Sylwia S. McMichael
problem A1

Let $X \sim$ Poisson (1). Calculate the following.

a) $P(X \geq 2)$

b) $E\left(\frac{1}{X+1}\right)$

Solution:

(a) $X \sim$ Poisson (1) $\Rightarrow$ $P(X=k) = \dfrac{e^{-1}}{k!}$ $\quad k = 0, 1, 2, \ldots$

$$P(X \geq 2) = P(X=2) + P(X=3) + \ldots$$

$$= \sum_{k=2}^{\infty} P(X=k) = \sum_{k=0}^{\infty} P(X=k) - P(X=0) - P(X=1)$$

$$= 1 - \frac{e^{-1}}{0!} - \frac{e^{-1}}{1!} = 1 - \frac{2}{e}$$

(b) $$E\left(\frac{1}{X+1}\right) = \sum_{k=0}^{\infty} \frac{1}{k+1} P(X=k)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{e^{-1}}{k!} = \sum_{k=0}^{\infty} \frac{e^{-1}}{(k+1)!} =$$

$$= \sum_{l=1}^{\infty} \frac{e^{-1}}{l!} = \sum_{l=0}^{\infty} P(X=l) - P(X=0) = 1 - \frac{e^{-1}}{0!} = 1 - \frac{1}{e}$$

Sonya Bowens
Exercises A
Problem 2

X has a pdf given by $f(x) = \begin{cases} cx & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

a) c? $\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \int_0^1 cx\,dx = 1 \quad \left.\frac{cx^2}{2}\right|_0^1 = \frac{1}{2}c = 1, \; c = 2$

b) $P\{1/2 < x < 3/4\} = \int_{1/2}^{3/4} 2x\,dx = \left. x^2 \right|_{1/2}^{3/4} = 3/4^2 - 1/2^2$

$= 9/16 - 1/4 = 9/16 - 4/16 = 5/16$

c) $E(\log X) = \int_0^1 \log x \, 2x\,dx$

$Eg(X) = \int g(x) f(x)\,dx$

$u = \log X \quad v = x^2$

$du = 1/x\,dx \quad dv = 2x\,dx$

$\int u\,dv = uv - \int v\,du$

$\int_0^1 \log x \, 2x\,dx = x^2 \log x - \int_0^1 x^2 \, 1/x\,dx$

$= x^2 \log x - \left.\frac{x^2}{2}\right|_0^1$

$= 1 \log 1 - 1/2 = 0 - 1/2 = -1/2$

Sonya Bowens
Exercises A
Problem 2 cont

d) $Var(x^2)$?

$$Var(x) = EX^2 - (EX)^2$$

$$Var(x^2) = EX^4 - (EX^2)^2$$

$$EX^4 = \int_0^1 x^4 \, 2x \, dx = \int_0^1 2x^5 dx = \frac{x^6}{3}\Big|_0^1 = 1/3$$

$$EX^2 = \int_0^1 x^2 \, 2x \, dx = \int_0^1 2x^3 dx = \frac{x^4}{2}\Big|_0^1 = 1/2$$

$$Var(x^2) = 1/3 - (1/2)^2 = 1/3 - 1/4 = \frac{4}{12} - \frac{3}{12} = 1/12$$

EXERCISES A

PROBLEM 3

Suppose the random variables Y_1, Y_2, Y_3 ----- are independent with $P(Y_i=0)=P(Y_i=1)=\frac{1}{2}$ Define,

$$X = \sum_{i=1}^{\infty} \frac{Y_i}{2^i} = \frac{Y_1}{2} + \frac{Y_2}{4} + \frac{Y_3}{8} + \text{-----}$$

Calculate the following:

