[7.22] (extra part)

As $n \to \infty$ (of order constant $\times \frac{1}{n}$)

$$\left(\frac{\sigma^2/n}{\sigma^2/n + \tau^2} \right) \mu = \underbrace{O\left(\tfrac{1}{n}\right)}$$

and $$\frac{\tau^2}{\tau^2 + (\sigma^2/n)} = 1 + O\left(\tfrac{1}{n}\right)$$

so that $E(\theta | \bar{x}) = \bar{x} + O\left(\frac{1}{n}\right)$.

(This means $E(\theta | \bar{x}) = \bar{x} +$ "error" where "error" goes to zero (as $n \to \infty$) at a rate like $\underline{\frac{\text{constant}}{n}}$.)

Similarly

$$\frac{\tau^2}{(\sigma^2/n) + \tau^2} = 1 + O\left(\tfrac{1}{n}\right) \quad \text{as } n \to \infty$$

so that $Var(\theta | \bar{x}) = \frac{\sigma^2}{n}\left(1 + O\left(\frac{1}{n}\right)\right)$.

For this problem (where σ^2 is assumed known), the MLE for θ is $\bar{X}$ and the estimated variance (which is <u>exact</u> in this case) is σ^2/n. So, we have asymptotic agreement.

[7.24] (extra part)

Let $\bar{x} = \frac{y}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Then (as $n \to \infty$)

$$E(\lambda|y) = \frac{n\beta}{n\beta+1}\bar{x} + \frac{1}{n\beta+1}\alpha\beta$$

$$= \left[1 + O\left(\tfrac{1}{n}\right)\right]\bar{x} + O\left(\tfrac{1}{n}\right)$$

$$= \bar{x} + O\left(\tfrac{1}{n}\right)$$

$$Var(\lambda|y) = \frac{\left(\bar{x} + \frac{\alpha}{n}\right)}{n}\frac{n^2\beta^2}{(n\beta+1)^2}$$

$$= \frac{\bar{x}}{n}\left(1 + \frac{\alpha}{n\bar{x}}\right)\left(\frac{n\beta}{n\beta+1}\right)^2$$

$$= \frac{\bar{x}}{n}\left(1 + O\left(\tfrac{1}{n}\right)\right).$$

The MLE is $\hat{\lambda} = \bar{x}$ and its estimated variance is $Var\,\bar{X} = \frac{Var\,X_1}{n} = \frac{\lambda}{n}$ with λ estimated by $\bar{x}$ to give $\frac{\bar{x}}{n}$.

So, we have asymptotic agreement.

[7.42]

Do part (b) first. (It is easier.)

For convenience, let $\tau_i = \frac{1}{\sigma_i^2}$.

$$W^* = \frac{\sum \tau_i W_i}{\sum \tau_i}.$$

$$\mathrm{Var}(W^*) = \left(\frac{1}{\sum \tau_i}\right)^2 \sum \tau_i^2 \underbrace{\mathrm{Var}\, W_i}_{= \sigma_i^2 = \frac{1}{\tau_i}}$$

$$= \left(\frac{1}{\sum \tau_i}\right)^2 \left(\sum \tau_i\right) = \frac{1}{\sum \tau_i}$$

as desired.

(a) There are many ways to prove W^* has the minimum variance. One way is to use Lagrange multipliers. Another approach is to try and reduce the problem to a well-known inequality. We take the second approach. We shall reduce the problem to a special case of the following lemma.

[7.42 continued]

Lemma: Let $p_1, p_2, \ldots, p_K$ satisfy $p_i \geq 0$ and $\sum p_i = 1$. The minimum value of $\sum x_i^2 p_i$ subject to the constraint $\sum x_i p_i = b$ is attained by setting $x_i = b$ for $i = 1, 2, \ldots, k$.

