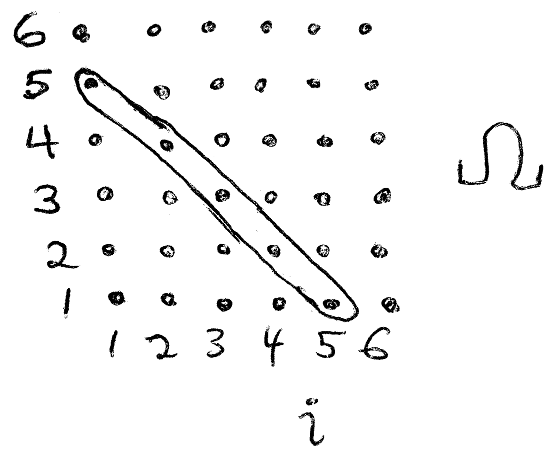
$\omega = (i,j)$

X = # on first die $\quad X(\omega) = i$

Y = # on second die $\quad Y(\omega) = j$

Z = total on both die $\quad Z(\omega) = i + j$

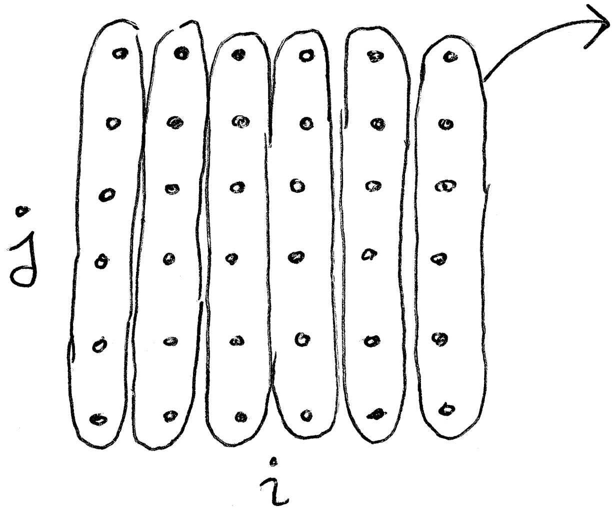
Notation for events



$\{Z = 6\}$ means $\{\omega : Z(\omega) = 6\} \subset \Omega$

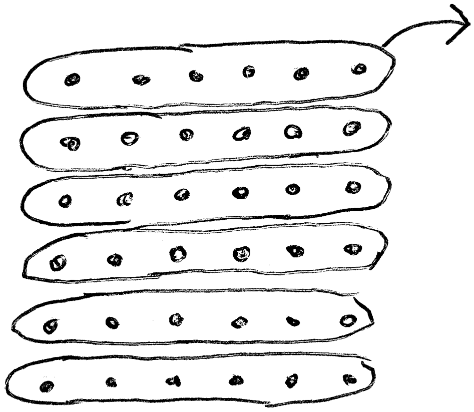
Every rv gives a partition of Ω.

In our example,



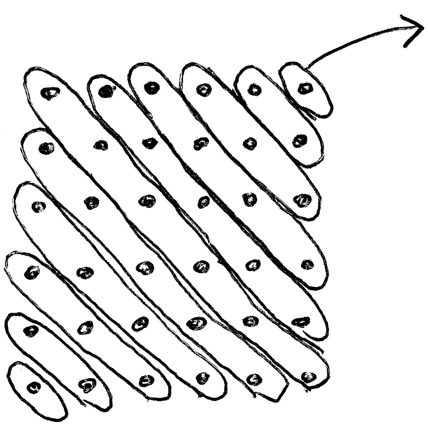
$\{X=6\}$

partition induced by X,



$\{Y=6\}$

partition induced by Y,



$\{Z=12\}$

partition induced by Z.

Definition: The induced probability function of r.v. X is P_X defined by

$$\underbrace{P_X(A)}_{\substack{\text{a set of}\\ \text{real numbers}}} = P(\underbrace{X \in A}_{\substack{\text{event}\\ \text{subset of } \Omega}}) = P(\omega : X(\omega) \in A).$$

In particular
$$P_X(\{x\}) = P(X = x).$$

Notation:
$$\mathcal{X} = \text{range of } X = \text{set of all possible values of } X.$$

$P_X(\cdot)$ is a probability function defined for $A \subset \mathcal{X}$. (or $A \subset \mathbb{R}$)

$P(\cdot)$ is a prob. fn. defined for $A \subset \Omega$.