1) $E(X)$ 2) $Var(X)$

Solution 1) $E(X) = E\left(\sum_{i=1}^{\infty} \frac{Y_i}{2^i}\right)$

$$= \sum_{i=1}^{\infty} E\left(\frac{Y_i}{2^i}\right) \quad \text{since } E\left(\sum_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} E(X_i)$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} E(Y_i)$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{1}{2} \quad \text{since } E(Y_i) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$= \frac{1}{2}$$

since $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is an infinite geometric series which converges to 1

ie $\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

2) $Var(X) = V\left(\sum_{i=1}^{\infty} \frac{Y_i}{2^i}\right)$

$$= \sum_{i=1}^{\infty} V\left(\frac{Y_i}{2^i}\right) \quad \text{since } Y_i \text{ are independent}$$

$$= \sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right)^2 V(Y_i) \quad \text{since } V(aX_i) = a^2 V(X_i)$$

$$= \sum_{i=1}^{\infty} \frac{1}{4^i} \cdot \frac{1}{4} \quad \text{since } V(Y_i) = \left(0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$= \frac{1}{4} \sum_{i=1}^{\infty} \frac{1}{4^i}$$

$$= \frac{1}{4} \cdot \frac{1}{3}$$

$$= \frac{1}{12}$$

since $\sum_{i=1}^{\infty} \frac{1}{4^i}$ is an infinite geometric series which converges to $1/3$ ie $\sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{\frac{1}{4}}{1-1/4} = \frac{1}{3}$

Exercises B

Minton

Problem 1

A monkey types 6 letters at random. Each keystroke is independent of the others with all 26 letters equally likely.

(a) P(monkey types the sequence AHA) = ?

Let $a_1 = AHA???$ (? = any possible letter)

$a_2 = ?AHA??$

$a_3 = ??AHA?$

$a_4 = ???AHA$. ($a_1, \dots, a_4$ are events)

$P(a_1) = P(a_2) = P(a_3) = P(a_4) = \left(\frac{1}{26}\right)^3$.

$a_1 \cap a_2 = \emptyset$ $\quad a_1 \cap a_3 = AHAHA?$ $\quad a_1 \cap a_4 = AHAAHA$

$a_2 \cap a_3 = \emptyset$ $\quad a_2 \cap a_4 = ?AHAHA$ $\quad a_3 \cap a_4 = \emptyset$.

$a_1 \cap a_2 \cap a_3 = \emptyset$ $\quad a_1 \cap a_3 \cap a_4 = \emptyset$ $\quad a_1 \cap a_2 \cap a_4 = \emptyset$

$a_2 \cap a_3 \cap a_4 = \emptyset$ $\quad a_1 \cap a_2 \cap a_3 \cap a_4 = \emptyset$

P(monkey types sequence AHA) = $P(a_1 \cup a_2 \cup a_3 \cup a_4)$

$= P(a_1) + \dots + P(a_4) - P(a_1 \cap a_2) - \dots -$

$P(a_3 \cap a_4) + P(a_1 \cap a_2 \cap a_3) + \dots +$

$P(a_2 \cap a_3 \cap a_4) - P(a_1 \cap a_2 \cap a_3 \cap a_4)$

$= \boxed{4\left(\frac{1}{26}\right)^3 - 2\left(\frac{1}{26}\right)^5 - \left(\frac{1}{26}\right)^6}$

Minton

(b) $P(\text{Monkey types XXXX}) = ?$

Let $a_1 = XXXX??$ (? = any possible letter)

$a_2 = ?XXXX?$

$a_3 = ??XXXX$. (a_1, a_2, a_3 are events)

$P(a_1) = P(a_2) = P(a_3) = \left(\frac{1}{26}\right)^4$.

