Ch. 2

Transformations

X is a r.v.
g is some function
$Y = g(X)$ is a r.v.
What is distn. of Y?

Notation

$\mathcal{X}$ is the $\underbrace{\text{range of } X}_{\text{the set of all possible values of } X}$

$= \{x : f_X(x) > 0\}$ if X has a pdf or pmf

$\mathcal{Y}$ is the range of Y

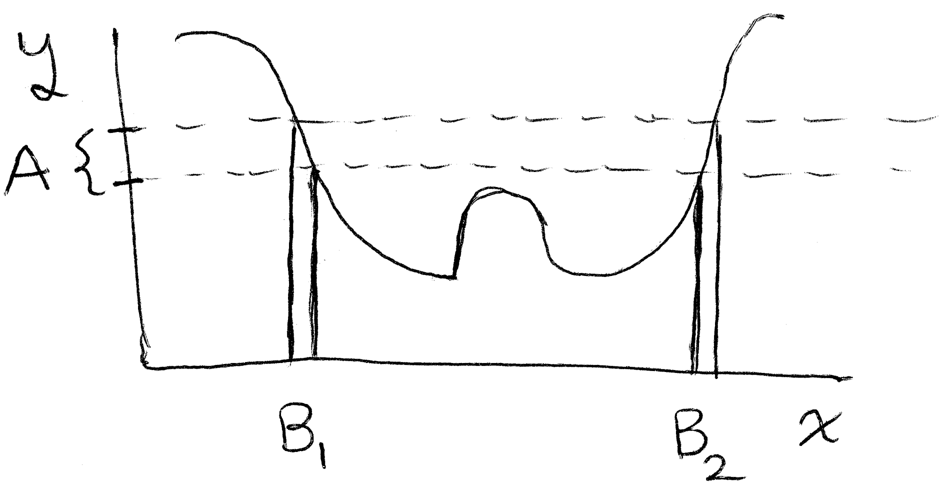
$= \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$

Finding distn. of $Y = g(X)$

Various approaches depending on properties of g and X.

General Rule (Calculation from First principles)

If $Y = g(X)$, then $P_Y(A) = P_X(g^{-1}(A))$ where for any set A we define $g^{-1}(A) = \{x : g(x) \in A\}$.



$$g^{-1}(A) = B_1 \cup B_2$$

Problem: Suppose U has a Uniform$(0,1)$ distribution with

$$\text{pdf } f_U(u) = \begin{cases} 1 & \text{for } 0<u<1 \\ 0 & \text{otherwise} \end{cases},$$

$$\text{cdf } F_U(u) = \begin{cases} u & \text{for } 0 \leq u < 1 \\ 0 & \text{for } u<0 \\ 1 & \text{for } u \geq 1 \end{cases}.$$

Define the r.v. Y by

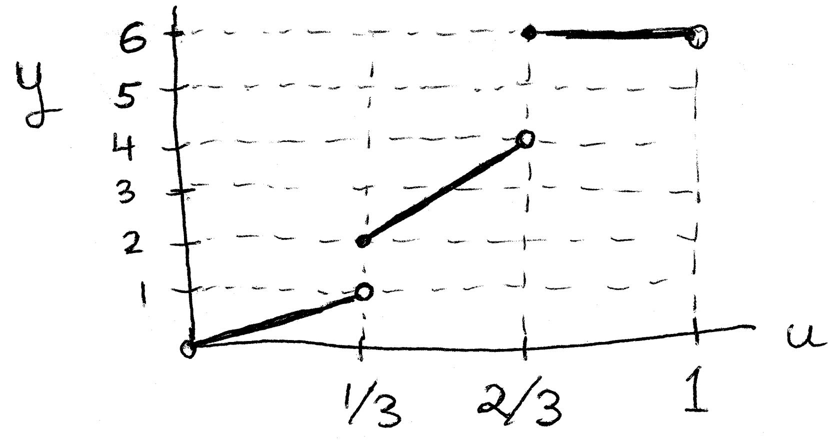
$$Y = \begin{cases} 3U & \text{for } U < 1/3 \\ 6U & \text{for } 1/3 \leq U < 2/3 \\ 6 & \text{for } U \geq 2/3. \end{cases}$$

Find the cdf of Y.

(Solution on following pages.)

Solution:

Y as a function of U looks like this:



As U ranges over $[0,1]$, the possible values of Y are

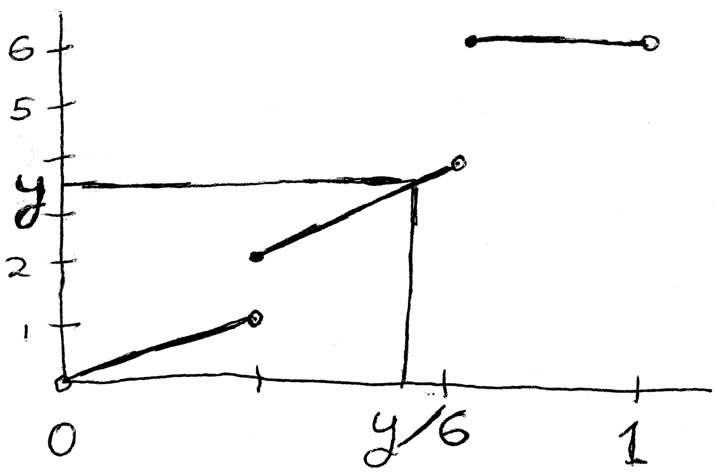
$$\mathcal{Y} = (0,1) \cup [2,4) \cup \{6\} \subset [0,6].$$

Thus $F_Y(y) = 0$ for $y < 0$

and $F_Y(y) = 1$ for $y \geq 6$.