Example 1 (Dice)

X

x	$P_X(\{x\})$
1	1/6
$\vdots$	$\vdots$
6	1/6

r.v. Y

y	$P_Y(\{y\})$
1	1/6
$\vdots$	$\vdots$
6	1/6

r.v. Z

z	$P_Z(\{z\})$
2	1/36
3	2/36
$\vdots$	
11	2/36
12	1/36

<u>Definition</u> If $P(X \in A) = P(Y \in A)$ for all $A \subset \mathbb{R}$, (that is, $P_X = P_Y$), we say X and Y are <u>identically distributed</u> (or have <u>same distribution</u>).

X and Y above have $P_X = P_Y$.

$$P_Z((3,\infty)) = P(Z>3) = 33/36$$

Example 2

Experiment: Toss dart (uniformly) at random on circle of radius R.

$$\Omega = \{(x,y) : x^2+y^2 \le R^2\}$$

$$P(A) = \frac{\text{Area}(A)}{\text{Area}(\Omega)} \quad (\text{for } A \subset \Omega).$$

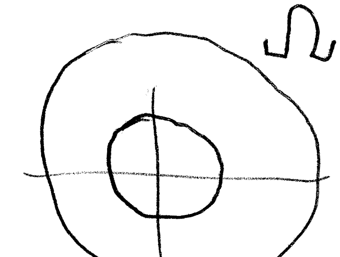
some RV's

$$\omega = (x,y)$$

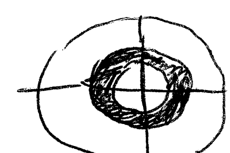
Z = distance from center $\qquad Z(\omega) = \sqrt{x^2+y^2}$

Q = quadrant dart lies in $\qquad Q(\omega) = \begin{cases} 1 & \text{if } x \ge 0, y \ge 0 \\ \vdots & \\ 4 & \text{if } x \ge 0, y < 0 \end{cases}$

some events

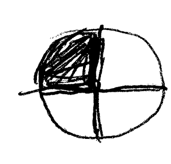


$$\{Z=1\} = \{\omega : Z(\omega)=1\}$$



$$\{1 \le Z \le 2\} = \{\omega : 1 \le Z(\omega) \le 2\}$$

$$\{Q=2\} = \{\omega : Q(\omega)=2\}$$



Example 2 (Dart)

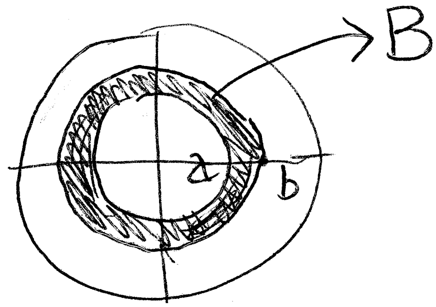
Q

q	$P_Q(\{q\})$
1	1/4
2	1/4
3	1/4
4	1/4

$P_Q(A)$ computed from table.

Z

$$P_Z(\underbrace{(a,b)}_{\subset \mathbb{R}}) = P(\underbrace{a<Z<b}_{B\subset\Omega}) = \frac{\text{Area}(B)}{\text{Area}(\Omega)}$$



$$= \frac{\pi b^2 - \pi a^2}{\pi R^2}$$

The distribution of <u>any</u> r.v. may be described by giving its

<u>induced prob. fn.</u> $P_X(A) = P(X \in A)$,

or its

<u>cdf</u> (cumulative dist fn.) $F_X(t) = P(X \le t)$

$(= P_X((-\infty, t]))$.

<u>Types of r.v.'s</u>

X is <u>discrete</u> r.v. $\Rightarrow$ range of X (called $\mathcal{X}$) is <u>finite</u> or $\underbrace{\text{countable}}$.