$a_1 \cap a_2 = XXXXX?$ $\quad a_1 \cap a_3 = XXXXXX$

$a_2 \cap a_3 = ?XXXXX$ $\quad a_1 \cap a_2 \cap a_3 = XXXXXX$

$$P(\text{Monkey types XXXX}) = P(a_1 \cup a_2 \cup a_3)$$

$$= P(a_1) + P(a_2) + P(a_3) - P(a_1 \cap a_2) - P(a_2 \cap a_3) - P(a_1 \cap a_3) + P(a_1 \cap a_2 \cap a_3)$$

$$= 3\left(\frac{1}{26}\right)^4 - 2\left(\frac{1}{26}\right)^5 - \left(\frac{1}{26}\right)^6 + \left(\frac{1}{26}\right)^6$$

$$= \boxed{3\left(\frac{1}{26}\right)^4 - 2\left(\frac{1}{26}\right)^5}$$

Minton

(C) $P(\text{monkey types ART or ALL}) = ?$

Let $a_1 = ART???$ $\quad a_5 = ALL???$ $\quad$ (? = any possible letter)

$a_2 = ?ART??$ $\quad a_6 = ?ALL??$ $\quad$ ($a_1, \ldots, a_8$ are events)

$a_3 = ??ART?$ $\quad a_7 = ??ALL?$

$a_4 = ???ART$ $\quad a_8 = ???ALL$.

$$P(a_1) = \ldots = P(a_8) = \left(\frac{1}{26}\right)^3$$

$a_1 \cap a_4 = ARTART$ $\quad a_4 \cap a_5 = ALLART$

$a_1 \cap a_8 = ARTALL$ $\quad a_5 \cap a_8 = ALLALL$

All other intersections not written above are equal to the empty set.

$$P(\text{monkey types ART or ALL}) = P(a_1 \cup \cdots \cup a_8)$$

$$= P(a_1) + \ldots + P(a_8) - P(a_1 \cap a_4)$$

$$- P(a_1 \cap a_8) - P(a_4 \cap a_5) - P(a_5 \cap a_8)$$

$$= \boxed{8\left(\frac{1}{26}\right)^3 - 4\left(\frac{1}{26}\right)^6}$$

Rommel Bain
B2

2a. A monkey types 6 letters at random. Each stroke is independent of the other with probability $\frac{1}{26}$.

Let E_i = the i^{th} selected letter, $i = 1, .., 6$

# of possible ways to choose 6 distinct letters from 26 letters $= \binom{26}{6}$

# of possible ways to choose 6 letters with repetition $= 26^6$

If $P(E_1 < E_2 < \dots < E_6)$ = probability chosen are distinct and in Alpha order

then $P(E_1 < .. < E_6) = \binom{26}{6} / 26^6$

Rommel Bain

B2

2b. Probability the 6 typed letters are in Alpha order, with repetitions allowed. Probability of selecting letter $= \frac{1}{26}$.

Let $A = \{ E_1 \le E_2 \le .. \le E_6 \}$ then

$\#(A)$ = the number of unordered, with replacement arrangements of 6 letters from 26. For every combination of 6 letters, there is only one of placing the letters in Alpha order. Therefore, $\#(A) = \binom{26+6-1}{6} = \binom{31}{6}$.

$P(A) = \frac{\#(A)}{26^6} = \binom{31}{6} / 26^6$

[B2] The Monkey types 6 letters.

Let $A = \{$the letters are in alphabetic order (repetitions allowed)$\}$.

$$P(A) = \frac{\#(A)}{26^6} = \frac{\binom{31}{6}}{26^6}$$

To determine $\#(A)$:

Let N_A = # of A's typed by monkey

$\vdots$

N_Z = # of Z's typed by monkey.

$(N_A, N_B, \ldots, N_Z)$ is a 26-tuple of nonnegative integers with $N_A + N_B + \cdots + N_Z = 6$.

The number of such 26-tuples is

$$\binom{26+6-1}{6} = \binom{31}{6}$$ (see solution to 1.19 in Misc. Exercises)

We can now view counting $\#(A)$ as the following:

Task 1: Choose a 26-tuple $(N_A, N_B, \ldots, N_Z)$ — $\binom{31}{6}$ ways

Task 2: Place the N_A A's, N_B B's, N_C C's $\cdots$ in alphabetical order — 1 way

---

$$\#(A) = \binom{31}{6} \times 1 = \binom{31}{6}$$

Exercise B

Yoonjung Lee.

