

A Location-Scale Family of distributions has densities (pdf's) of the form

$$g(x|\mu, \sigma) = \frac{1}{\sigma} \psi\left(\frac{x-\mu}{\sigma}\right) \quad \text{where}$$

ψ is a pdf, $\sigma > 0$,
 $-\infty < \mu < \infty$.

Properties

1. $g(x|0, 1) = \psi(x)$
2. If $X \sim g(\cdot|\mu, \sigma)$, then $\frac{X-\mu}{\sigma} \sim g(\cdot|0, 1)$.
3. If $X \sim g(\cdot|0, 1)$, then $\sigma X + \mu \sim g(\cdot|\mu, \sigma)$.

(Prove 2 and 3 by the material in Section 2.1.)

Example: The normal distributions are a Location-scale family.

Take $\psi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. (The standard normal distn.)

$$\begin{aligned} \text{Then } g(x|\mu, \sigma) &= \frac{1}{\sigma} \psi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2} \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \end{aligned}$$

is the pdf of a $N(\mu, \sigma^2)$ distn.

Example: The Cauchy Location-Scale family.

$$\text{Take } \psi(x) = \frac{1}{\pi} \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

$$\begin{aligned} \text{Then } g(x|\mu, \sigma) &= \frac{1}{\sigma} \psi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma} \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty \end{aligned}$$

defines the Cauchy L-S family.

Note: For this family of distns, μ is not the mean and σ is not the standard deviation.
(Same remark applies in next example.)

Example: The Uniform distn's form a L-S family.

$$\text{Take } \psi(x) = I_{(0,1)}(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

ψ is the pdf of the Uniform(0,1) distn.

$$\begin{aligned} \text{Then } g(x|\mu, \sigma) &= \frac{1}{\sigma} \psi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma} I_{(0,1)}\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

$$\left(\begin{array}{l} \text{Note: } \frac{x-\mu}{\sigma} \in (0,1) \text{ iff } x-\mu \in (0,\sigma) \\ \text{iff } x \in (\mu, \mu+\sigma) \end{array} \right)$$

$$= \frac{1}{\sigma} I_{(\mu, \mu+\sigma)}(x)$$

which is the pdf of the Uniform($\mu, \mu+\sigma$) distn.

Let ψ be a pdf.

A scale family of distns. has densities of the form $g(x|\sigma) = \frac{1}{\sigma} \psi\left(\frac{x}{\sigma}\right)$

where $\sigma > 0$.

(σ is the scale parameter.)

A location family of distns. has densities of the form $g(x|\mu) = \psi(x-\mu)$

where $-\infty < \mu < \infty$.

Example: The $N(\mu, 1)$ distns. form a location family.

The $N(0, \sigma^2)$ distns. form a scale family.

Take $\psi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Then

$\psi(x-\mu) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}$ which is the pdf of $N(\mu, 1)$,

$\frac{1}{\sigma} \psi\left(\frac{x}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$ which is the pdf of $N(0, \sigma^2)$.

Example : The family of Gamma(α_0, β) distns (where α_0 is any fixed value of α) forms a scale family.

$$\text{Take } \psi(x) = \frac{x^{\alpha_0-1} e^{-x}}{\Gamma(\alpha_0)}, \quad x > 0.$$

$$\text{Then } \frac{1}{\sigma} \psi\left(\frac{x}{\sigma}\right) = \frac{\frac{1}{\sigma} \left(\frac{x}{\sigma}\right)^{\alpha_0-1} e^{-x/\sigma}}{\Gamma(\alpha_0)}$$

$$= \frac{x^{\alpha_0-1} e^{-x/\sigma}}{\sigma^{\alpha_0} \Gamma(\alpha_0)}$$

which is the pdf of the Gamma(α_0, σ) distn.

Note: If we permit both α and β to vary, the family of Gamma(α, β) distns. does not form a Location-Scale family. (α is a "shape" parameter.)

Exponential Families

The family of pdf's or pmf's

$$\{f(x|\theta) : \theta \in \Theta\}$$

↑ The parameter space.

(θ might represent a single parameter or a vector of parameters.)

is an exponential family if we can write

$$f(x|\theta) = h(x)c(\theta) \exp\left(\sum_{i=1}^k w_i(\theta)t_i(x)\right)$$

valid for all x and all $\theta \in \Theta$.

This is the general K parameter exponential family (kpef).

For $k=1$, the general one parameter exponential family (1pef) has the form

$$f(x|\theta) = h(x)c(\theta) \exp\{w(\theta)t(x)\}$$

valid for all x and all $\theta \in \Theta$.

Note: We allow h to be degenerate (constant), but require all the other functions to be nondegenerate (nonconstant).

Examples of 1pef's

Exponential distributions

$$f(x|\beta) = \frac{1}{\beta} e^{-x/\beta}, x > 0, \beta > 0. \text{ (pdf)}$$

In this example

$$\theta = \beta, \Theta = (0, \infty).$$

$$f(x|\beta) = \underbrace{I_{(0,\infty)}(x)}_{h(x)} \cdot \underbrace{\frac{1}{\beta}}_{c(\theta)} \cdot \exp\left\{ \underbrace{-\frac{1}{\beta}}_{w(\theta)} \cdot \underbrace{x}_{t(x)} \right\}$$

Thus $f(x|\beta)$ forms a 1pef with the parts as identified above.