Also $P\{1 < Y < 2\} = 0$ and $P\{4 < Y < 6\} = 0$ so that $F_Y(y)$ is flat in the intervals $(1,2)$ and $(4,6)$.

Also note that $P\{Y=6\} = P\{\frac{2}{3} \leq U < 1\} = \frac{1}{3}$ so that $F_Y(y)$ jumps by $1/3$ at the value $y = 6$.



For $2 < y < 4$,

$$\{Y \leq y\} = \{U \leq y/6\}$$

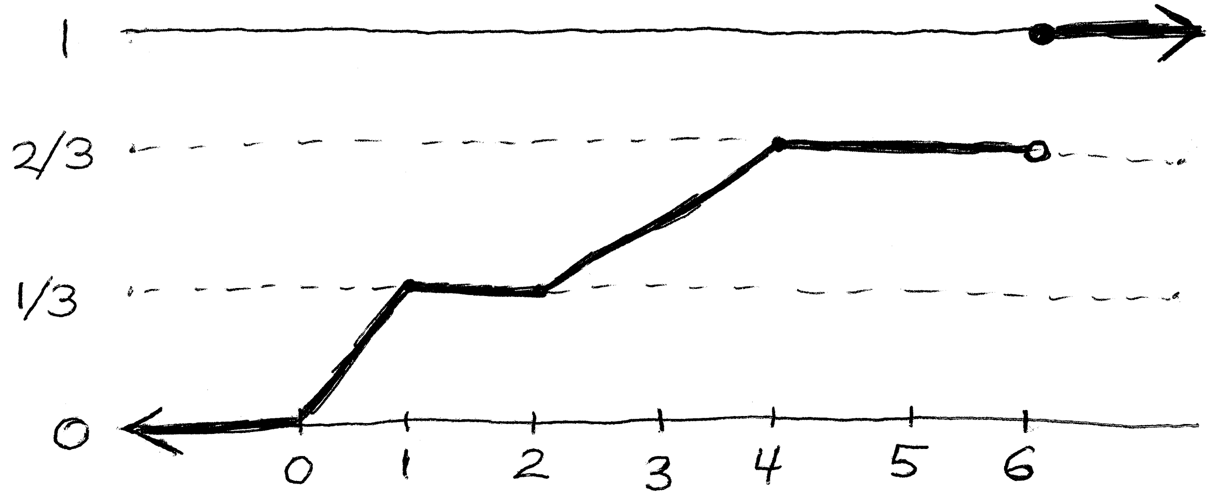
so that $F_Y(y) = F_U(y/6) = y/6$.

Similarly, for $0 < y < 1$,

$$\{Y \leq y\} = \{U \leq y/3\}$$

so that $F_Y(y) = F_U(y/3) = y/3$.

Piecing this all together gives us:



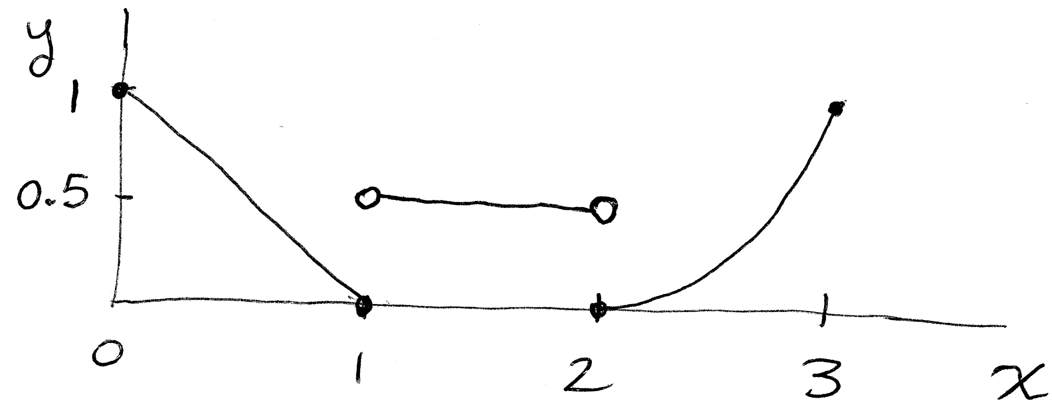
More formally

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y/3 & \text{for } 0 \le y < 1 \\ 1/3 & \text{for } 1 \le y < 2 \\ y/6 & \text{for } 2 \le y < 4 \\ 2/3 & \text{for } 4 \le y < 6 \\ 1 & \text{for } y \geq 6 \end{cases}$$

Example: Suppose X has pdf

$$f_X(x) = \begin{cases} \frac{2}{9}x & \text{for } 0 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

Define the function $y = g(x)$ by



$$g(x) = \begin{cases} 1-x & \text{for } 0 \le x \le 1 \\ 1/2 & \text{for } 1 < x < 2 \\ (x-2)^2 & \text{for } 2 \le x \le 3 \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

Define the r.v. $Y = g(X)$.

