[Solution to 4.19(b)]

$$f_{X_1,X_2}(x_1,x_2) = \frac{x_1^{\alpha_1-1}e^{-x_1}}{\Gamma(\alpha_1)} \cdot \frac{x_2^{\alpha_2-1}e^{-x_2}}{\Gamma(\alpha_2)} \quad \text{for } 0 < x_1 < \infty, 0 < x_2 < \infty\,.$$

Let $Y_1 = X_1/(X_1+X_2)$ and $Y_2 = X_1 + X_2$. The inverse transformation is $X_1 = Y_1Y_2$ and $X_2 = Y_2 - X_1 = Y_2 - Y_1Y_2 = Y_2(1-Y_1)$. Thus, the transformation is 1-1. The support of (X_1, X_2) is $\mathcal{A} = (0,\infty)\times(0,\infty)$. The support of (Y_1,Y_2) is $\mathcal{B} = (0,1)\times(0,\infty)$. (Clearly the possible values of Y_2 are 0 to ∞. For any fixed value of $Y_2 = X_1 + X_2$, the value of X_1 can range from 0 to Y_2, and thus $Y_1 = X_1/Y_2$ can range from 0 to 1.) The Jacobian of the inverse transformation $x_1 = y_1y_2, x_2 = y_2(1-y_1)$ is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2(1-y_1) + y_1y_2 = y_2\,.$$

Thus

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(y_1y_2, y_2(1-y_1))|y_2| \quad \text{for } (y_1,y_2) \in (0,1)\times(0,\infty)\,. \\ &= \frac{(y_1y_2)^{\alpha_1-1}e^{-y_1y_2}}{\Gamma(\alpha_1)} \cdot \frac{(y_2(1-y_1))^{\alpha_2-1}e^{-y_2(1-y_1)}}{\Gamma(\alpha_2)} \cdot y_2 \\ &= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1} \cdot \frac{y_2^{\alpha_1+\alpha_2-1}e^{-y_2}}{\Gamma(\alpha_1+\alpha_2)} \\ &\quad \text{for } 0 < y_1 < 1 \text{ and } 0 < y_2 < \infty. \end{aligned}$$

We now see that the joint density factors into a function of y_1 times a function of y_2 valid for all y_1, y_2. Thus Y_1 and Y_2 are independent. We have factored the joint density as a Beta(α_1,α_2) density for Y_1 times a Gamma$(\alpha_1+\alpha_2)$ for Y_2. Thus $Y_1 \sim$ Beta(α_1,α_2). In a similar way we can show that $X_2/(X_1+X_2) \sim$ Beta(α_2,α_1).

Note: If we used some other choice of Y_2 instead of $Y_2 = X_1 + X_2$ (say, $Y_2 = X_1$), then Y_1 and Y_2 would (probably) not be independent, and we could not obtain the marginal distribution of Y_1 by "inspection" as we did above. We would have to obtain the marginal density $f_{Y_1}(y_1)$ by integrating over y_2 in the joint density.