Problem 3. Suppose n people play Russian roulette. Each person has a gun which fires with probability π when the trigger is pulled. (Assume the guns are independent of each other and successive shots of the same gun are independent). A round of play consists of every one who is still alive raising the guns to their temples and firing simultaneously.

Let n people: $x_1, x_2, \cdots x_n$.

(a). What is the probability that one or more people are still alive after k rounds of play?

P(one or more people are still alive after k rounds)

$= 1 - P($ n people die before k^{th} round or at k^{th} round$)$.

$= 1 - [\, 1 - P(\, x_1$ is still alive after k^{th} round$)\,]^n$

(by the independence assumption).

$= 1 - [\, 1 - P($ successive k shots fail$)\,]^n$

$= 1 - [\, 1 - (1-\pi)^k \,]^n$

Comment: Some people may benefit from seeing this same argument stated more formally, so here it is.

Let $A_i = \{$ person i still alive after k rounds $\}$.

$$\begin{aligned}
P(A_1 \cup A_2 \cup \cdots \cup A_n) &= 1 - P(A_1^c \cap A_2^c \cap \cdots A_n^c) \text{ (by DeMorgan's Law)} \\
&= 1 - \prod_{i=1}^{n} P(A_i^c) \\
&= 1 - \prod_{i=1}^{n} (1 - P(A_i)) \\
&= 1 - (1 - (1-\pi)^k)^n
\end{aligned}$$

Yoonjung Lee.

(b). The last person (or persons) to die receives a prize (flowers on the grave). What is the probability this prize goes to only one person?

< solution >

Let E : the event that this prize goes to only one person.

F_i : the event that only the X_i receives the prize. $(i=1,2,\dots n)$.

Since everyone has equal chance to be a winner, $P(F_1) = P(F_2) = \cdots P(F_n)$.

Since F_i's are disjoint, and $\bigcup_{i=1}^{n} F_i = E$, we obtain ;

$P(E) = n \cdot P(F_1)$.

To compute $P(F_1)$

let D_k : the event that x_1 die k^{th} round.

Then $P(F_1) = \sum_{k=2}^{\infty} P(D_k) \cdot P(F_1 | D_k)$. $\quad P(D_k) = (1-\pi)^{k-1} \cdot \pi$.

$P(F_1 | D_k) = P($ $n-1$ people die before k^{th} round$)$

$= [P(X_2$ die before k^{th} round$)]^{n-1}$

$= [1 - P(X_2$ die after $(k-1)^{th}$ round$)]^{n-1}$

$= [1 - P($successive $k-1$ shots fail$)]^{n-1}$

$= [1-(1-\pi)^{k-1}]^{n-1}$.

So $P(E) = n \cdot \sum_{k=2}^{\infty} (1-\pi)^{k-1} \pi [1-(1-\pi)^{k-1}]^{n-1}$.

B PROBLEM 4

SHOW THAT:

$$P(A \cap B^c \cap C^c) = P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Solution: OUR PROOF IS BASED ON REPEATEDLY USING THE FACT:

$$P(A \cap B^c) = P(A) - P(A \cap B)$$ FROM THEOREM 1.2.2(a) ON PAGE 10

× × ×

$$P(A \cap B^c \cap C^c) = P((A \cap B^c) \cap C^c)$$

$$\begin{aligned} P((A \cap B^c) \cap C^c) &= P(A \cap B^c) - P((A \cap B^c) \cap C) \\ &= P(A \cap B^c) - P((A \cap C) \cap B^c) \\ &= P(A) - P(A \cap B) - [P(A \cap C) - P(A \cap C \cap B)] \\ &= P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) \end{aligned}$$

$\blacksquare$

Exercises B

Problem 5

note:

$$P(A \cap B^c \cap C^c) = P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

To have a worthless hand in poker, there should be 1) no repeated values, 2) no runs of 5, ~~and~~ or 3) not all cards have the same suit. We set A, B^c, C^c accordingly.