Problem: Calculate $P(\frac{1}{4} < Y < \frac{3}{4})$

$$= P_Y((\tfrac{1}{4}, \tfrac{3}{4})).$$

Let $A = (\frac{1}{4}, \frac{3}{4})$. $P_Y(A) = P_X(g^{-1}(A))$.

From the diagram and definition of g,

$$g^{-1}(A) = \left(\tfrac{1}{4}, \tfrac{3}{4}\right) \cup (1,2) \cup \left(2.5, 2+\sqrt{\tfrac{3}{4}}\right).$$

This just means

$$Y \in \left(\tfrac{1}{4}, \tfrac{3}{4}\right) \text{ iff } X \in \left(\tfrac{1}{4}, \tfrac{3}{4}\right) \cup (1,2) \cup \left(2.5, 2+\sqrt{\tfrac{3}{4}}\right).$$

Thus

$$P_Y(A) = P_X(g^{-1}(A))$$

$$= P_X\left(\left(\tfrac{1}{4}, \tfrac{3}{4}\right)\right) + P_X((1,2)) + P_X\left(\left(2.5, 2+\sqrt{\tfrac{3}{4}}\right)\right)$$

$$= \frac{1}{9}\left[\left(\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right] + \frac{1}{9}\left[2^2 - 1^2\right]$$

$$+ \frac{1}{9}\left[\left(2+\sqrt{\frac{3}{4}}\right)^2 - (2.5)^2\right]$$

since $P_X((a,b)) = \int_a^b f_X(x)\,dx = \int_a^b \frac{2}{9}x\,dx$

$$= \left.\frac{x^2}{9}\right|_a^b = \frac{1}{9}(b^2 - a^2)$$

so long as $(a,b) \subset \mathcal{X}$,
that is, $0 \leq a < b \leq 3$.

Review

r.v. X with $\underbrace{\text{range } \mathcal{X}}_{\substack{\text{set of possible values} \\ = \{x : f_X(x) > 0\} \text{ if } X \text{ has a pdf } f_X}}$.

$Y = g(X)$.

Y has range $\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$.

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① If g is strictly increasing and continuous (on some interval containing $\mathcal{X}$), then

$$F_Y(y) = F_X(g^{-1}(y)) \quad \text{for } y \in \mathcal{Y}.$$

② If, in addition, X has a pdf and g^{-1} has a continuous derivative (on some interval containing $\mathcal{Y}$), then

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) \quad \text{for } y \in \mathcal{Y}.$$

$(= 0 \text{ outside } \mathcal{Y})$

Note: $\dfrac{d}{dy} g^{-1}(y) = \dfrac{1}{g'(g^{-1}(y))}$.

(1′) If g is strictly <u>decreasing</u> and continuous (on $\mathcal{X}$), then

$$\underbrace{F_Y(y-)}_{= P\{Y<y\}} = 1 - F_X(g^{-1}(y))$$

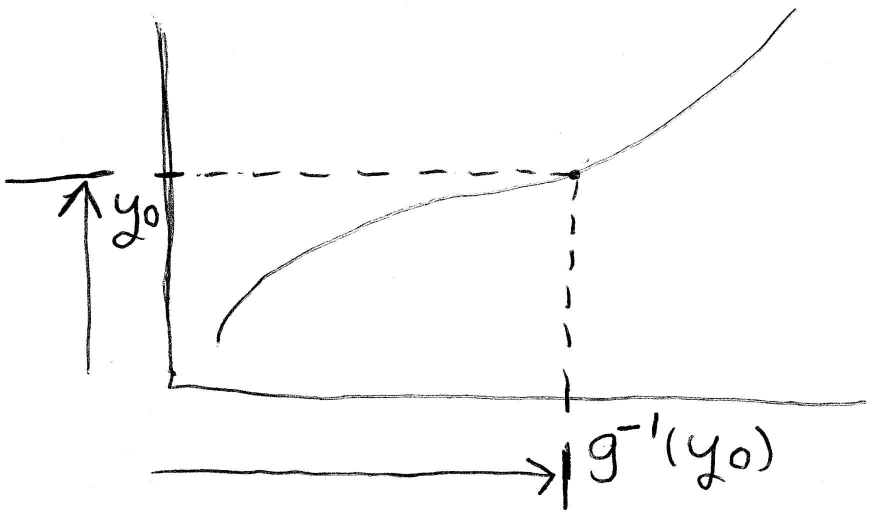
for $y \in \mathcal{Y}$.