Binomial distributions

The family of Binomial(n, p) distributions with n known (fixed) is a 1pef. The pmf is

$$f(x|p) = \underbrace{\binom{n}{x} p^x (1-p)^{n-x}}_{(1-p)^n \left(\frac{p}{1-p}\right)^x}, x = 0, 1, \dots, n, 0 < p < 1$$

In this example $\Theta = p$, $\Theta = (0, 1)$, and
 $f(x|p) =$

$$\underbrace{\binom{n}{x} I_{\{0, 1, \dots, n\}}}_{h(x)} \underbrace{(1-p)^n}_{c(\theta)} \exp \left\{ \underbrace{x}_{t(x)} \cdot \underbrace{\log\left(\frac{p}{1-p}\right)}_{w(\theta)} \right\}.$$

Examples of 2 pef's

The family of $N(\mu, \sigma^2)$ distributions

The $N(\mu, \sigma^2)$ pdf is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

valid for $\sigma^2 > 0$ and $-\infty < \mu < \infty$.

Here $\theta = (\mu, \sigma^2)$ and

$$\Theta = \{(\mu, \sigma^2) : \sigma^2 > 0 \text{ and } -\infty < \mu < \infty\}.$$

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left\{-\frac{x^2}{2\sigma^2} + \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right\}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}}_{h(x)} \underbrace{\frac{1}{\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right)}_{C(\theta)} \exp\left\{\underbrace{-\frac{1}{2\sigma^2} \cdot x^2}_{w_1(\theta)t_1(x)} + \underbrace{\frac{\mu}{\sigma^2} \cdot x}_{w_2(\theta)t_2(x)}\right\}$$

We have a 2 pef with the parts as identified above.

Note: In writing an exponential family, h is allowed to be degenerate (constant), but not any of the other parts. Also, we require $h(x) \geq 0$ for all x .

Non-exponential families

There are many families of distributions which are not exponential families.

The Cauchy Location-Scale family

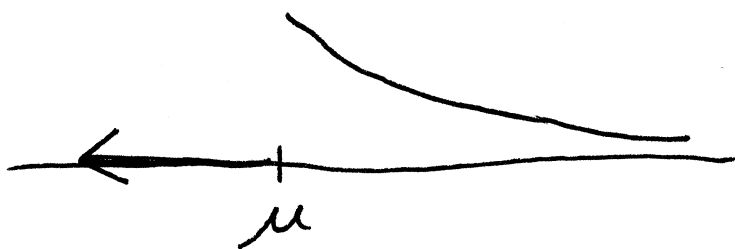
$$f(x|\mu, \sigma) = \frac{1}{\sigma} \cdot \frac{1}{\pi(1 + (\frac{x-\mu}{\sigma})^2)} \text{ for all } x$$

cannot be written as an exponential family.
(Try it!)

A trickier example:

Consider this shifted exponential distribution with pdf

$$f(x|\mu, \beta) = \frac{1}{\beta} e^{-(x-\mu)/\beta} \text{ for } x > \mu$$



Plot of pdf

$$?? \quad f(x|\mu, \beta) = \underbrace{1}_{\theta} \cdot \underbrace{\frac{e^{\mu/\beta}}{\beta}}_{h(x)} e^{-\frac{1}{\beta} \cdot x} \quad w(\theta) t(x)$$

This is not valid for all x , but only for $x > \mu$. To get an expression valid for all x , we need an indicator function.

$$f(x|\mu, \beta) = \underbrace{I_{(\mu, \infty)}(x)}_{\substack{\text{not a} \\ \text{function} \\ \text{of } x \text{ alone!}}} \underbrace{e^{\mu/\beta}}_{c(\theta)} e^{\underbrace{-\frac{1}{\beta} \cdot x}_{w(\theta)t(x)}}$$

This is not an exponential family.

Definition: The support of a pdf or pmf $f(x)$ is the set $\{x : f(x) > 0\}$.

Fact: The support of an exponential family of pdf's (pmf's) $f(x|\theta)$ is the same for all θ .

Proof (for 1pdf): The support of $f(x|\theta) = h(x)c(\theta)\exp\{w(\theta)t(x)\}$ is $\{x : h(x) > 0\}$ which does not involve θ .

Return to Previous Example

The pdf

$$f(x|\mu, \beta) = \frac{1}{\beta} e^{-(x-\mu)/\beta} \text{ for } x > \mu$$

has support $\{x: x > \mu\}$ which depends on $\theta = (\mu, \beta)$ through the value μ .

Thus (without further work) we know this is not an exponential family.

Example: The family of Uniform(a, b) distributions with $-\infty < a < b < \infty$ is not an exponential family.

The Uniform(a, b) density

$$\underbrace{f(x|a, b)}_{\theta} = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

has support $\{x: a < x < b\}$ which depends on $\theta = (a, b)$. Thus (without further work) we know this is not an exponential family.

Example: The Cauchy location-scale family is not an exponential family, but its support is the same for all $\theta = (\mu, \sigma)$.