Proof of Lemma: We see that

$$\sum x_i^2 p_i = EX^2 \text{ and } \sum x_i p_i = EX$$

where X is a r.v. with $P(X = x_i) = p_i$ for $i = 1, \ldots, k$. So, the problem is equivalent to choosing $x_1, \ldots, x_K$ to minimize EX^2 subject to $EX = b$. Since $\text{Var}\, X \geq 0$, we know that $EX^2 \geq (EX)^2 = b^2$ so that $\sum x_i^2 p_i \geq b^2$. Clearly, this minimum value of b^2 is achieved by setting $x_i = b$ for all i.

[7.42 continued]

We continue to use $\tau_i = 1/\sigma_i^2$.

$Var(\sum a_i W_i) = \sum (a_i^2/\tau_i)$ and the condition $E_\theta(\sum a_i W_i) = \theta$ is equivalent to $\sum a_i = 1$. Thus our problem is to

minimize $\sum \frac{a_i^2}{\tau_i}$ subject to $\sum a_i = 1$

equivalently

minimize $\sum \frac{a_i^2}{\tau_i^2} \cdot \tau_i$ subject to $\sum \frac{a_i}{\tau_i} \cdot \tau_i = 1$

which (dividing both sides by $\sum \tau_j$) is equivalent to

[7.42 continued]

minimize $\sum \frac{a_i^2}{\tau_i^2} \cdot \left(\frac{\tau_i}{\sum_j \tau_j}\right)$

subject to $\sum \frac{a_i}{\tau_i} \cdot \left(\frac{\tau_i}{\sum_j \tau_j}\right) = \frac{1}{\sum_j \tau_j}$

which is equivalent to

minimize $\sum x_i^2 p_i$

subject to $\sum x_i p_i = b$

where $x_i = \frac{a_i}{\tau_i}$, $p_i = \frac{\tau_i}{\sum_j \tau_j}$, $b = \frac{1}{\sum_j \tau_j}$.

By the lemma, the minimum is achieved by taking

$$x_i = b \quad \text{for all } i$$

or $\frac{a_i}{\tau_i} = \frac{1}{\sum_j \tau_j}$ or $a_i = \frac{\tau_i}{\sum_j \tau_j}$

which is the desired answer.

[7.60] $X_1, \ldots, X_n$ are iid Gamma(α, β) with α known.

We know that a complete, sufficient statistic for this problem is $T = \sum X_i$ and that $T \sim$ Gamma$(n\alpha, \beta)$. Thus

$$E T^k = \int \frac{t^{k+n\alpha-1}}{\beta^{n\alpha}\Gamma(n\alpha)} e^{-t/\beta} dt$$

$$= \frac{\beta^k \Gamma(n\alpha+k)}{\Gamma(n\alpha)} \quad (\text{valid for } k > -n\alpha)$$

Setting $k=-1$ gives (recall $\Gamma(x+1) = x\Gamma(x)$)

$$E\frac{1}{T} = \frac{1}{\beta(n\alpha-1)} \quad (\text{valid if } n\alpha > 1)$$

Thus $S = \frac{n\alpha - 1}{T}$ is an unbiased estimator of $1/\beta$. It is best unbiased because it is a function of a complete, sufficient statistic.

[Extra Problem] Find the Fisher information matrix for a k-parameter exponential family with the natural parameter:

$$f(\underset{\sim}{x} \mid \underbrace{\theta_1, \dots, \theta_k}_{\theta}) = c(\theta) h(\underset{\sim}{x}) \exp\left(\sum \theta_i t_i(\underset{\sim}{x})\right).$$

The easiest approach is to use

$$I(\theta) = E\left(-\frac{\partial^2}{\partial \theta^2} \log f(\underset{\sim}{X} \mid \theta)\right)$$

which means

$$I_{ij} = E\left(-\frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(\underset{\sim}{X} \mid \theta)\right).$$

Clearly $-\dfrac{\partial^2}{\partial \theta_i \partial \theta_j} \log f = -\dfrac{\partial^2}{\partial \theta_i \partial \theta_j} \log c(\theta)$.

Thus, I is a $k \times k$ matrix with entries

$$I_{ij} = -\frac{\partial^2}{\partial \theta_i \partial \theta_j} \log c(\theta).$$