(2′) If, in addition, X has a pdf and g^{-1} has a continuous derivative (on $\mathcal{Y}$), then

$$f_Y(y) = f_X(g^{-1}(y))\left(-\frac{d}{dy} g^{-1}(y)\right)$$

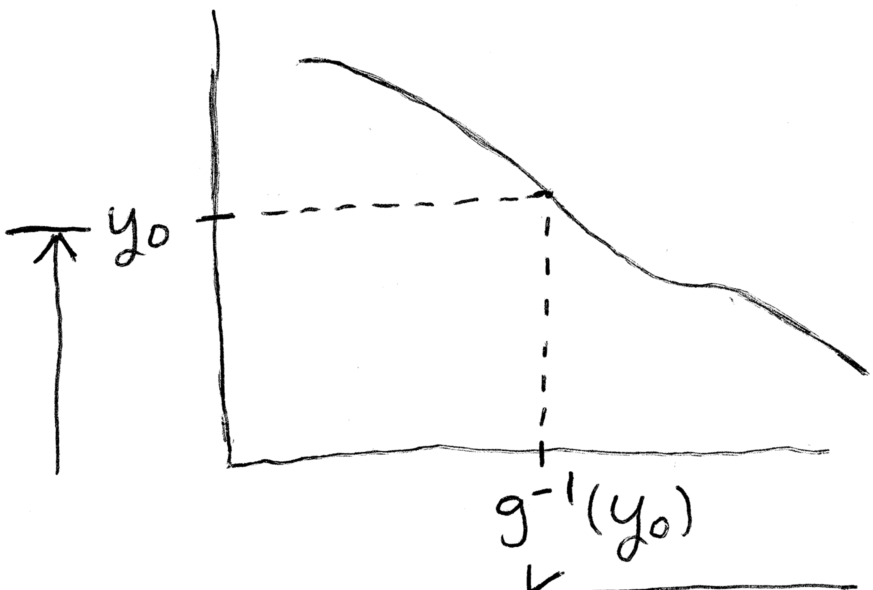
for $y \in \mathcal{Y}$

($= 0$ outside)



Plot of $y = g(x)$, continuous and strictly <u>increasing</u>

$\{Y \le y_0\}$ iff $\{X \le g^{-1}(y_0)\}$
so that $F_Y(y_0) = F_X(g^{-1}(y_0))$.



Plot of $y = g(x)$, continuous and strictly <u>decreasing</u>.

$\{Y < y_0\}$ iff $\{X > g^{-1}(y_0)\}$ so that

$$F_Y(y_0-) = P(X > g^{-1}(y_0)) = 1 - P(X \le g^{-1}(y_0))$$
$$= 1 - F_X(g^{-1}(y_0)).$$

In the earlier notation:

Let $A = (-\infty, y_0]$.

$$F_Y(y_0) = P_Y(A) = P_X(g^{-1}(A)).$$

If g is continuous and strictly increasing (with $\lim_{x \to -\infty} g(x) = -\infty$),

then $g^{-1}(A) = (-\infty, g^{-1}(y_0)]$

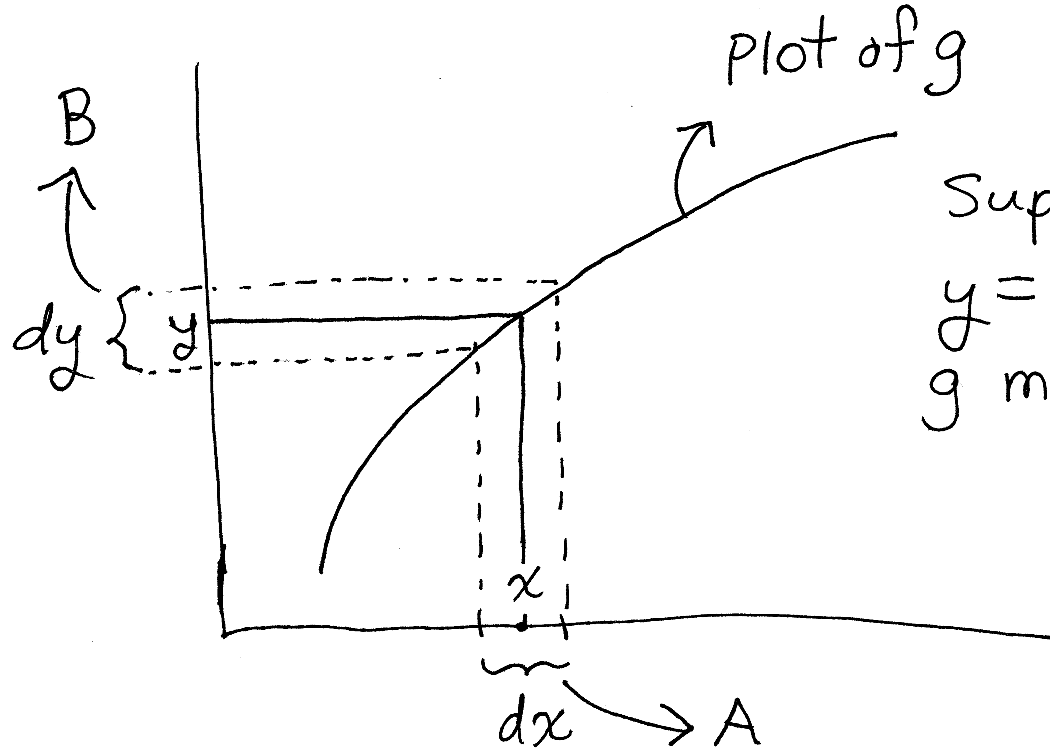
and $P_X(g^{-1}(A)) = F_X(g^{-1}(y_0))$.