A: no repeated values

B^c: no runs of 5 $\Rightarrow$ B: having a run of 5

C^c: not all 5 cards have same suit $\Rightarrow$ C: all 5 have same suit.

We are looking for $P(A \cap B^c \cap C^c)$.

Calculate

$$P(A) = \frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = .5071 \approx .51$$

note: lose 4 choices everytime you pick a card, so as not to repeat the value.

$$P(A \cap B) = P(B) = \frac{\underbrace{10}_{\text{\# of possible runs of 5}} \times \underbrace{4^5}_{\text{choices for suit of each card}}}{\binom{52}{5}} = .004$$

= Holds since having a run of five means holding 5 distinct values

$$P(A \cap C) = \frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \approx .002$$

$$P(A \cap B \cap C) = \frac{\underbrace{10}_{\text{\# of possible runs}} \times \underbrace{4}_{\text{choices for repeated suit}}}{\binom{52}{5}} = .000015$$

cont. →

$$\therefore \quad P(A \cap B^c \cap C^c) = P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

$$= .5071 - .004 - .002 + .000015$$

$$= .501115$$

Sonya Bowens
Exercises B
Problem 6

Consider the "Gambler's Ruin" problem with a fair coin as discussed in class. Suppose your initial fortune is \$10 dollars and your goal is to reach \$100 dollars.

As discussed in class:

$A = \{$event that you reach your goal (g) given initial fortune of $(z)\}$

$P(A) = f(z) = z/g$

a) What is the probability of achieving that goal?

$P(A) = 10/100 = 1/10$

b) Given that you reached your goal, what is the probability you lost the first 3 tosses?

$C = \{$event 1^{st} 3 tosses is Tails$\}$

$$P(C|A) = \frac{P(C)\,P(A|C)}{P(A)}$$

$P(A|C) = f(z-3)$ $\{$from $P(A|B_2) = z-1$ in class$\}$

$P(C) = 1/2 \cdot 1/2 \cdot 1/2 = 1/8$

$$P(C|A) = \frac{1/8\ f(z-3)}{f(z)} = \frac{1/8\left(\frac{z-3}{g}\right)}{z/g}$$

$$= \frac{1/8\,(7/100)}{1/10} = 7/80 = .0875$$

(B6) (c) Given that you reached the goal, what is the
(cont.) probability your holdings never drop below \$10 dollars?

Let $\underline{a}$ be the amount that you cannot drop below.

Suppose the game is over if you ever reach $\underline{a}$ or $\underline{g}$, where $a < g$. Starting with an initial fortune of z, where $a < z < g$, the probability that you reach your goal g is $h(z) = \frac{z-a}{g-a}$.

Let $\underline{B}$ be the event that you never drop below initial fortune.

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{h(z)}{f(z)} = \frac{\frac{z-a}{g-a}}{z/g}$$

($a = 9 \Rightarrow$ you should never achieve this value)

$$\frac{\frac{z-a}{g-a}}{z/g} = \frac{\frac{10-9}{100-9}}{1/10} = \frac{1/91}{1/10} = 10/91$$

$$= .10989109\overline{9} \approx .11$$

Sonya Bowens
Exercise B
Prob 6 cont

Problem B7

(a) The Law of Total Probability leads to

$$\psi(z) = p\,\psi(z+1) + (1-p)\,\psi(z-1)$$

(for integer values of z with $1 \le z \le g-1$).

(b) Clearly $\psi(0) = 0$ and $\psi(g) = 1$.

Let $c = 1 - \left(\frac{1-p}{p}\right)^g$.

$$p\,\psi(z+1) + (1-p)\,\psi(z-1)$$

$$= \frac{1}{c}\left[p - \frac{(1-p)^{z+1}}{p^z} + (1-p) - \frac{(1-p)^z}{p^{z-1}}\right]$$

$$= \frac{1}{c}\left[1 - \frac{(1-p)^z}{p^z}\Big((1-p) + p\Big)\right] = \psi(z)$$

(c) $\psi(10) \doteq .00041$ compared with $.10$ in problem 6(a). A little bias makes a big difference!