(infinite, but can be arranged in a sequence

$\mathcal{X} = \{x_1, x_2, x_3, \ldots\}$.)

The distn. of a discrete r.v. is usually described by its <u>probability mass function</u> (pmf):

$f_X(x) = P(X = x) = P_X(\{x\})$.

The cdf of a discrete rv is a

step function which has steps (jumps) at the values in $\mathcal{X}$ and is flat between these values.

X is a (absolutely) continuous r.v. means

X has a probability density function (pdf)

denoted $f_X(x)$ satisfying

$$P(a<X<b) = P_X((a,b)) = \int_a^b f_X(x)\,dx,$$

$$P(X\in A) = P_X(A) = \int_A f_X(x)\,dx.$$

For a continuous r.v., the cdf is a continuous function (no jumps),

Range $\mathcal{X}$ is uncountable, and $P(X=x)=0$ for all x.

Other types of r.v.'s

Mixed Continuous/Discrete : the distn. has both a discrete and a continuous component.

Exotic cases such as rv's which are $\underbrace{\text{continuous}}_{P(X=x)=0 \text{ for all } x}$, but $\underbrace{\text{not absolutely continuous}}_{X \text{ does not have a pdf.}}$.

Synopsis

Experiment (Ω, P)

Random Variable $X : \Omega \to \mathbb{R}$ (real numbers)

$\omega \mapsto X(\omega)$

$$P_X(A) \equiv P\{X \in A\}$$

$$F_X(t) \equiv P_X((-\infty, t]) = P\{X \leq t\} \quad \underline{\underline{\text{cdf}}}$$