If g is continuous and strictly decreasing (with $\lim_{x \to +\infty} g(x) = -\infty$),

then $g^{-1}(A) = [g^{-1}(y_0), \infty)$

and
$$\begin{aligned} P_X(g^{-1}(A)) &= P(X \geq g^{-1}(y_0)) \\ &= 1 - P(X < g^{-1}(y_0)) \\ &= 1 - F_X(g^{-1}(y_0)-). \end{aligned}$$

Formula 2: Heuristic argument



Suppose $y = g(x)$ and g maps A to B.

$$P(X \in A) \approx f_X(x)\,dx$$

↳ small interval of length dx around x

$$P(Y \in B) \approx f_Y(y)\,dy$$

↳ small interval of length dy around y

$$f_Y(y)\,dy = f_X(x)\,dx$$

implies

$$f_Y(y) = f_X(x)\frac{dx}{dy} = f_X(g^{-1}(y))\frac{d\,g^{-1}(y)}{dy}$$

Note that

$$g \text{ increasing} \Rightarrow g^{-1} \text{ increasing}$$
$$\Rightarrow \frac{d}{dy} g^{-1}(y) > 0,$$

$$g \text{ decreasing} \Rightarrow g^{-1} \text{ decreasing}$$
$$\Rightarrow \frac{d}{dy} g^{-1}(y) < 0.$$

Thus combining (2) and (2') gives

(3) If g is strictly monotonic with a continuous derivative and X has a pdf, then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \quad \text{for } y \in \mathcal{Y}$$
$$= 0 \quad \text{for } y \notin \mathcal{Y}.$$

(4) If g is <u>not</u> monotonic, break down its domain into disjoint subsets where it is monotonic. Then apply previous cases.

(Theorem 2.1.8 (p. 53) formalizes this approach.)

<u>OR</u> work from "first principles".

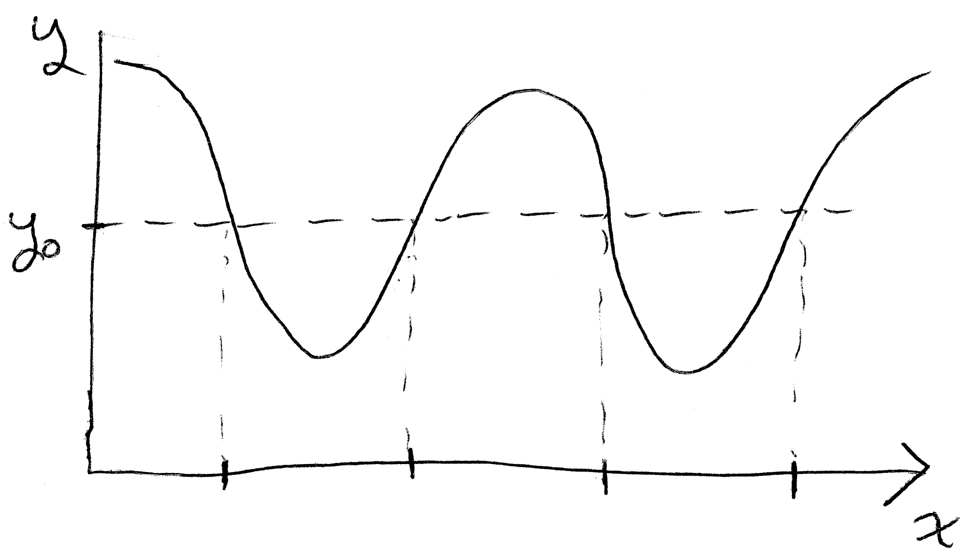
(Perhaps compute the cdf and differentiate.)

Formula in Thm. 2.1.3 is equivalent to

$$f_Y(y) = \sum_{x \in g^{-1}(\{y\})} f_X(x) \left| \frac{1}{g'(x)} \right|$$

for $y \in \mathcal{Y}$.

(obvious assumptions:
X has density.
g has derivative.
g has no flat spots.)



Example: Suppose X has a pdf.

If $Y = aX + b$, $a > 0$,

then $f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$.

From "formula" approach:

$Y = g(X)$ with $g(x) = ax + b$.

Solve $y = g(x)$ to get $x = g^{-1}(y)$:

$$y = ax + b \Rightarrow x = \frac{y-b}{a}.$$

Thus $g^{-1}(y) = \frac{y-b}{a}$, $\frac{d}{dy} g^{-1}(y) = \frac{1}{a} > 0$

leading to $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$

and $f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$.

