[4.45(a)] If $f_X(x)$ is known to be pdf,

to show

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2},$$

it suffices to show

$$f_X(x) \propto e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} = e^{-\frac{1}{2}\omega^2}.$$

$\uparrow$

proportional up to constant (not involving x)

Following the solution manual:

$$f_X(x) = \frac{e^{-\omega^2/2(1-\rho^2)}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(z^2-2\rho\omega z)}\,\sigma_Y\,dz$$

[Drop multiplicative constants (not involving x), but recall $\omega = \omega(x) = \frac{x-\mu_X}{\sigma_X}$]

$$\propto e^{-\frac{\omega^2}{2(1-\rho^2)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(z^2-2\rho\omega z)}\,dz$$

[Complete the square in the exponent
$z^2 - 2\rho\omega z = (z-\rho\omega)^2 - \rho^2\omega^2$]

[Factor out $e^{\rho^2 w^2/2(1-\rho^2)}$]

$$= \underbrace{e^{-\frac{w^2}{2(1-\rho^2)}} e^{\frac{\rho^2 w^2}{2(1-\rho^2)}}}_{\text{combine}} \int_{-\infty}^{\infty} \underbrace{e^{-\frac{1}{2(1-\rho^2)}(z-\rho w)^2} dz}_{\text{change of variable}}$$

change of variable
$u = z - \rho w$
$du = dz$

$$= e^{-w^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-u^2/2(1-\rho^2)} du}_{\text{constant}}$$

constant
(No longer any x since w is now gone.)

$$\propto e^{-w^2/2}$$

Done!

[4.45(b)] Since $f_{Y|X}(y|x)$ is a pdf in y for every fixed value of x, to show

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_Y\sqrt{1-\rho^2}}\; e^{-\frac{\left(y-\mu_Y-\rho\sigma_Y\left(\frac{x-\mu_X}{\sigma_X}\right)\right)^2}{2\sigma_Y^2(1-\rho^2)}}$$

it suffices to show

$$f_{Y|X}(y|x) \propto e^{-\frac{(y-\mu_Y-\rho\sigma_Y w)^2}{2\sigma_Y^2(1-\rho^2)}}$$

$\uparrow$ proportional up to constant (not involving y)

[using $w = \frac{x-\mu_X}{\sigma_X}$]

$$\propto \exp\left[-\frac{(y-\mu_Y)^2-2\rho\sigma_Y w(y-\mu_Y)}{2\sigma_Y^2(1-\rho^2)}\right]$$

[by expanding the square and dropping constant]

$$= \exp\left[\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho w\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right\}\right]$$

As in the solution manual:

$$f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x)$$

$$= \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}}{\frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-(x-\mu_X)^2/2\sigma_X^2}}$$

[Drop multiplicative constants (not involving y).
Note: The entire denominator is a constant, and the $\left(\frac{x-\mu_X}{\sigma_X}\right)^2$ term in the exponent of the numerator gives another constant.]

$$\propto \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right\}$$

Done!

[4.45 (c)]

Start from the initial expression for the joint pdf of

$U = aX + bY$

$V = Y$

given in the solution manual.

By expanding everything in the exponent it clearly has the form

$$\text{pdf } f_{U,V}(u,v) \propto \exp\{cu^2 + dv^2 + euv + fu + gv\}$$

Thus, by Lemma stated in lecture, (U, V) has a bivariate normal distribution.

By part (a), the marginals are normal.

Thus U is Normal with some mean μ_U and variance σ_U^2.

The values μ_U and σ_U^2 are calculated as in the solution manual:

$\mu_U = E(aX + bY) = a\mu_X + b\mu_Y$

$\sigma_U^2 = Var(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$.