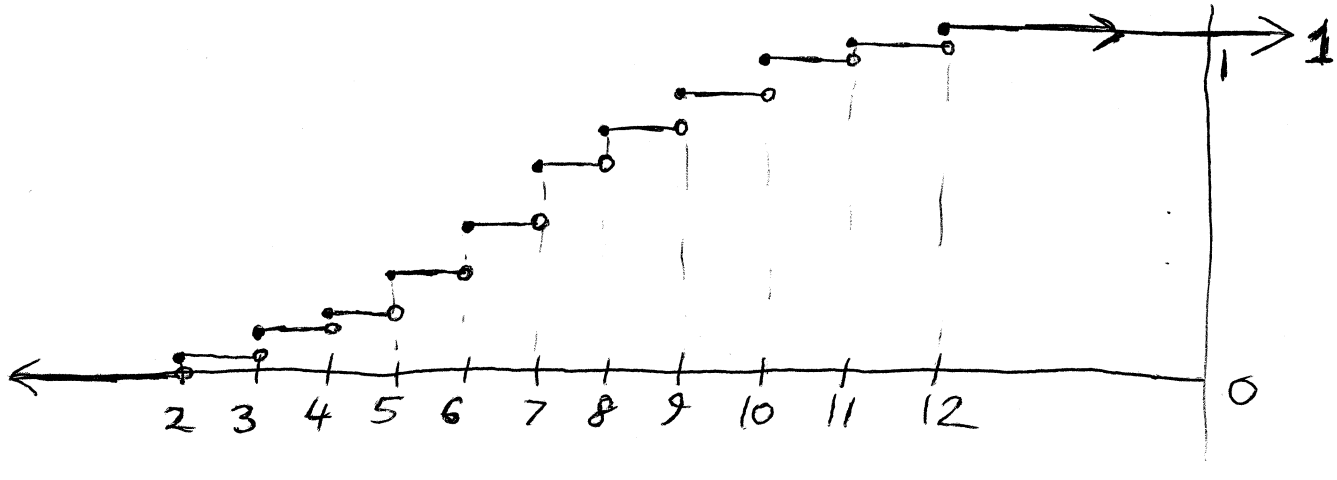
Examples

z = total on 2 dice

pmf

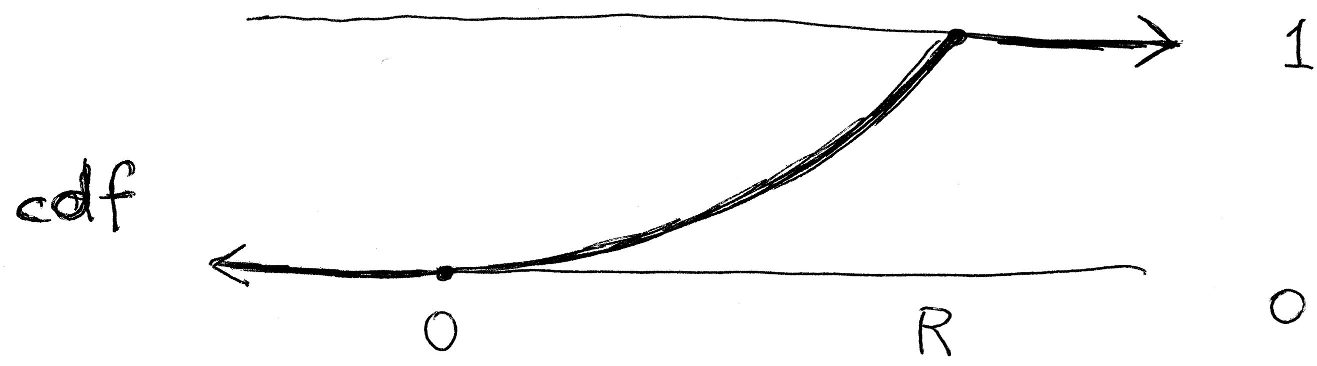
z	2	3	4	5	6	7	8	9	10	11	12
$f_Z(z)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

cdf



X = distance of ω from center

dart location



X and Z are bounded random variables.

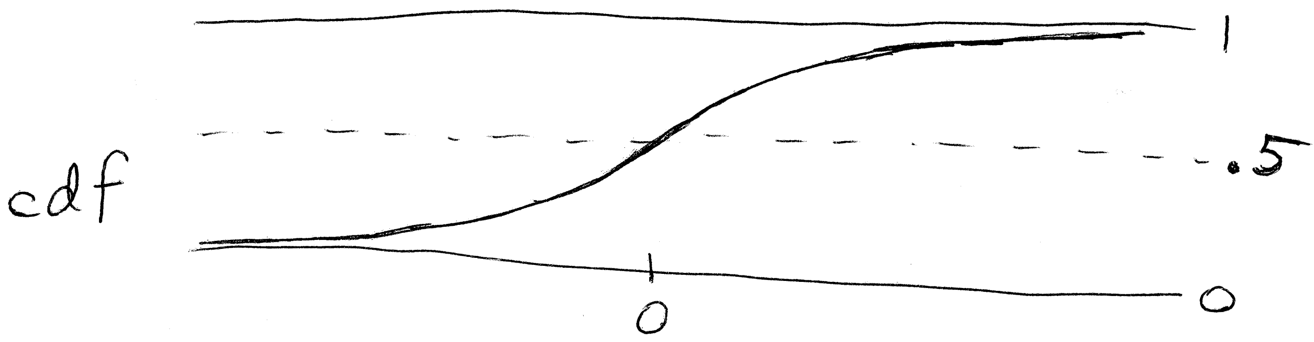
(range of X is bounded $[0,R]$
range of $Z = \{2,3,\ldots,12\}$ is bounded.)

some unbounded examples are:

Standard Normal Random Variable

density (pdf) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left(\equiv \varphi(x) \right)$