From "first principles":

$$\{Y \le y\} = \{aX + b \le y\} = \left\{X \le \frac{y-b}{a}\right\}.$$

Thus $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$ (by taking probabities above).

Now differentiate to get the density.

<u>Example</u>: Special case of log-normal dist.

Suppose $X \sim N(0,1)$.
Find the cdf and pdf of $Y = e^X$.

First note

$$\mathcal{X} = (-\infty, \infty)$$

$$\mathcal{Y} = (0, \infty)$$

pdf of X is $\varphi(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

cdf of X is $\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,dt$.

Solution by "formula" approach:

$Y = g(X)$ with $g(x) = e^x$.

$$g^{-1}(y) = \log y \quad \text{for } y \in \mathcal{Y}.$$

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{y} \quad \text{for } y \in \mathcal{Y}.$$

$$F_Y(y) = F_X(g^{-1}(y)) = \Phi(\log y) \quad \text{for } y > 0.$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \quad \text{for } y \in \mathcal{Y}$$

$$= \frac{e^{-(\log y)^2/2}}{\sqrt{2\pi}} \cdot \frac{1}{y} \quad \text{for } y > 0.$$

Suppose H is a continuous cdf.

① If the r.v. X has cdf H, then $H(X) \sim \text{Uniform}(0,1)$.

(This is called the <u>probability integral transformation</u>.)

② If $U \sim \text{Uniform}(0,1)$, then the rv $H^{-1}(U)$ has cdf H.

(This is useful for generating rv's for simulations.)

<u>Proofs</u>: For simplicity assume also that H is <u>strictly</u> increasing. Then H^{-1} exists.

① Recall: If $Y = g(X)$ and g is strictly increasing and continuous, then $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.

In our situation

$$Y = H(X) \quad (\text{so } g = H)$$
$$\mathcal{Y} = (0,1)$$
$$F_X = H.$$

Thus $F_Y(y) = H(H^{-1}(y)) = y$ for $y \in (0,1)$

and $Y = H(X)$ has the cdf of a Uniform(0,1) dist.

Proof of 2 :

Now let $Y = H^{-1}(U)$.

Use $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$

with $X = U$ (so $F_X = F_U$)

$g = H^{-1}$ (so $g^{-1} = H$)

$\mathcal{Y} = (-\infty, \infty)$.

Thus

$$F_Y(y) = F_U(H(y)) = H(y)$$

since $F_U(u) = u$ for $u \in (0,1)$

and $H(y) \in (0,1)$.

Notation: $X \overset{d}{=} Y$ means

$$P(X \in A) = P(Y \in A) \text{ for all } A$$

or equivalently

$$P_X(A) = P_Y(A) \text{ for all } A$$

or equivalently

$$F_X(t) = F_Y(t) \text{ for all } t$$

or equivalently (if both X and Y have a pdf or pmf)

$$f_X(t) = f_Y(t) \text{ for all } t.$$

The symbol $\overset{d}{=}$ is read as "equal in distribution".

Abbreviated Version:

Suppose F is a continuous cdf.

If $X \sim F$ and $U \sim \text{Uniform}(0,1)$,

then $F(X) \overset{d}{=} U$ and $F^{-1}(U) \overset{d}{=} X$.

Example: The Cauchy distribution has

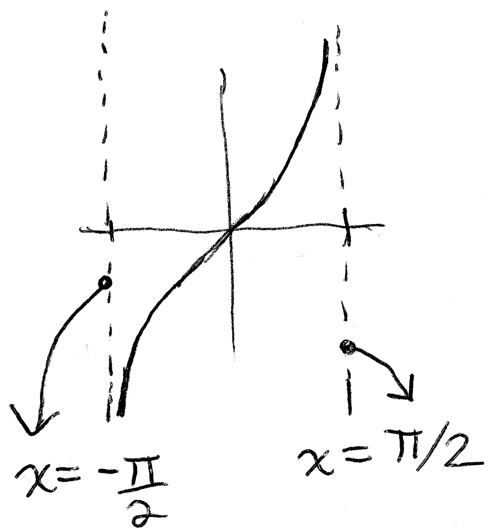
$$\text{cdf} \quad F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x ,$$

$$\text{pdf} \quad f(x) = \frac{1}{\pi} \frac{1}{1+x^2} .$$

Note: You should verify that $F(x)$ is a continuous cdf and that

$$\frac{d}{dx} F(x) = f(x) \quad \text{for all } x.$$

Recall: Plot of $y = \tan x$ looks like this.



Thus

$$\lim_{u \to -\infty} \tan^{-1} u = -\frac{\pi}{2} ,$$

$$\lim_{u \to +\infty} \tan^{-1} u = +\frac{\pi}{2} ,$$

and $\tan^{-1} u$ is a continuous, strictly increasing function of u.