Tom Jagger
Problem B.8.

Fair coin, Tosses are independant $X_i = +1$ or $X_i = -1$
$P(A_i) = P(X_i = 1)$, 3 tosses X_1, X_2, X_3, 4rth toss $= X_1 \cdot X_2 \cdot X_3 \cdot X_4$. $P(A_i) = 1/2$ for $i = 1, 2, 3$.

1.) We must show the following for 4 events say A, B, C, D to be mutually independant.

1.) Pair wise intersections

$P(A \cap B) = P(A) \cdot P(B)$,
$P(A \cap C) = P(A) \cdot P(C)$
$P(A \cap D) = P(A) \cdot P(D)$
$P(B \cap C) = P(B) \cdot P(C)$
$P(B \cap D) = P(B) \cdot P(D)$
$P(C \cap D) = P(C) \cdot P(D)$

2.) Triple way intersections

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$
$P(A \cap B \cap D) = P(A) \cdot P(B) \cdot P(D)$
$P(B \cap C \cap D) = P(B) \cdot P(C) \cdot P(D)$

3.) 4 way intersection

$P(A \cap B \cap C \cap D) = P(A) \cdot P(B) \cdot P(C) \cdot P(D)$.

2.) and 3.). Here is a table of tosses

$w = \{$set of 3 tosses$\}$

Deliniation of all elements of $\Omega = (X_1, X_2, X_3)$

$P(\omega)$	X_1	X_2	X_3	X_4
1/8	-1	-1	-1	-1
1/8	-1	-1	1	1
1/8	-1	1	-1	1
1/8	-1	1	1	-1
1/8	1	-1	-1	1
1/8	1	-1	1	-1
1/8	1	1	-1	-1
1/8	1	1	1	1

Remarks

$P(A_4) = P(X_4 = 1) = 1/2$

Probabilities assume independance of X_1, X_2, X_3 random variables.

$P(A_1 \cap A_4) = P(A_2 \cap A_4) = P(A_3 \cap A_4) = 1/4$
$P(A_1 \cap A_2 \cap A_4) = P(A_1 \cap A_3 \cap A_4) = P(A_2 \cap A_3 \cap A_4) = 1/8$
$P(A_1 \cap A_2 \cap A_3 \cap A_4) = 1/8$

Tom Jagger Problem B.8 page 2

2.) Show that $\{A_1, A_2, A_3, A_4\}$ are not mutually independant.

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(X_1 = 1 \text{ and } X_2 = 1 \text{ and } X_3 = 1 \text{ and } X_4 = 1)$$
$$= 1/8 \text{ from table}$$

$P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) = 1/16$ since $P(A_4) = 1/2$ from table.

Since they are not equal, the sets $\{A_1, A_2, A_3, A_4\}$ are <u>NOT</u> independant

3.) Show that. A_1, A_2, A_4 are independant.

1.) $P(A_1 \cap A_2) = P(X_1 = 1 \text{ and } X_2 = 1) = 1/4 = P(A_1) \cdot P(A_2)$

$P(A_1 \cap A_4) = P(X_1 = 1 \text{ and } X_4 = 1) = 1/4 = P(A_1) \cdot P(A_4)$
From table.

$P(A_2 \cap A_4) = P(X_2 = 1 \text{ and } X_4 = 1) = 1/4 = P(A_2) \cdot P(A_4)$
From table.

2.) $P(A_1 \cap A_2 \cap A_4) = P(X_1 = 1 \text{ and } X_2 = 1 \text{ and } X_4 = 1) = 1/8$ from table.

$$= P(A_1) \cdot P(A_2) \cdot P(A_4).$$

QED