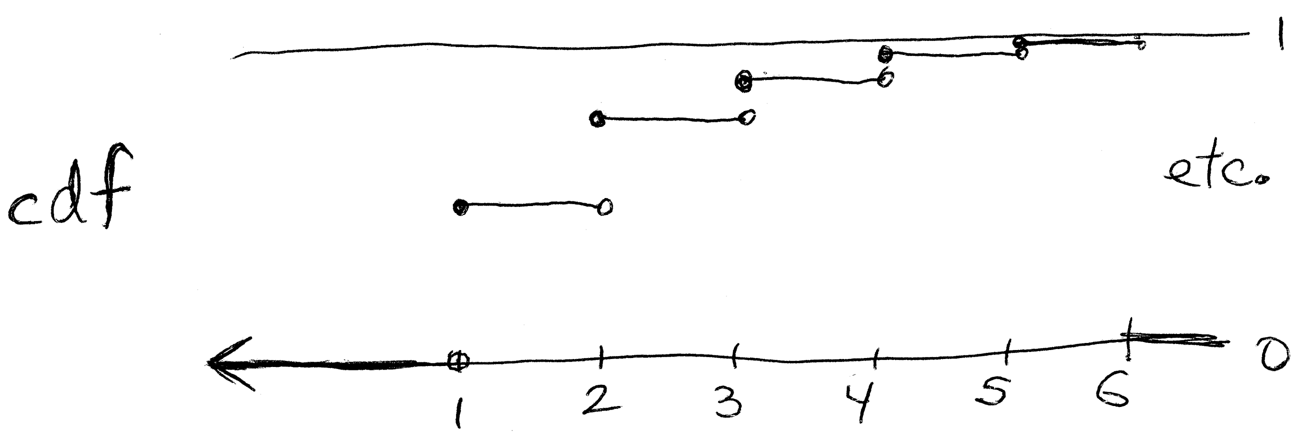
cdf $F(x) = \int_{-\infty}^{x} f(y)\,dy \left(\equiv \Phi(x) \right)$



Geometric Random Variable

pmf $f(x) = \begin{cases} (1-p)^{x-1} p & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$

For $p = 1/2$, $f(x) = \left(\frac{1}{2}\right)^x$.



More on cumulative distribution functions (cdf's)

Recall the definition:

For a r.v. X, the cdf $F_X(\cdot)$ is

$$F_X(t) = P_X((-\infty, t]) = P(X \le t).$$

Properties of cdf's

If F is the cdf of some r.v., then

(1) $F(-\infty) = 0$, $F(+\infty) = 1$.

(2) F is nondecreasing.

(3) F is right continuous.

Conversely, any function with properties 1, 2, 3 is the cdf of some random variable.

Details on 1, 2, 3

(1) says that

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to +\infty} F(x) = 1.$$

(2) says that

$x \le y$ implies $F(x) \le F(y)$.

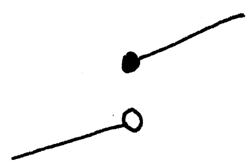
F cannot decrease, but can have flat spots.

(3) says that

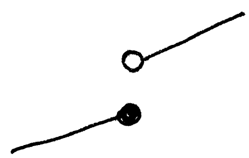
$$\lim_{x \downarrow y} F(x) = F(y) \quad \text{for all } y.$$

$x \downarrow y$ → x approaches y from above (from the right)

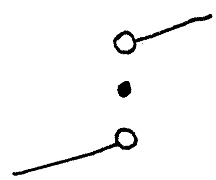
Jumps must look like this.



Not like this.



Not like this.



$$F(y-) \le F(y) = F(y+) \quad \text{for all } y$$

$$F(y-) = \lim_{x \uparrow y} F(x) \qquad\qquad F(y+) = \lim_{x \downarrow y} F(x)$$

The cdf $F_X(\cdot)$ $\underbrace{\text{contains all the info in}}_{\text{determines}}$ $P_X(\cdot)$

$$F_X(t) = P_X((-\infty, t]) = P(X \leq t)$$

$$F_X(t-) = P_X((-\infty, t)) = P(X < t)$$

$$F_X(b) - F_X(a) = P_X((a, b])$$
$$= P(a < X \leq b)$$

$$F_X(b-) - F(a) = P_X((a, b)) = P(a < X < b)$$

$$F_X(a) - F_X(a-) = P_X(\{a\}) = P(X = a)$$

Thus $F_X(\cdot)$ determines the probability assigned to any interval or point. Can build up from there.

$$1 - F_X(t) = P_X((t, \infty)) = P(X > t)$$

Properties of Densities:

If a r.v. X has a pdf $f(x)$, then

1. $f(x) \geq 0$ for all x,
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Conversely, any function $f(x)$ satisfying 1 and 2 can serve as a pdf (that is, there exists a r.v. X with $f(x)$ as its pdf).