Solving for x in $y = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x$ gives

$$x = F^{-1}(y) = \tan[\pi(y - \tfrac{1}{2})].$$

Therefore,

If $X \sim$ Cauchy, then $Y = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} X$

$\sim$ Uniform $(0, 1)$.

If $U \sim$ Uniform$(0, 1)$, then

$$Y = \tan\left[\pi\left(U - \frac{1}{2}\right)\right] \sim \text{Cauchy}.$$

Example : Suppose

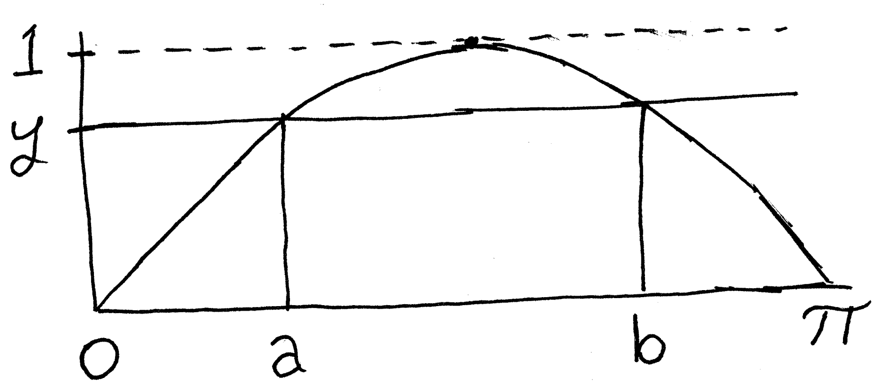
X has pdf $f_X(x) = \frac{3x^2}{\pi^3}$ for $0 < x < \pi$,

and $Y = \sin X$.

Find the pdf and cdf of Y.

Solution from "first principles".

We will obtain the cdf and differentiate this to get the pdf.



Plot of $y = \sin x$

First note that $\mathcal{X} = (0, \pi)$, $\mathcal{Y} = (0, 1]$,

and $F_X(x) = \frac{x^3}{\pi^3}$ for $0 < x < \pi$

since $F_X(x) = \int_{-\infty}^{x} f_X(u)du = \int_0^x \frac{3u^2}{\pi^3} du$

$$= \left.\frac{u^3}{\pi^3}\right|_0^x = \frac{x^3}{\pi^3}.$$

$$
\begin{aligned}
P(Y \le y) &= 1 - P(Y > y) \\
&= 1 - P(a < X < b) \quad \text{(from picture)} \\
&= 1 - (F_X(b) - F_X(a)) \\
&= 1 - \frac{b^3}{\pi^3} + \frac{a^3}{\pi^3} .
\end{aligned}
$$

Now plug in $a = \sin^{-1} y$, $b = \pi - \sin^{-1} y$:

$$F_Y(y) = 1 - \frac{(\pi - \sin^{-1} y)^3}{\pi^3} + \frac{(\sin^{-1} y)^3}{\pi^3}$$

for $0 < y < 1$,

$= 0$ for $y \le 0$,

$= 1$ for $y \ge 1$.

Now differentiate to get the pdf using

$$\frac{d}{dy}(\sin^{-1} y) = \frac{1}{\cos(\sin^{-1} y)} = \frac{1}{\sqrt{1-y^2}} .$$

This gives

$$f_Y(y) = \frac{3}{\pi^3}\left[(\pi - \sin^{-1} y)^2 + (\sin^{-1} y)^2\right]\frac{1}{\sqrt{1-y^2}}$$

for $0 < y < 1$

($= 0$ otherwise).

Continuation of Example

Now find pdf $f_Y(y)$ using formula:

If $Y = g(X)$, then (for $y \in \mathcal{Y}$)

$$f_Y(y) = \sum_{x \in g^{-1}(\{y\})} f_X(x) \left| \frac{1}{g'(x)} \right| .$$

In this example: $f_X(x) = \frac{3x^2}{\pi^3}$ for $0 < x < \pi$,

$g(x) = \sin x$, $g'(x) = \cos x$,

$g^{-1}(\{y\}) = \{a, b\} = \{\sin^{-1} y, \pi - \sin^{-1} y\}$,

$\cos a = \sqrt{1-y^2}$, $\cos b = -\sqrt{1-y^2}$.

Thus

$$f_Y(y) = f_X(a) \left| \frac{1}{g'(a)} \right| + f_X(b) \left| \frac{1}{g'(b)} \right|$$

$$= \frac{3a^2}{\pi^3} \left| \frac{1}{\cos a} \right| + \frac{3b^2}{\pi^3} \left| \frac{1}{\cos b} \right|$$

$$= \frac{3}{\pi^3} \left[(\sin^{-1} y)^2 + (\pi - \sin^{-1} y)^2 \right] \frac{1}{\sqrt{1-y^2}}$$

for $0 < y < 1$

(same as before).

Variation of Theorem 2.1.8

Suppose

X has pdf $f_X(x)$,

$\mathcal{X} = \{x : f_X(x) > 0\}$,

$Y = g(X)$

$\mathcal{X} = C \cup A_1 \cup A_2 \cup \cdots \cup A_K$

where $C, A_1, \ldots, A_K$ are disjoint,

$P(X \in C) = 0$,

$g_i(x) = g(x)$ for $x \in A_i$ $(i = 1, \ldots, K)$

g_i is "smooth" and strictly monotonic on A_i,

$B_i = \{y : y = g_i(x) \text{ for some } x \in A_i\}$

$= \text{range of } g_i$.

Then

$$f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| I_{B_i}(y)$$

where $I_{B_i}(y) = \begin{cases} 1 & \text{for } y \in B_i \\ 0 & \text{otherwise} \end{cases}$

Example: Suppose

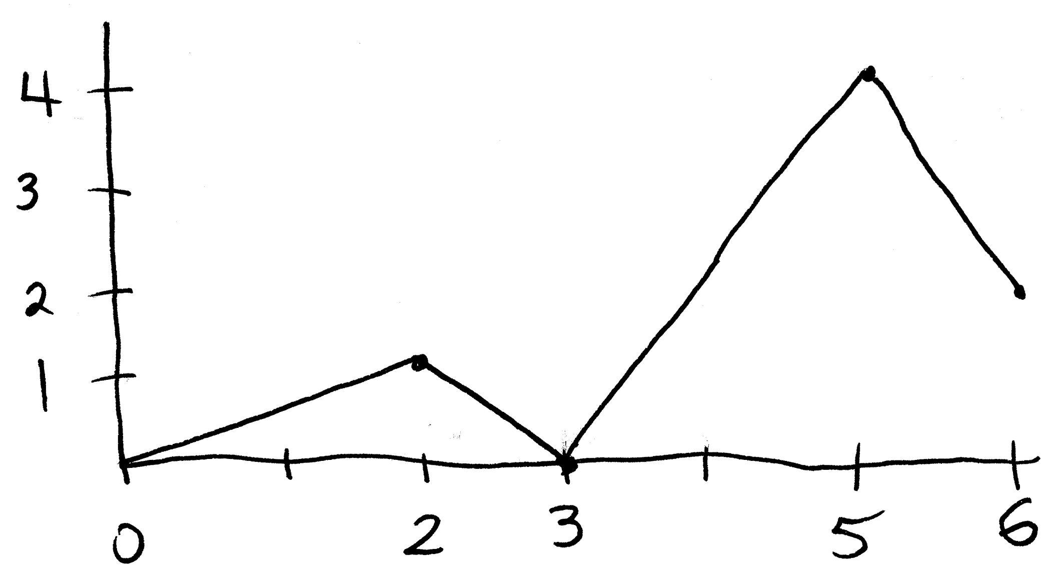
$f_X(x) = x^2/72$ for $0<x<6$

$Y = g(X)$ where

$$g(x) = \begin{cases} \frac{1}{2}x & \text{for } 0<x\le 2 \\ 3-x & \text{for } 2<x\le 3 \\ 2x-6 & \text{for } 3<x\le 5 \\ 14-2x & \text{for } 5<x<6 \end{cases}$$

Find the pdf of Y.

A plot of $g(x)$:



$A_1 = (0,2)\,,\; B_1 = (0,1)$

$y = g_1(x) = \frac{1}{2}x$

$\Rightarrow\; x = 2y = g_1^{-1}(y)$

$A_2 = (2,3)\,,\; B_2 = (0,1)$

$y = g_2(x) = 3 - x$

$\Rightarrow\; x = 3 - y = g_2^{-1}(y)$

$A_3 = (3,5)\,,\; B_3 = (0,4)$

$y = g_3(x) = 2x - 6$

$\Rightarrow\; x = \frac{1}{2}(y+6) = g_3^{-1}(y)$

$A_4 = (5,6)\,,\; B_4 = (2,4)$

$y = g_4(x) = 14 - 2x$

$\Rightarrow\; x = \frac{1}{2}(14 - y) = g_4^{-1}(y)$

$$\frac{(2y)^2}{72} |2| I_{(0,1)}(y) \qquad (1)$$

$$+ \frac{(3-y)^2}{72} |-1| I_{(0,1)}(y) \qquad (2)$$

$$+ \frac{\left(\frac{1}{2}(y+6)\right)^2}{72} \left|\tfrac{1}{2}\right| I_{(0,4)}(y) \qquad (3)$$

$$+ \frac{\left(\frac{1}{2}(14-y)\right)^2}{72} \left|-\tfrac{1}{2}\right| I_{(2,4)}(y) \qquad (4)$$

$$f_Y(y) = \begin{cases} (1)+(2)+(3) & \text{for } 0<y<1 \\ (3) & \text{for } 1<y<2 \\ (3)+(4) & \text{for } 2<y<4 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{73}{576}y^2 - \frac{y}{16} + \frac{3}{16} & \text{for } 0<y<1 \\ \frac{(y+6)^2}{576} & \text{for } 1<y<2 \\ \frac{y^2}{288} - \frac{y}{36} + \frac{29}{72} & \text{for } 2<y<4 \end{cases}$$

0 otherwise