Relationship between cdf and pdf:

Let $F(x)$ be the cdf of X.
If X has a pdf $f(x)$, then

(a) $F(x) = \int_{-\infty}^{x} f(y)\,dy$ for all x,

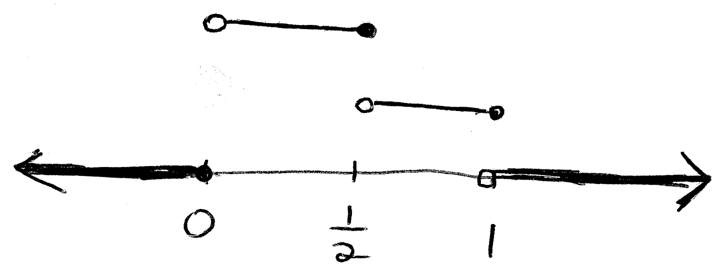
(b) $F'(x) = f(x)$ at all points x where $f(\cdot)$ is continuous.

(The text takes (a) as the definition of the pdf.)

Example:

The pdf $f(x) = \begin{cases} \frac{3}{2} & \text{for } 0 < x \le 1/2, \\ \frac{1}{2} & \text{for } \frac{1}{2} < x \le 1, \\ 0 & \text{otherwise} \end{cases}$

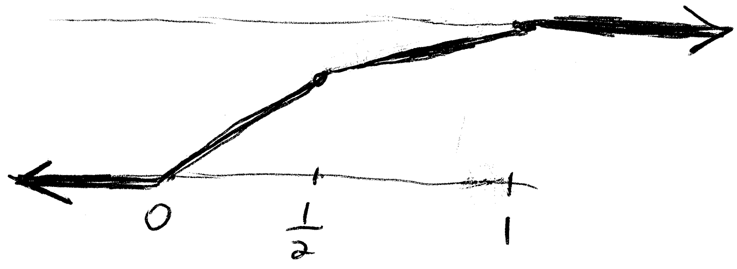
with graph



corresponds to the cdf

$$F(x) = \begin{cases} \frac{3}{2}x & \text{for } 0 \le x < 1/2, \\ \frac{1}{2}x + \frac{1}{2} & \text{for } \frac{1}{2} \le x < 1, \\ 1 & \text{for } x \ge 1, \\ 0 & \text{for } x < 0. \end{cases}$$

with graph

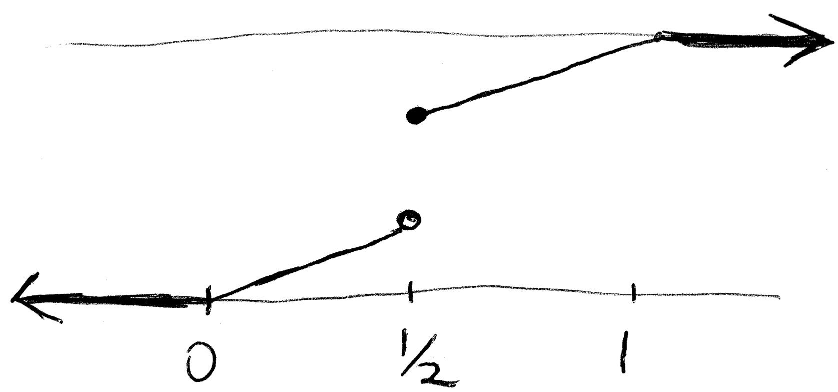


$F'(x) = f(x)$ except at $x = 0, \frac{1}{2}, 1$ where $F'(x)$ does not exist and $f(x)$ is not continuous.

Comment: If a cdf $F(x)$ has a jump (at any value of x), then there is no pdf.

Example:

The cdf $F(x) = \begin{cases} x/2 & \text{for } 0 \le x < 1/2 \\ x/2 + 1/2 & \text{for } 1/2 \le x < 1 \\ 1 & \text{for } x \ge 1 \\ 0 & \text{for } x < 0 \end{cases}$



has $F'(x) = \frac{1}{2}$ for $0 < x < 1$ except at the point $x = \frac{1}{2}$ where the derivative does not exist.

But $f(x) = \begin{cases} \frac{1}{2} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

is not a pdf.