Throughout this exam (X, d) is a metric space; $\mathbb{R}$ is equipped with the usual metric.

1 (22 points). Statements

a) Define what it means for an ordered set $(S, \leq)$ to have the *least upper bound property.*

b) Define what it means for a function $f : X \to X$ to be *continuous* at a point $p \in X$.

c) Present a statement involving sequences which is equivalent to *dis*continuity of $f : X \to X$ at the point $p \in X$.

Answer.

a) Every non-empty subset of S which has an upper bound also has a least upper bound.

b) For every $\epsilon > 0$, there is a $\delta > 0$ such that $d(f(x), f(p)) < \epsilon$ whenever $d(x, p) < \delta$.

c) There is a sequence (x_n) in X and a number $\epsilon > 0$ such that the sequence (x_n) converges to p, but $d(f(x_n), f(p)) > \epsilon$ for each $n \in J$. [I also accepted "there is a sequence (x_n) converging to p, so that the image sequence $(f(x_n))$ does not converge to $f(p)$".]

2 (24 points). Compute:

a) $\limsup (-1)^n \left(\dfrac{2n+3}{5n-1} \right)$

b) $\int_0^1 x^3 d\alpha(x)$ where $\alpha(x) = \begin{cases} x^2, & x \leq \frac{1}{2} \\ 2x^2, & x > \frac{1}{2}. \end{cases}$

c) $\displaystyle\lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{2n}$.

Solution.

a) $\frac{2}{5}$.

b) Write $\alpha = \alpha_1 + \frac{1}{4}\alpha_2$ where $\alpha_1(x) = \begin{cases} x^2, & x \leq \frac{1}{2} \\ 2x^2 - \frac{1}{4}, & x > \frac{1}{2} \end{cases}$

while $\alpha_2(x) = \begin{cases} 0, & x \leq \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$. Since α_1 is continuous, we have

$$\int_0^1 x^3 d\alpha_1(x) = \int_0^{\frac{1}{2}} 2x^4\, dx + \int_{\frac{1}{2}}^1 4x^4\, dx = \frac{1}{80} + \frac{4}{5} - \frac{1}{40} = \frac{63}{80},$$

whence $\int_0^1 x^3 d\alpha = \frac{63}{80} + \left(\frac{1}{4}\right)\left(\frac{1}{8}\right) = \frac{131}{160}$.

c) The quickest approach is to note that

$$\lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{2n} = \frac{1}{\left[\lim_{n\to\infty}(1 + \frac{1}{n})^n\right]^2} = \frac{1}{e^2}.$$

One can also take logarithms and apply l'hopital's rule.

3 (42 points). Give examples of the following:

a) a countable set of irrational numbers,
b) an open cover of $\mathbb{R}$ which does not admit a finite subcover,
c) a countable compact subset of $\mathbb{R}$,
d) a disconnected subset of $\mathbb{R}^2$ whose closure is connected,
e) a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ which does not attain a maximum value,
f) a continuous function $f : (0,1) \to \mathbb{R}$ which is not uniformly continuous,
g) a non-Riemann-integrable function $f : [0,1] \to \mathbb{R}$ such that f^2 is integrable,

Examples.

a) $\sqrt{2}J$
b) $\{(-n,n) : n \in J\}$
c) $\{\frac{1}{n} : n \in J\} \cup \{0\}$
d) $N_1(0,0) \cup N_1(2,0)$
e) $f = \arctan$
f) $f(x) = \frac{1}{x}$
g) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ -1, & x \in [0,1] \backslash \mathbb{Q} \end{cases}.$

4 (16 points). Let E be an uncountable set of real numbers.

a) Prove that for some $n \in J$, the closed interval $[-n, n]$ contains uncountably many points of E.
b) Prove that E must have a limit point in $\mathbb{R}$.

Proof. For each $n \in J$, write $E_n := E \cap [-n, n]$.

For Part a), note that if each E_n were at most countable, then their union E would also be countable, contrary to assumption.

For Part b), fix n as in Part a). Since the interval $[-n, n]$ is compact, its infinite subset E_n must have a limit point p; a fortiori, p is also a limit point of E.

5 (16 points). Suppose E and F are compact subsets of a metric space X. Prove that their union $E \cup F$ is also compact.

Proof. Let $\mathcal{V}$ be an open cover of $E \cup F$. In particular, $\mathcal{V}$ covers E, whence by compactness, there is a finite subcollection $\mathcal{E}$ of $\mathcal{V}$ which also covers E. Similarly, there is a finite subcollection $\mathcal{F}$ of $\mathcal{V}$ which covers F. Then $\mathcal{E} \cup \mathcal{F}$ is a finite subcollection of $\mathcal{V}$ which covers $E \cup F$.

6 (16 points). Suppose $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing and bounded. Prove that $\lim_{x\to\infty} f(x)$ exists.

Proof. The set $S := \{f(x) : x \in \mathbb{R}\}$ is non-empty and bounded so it has a least upper bound L.

Let $\epsilon > 0$. Since $L - \epsilon$ is not an upper bound of S, we can find a real number M such that $f(M) > L - \epsilon$.

Now suppose $x \geq M$. By monotonicity, $L - \epsilon < f(M) \leq f(x) \leq L < L + \epsilon$, whence $|f(x) - L| < \epsilon$ as desired.

7 (16 points). Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and f' is bounded. Prove that f is uniformly continuous.

Proof. Choose $M > 0$ so that $|f(x)| \le M$ for all $x \in \mathbb{R}$. Let $\epsilon > 0$. Take $\delta = \frac{\epsilon}{M}$. Now suppose $|x - y| < \delta$. Applying the Mean Value Theorem, we find a number c so that

$$|f(x) - f(y)| = |(x - y)f'(c)| \le M|x - y| < M\delta = \epsilon,$$

as desired.

8 (16 points). Suppose $f : [0, 1] \to \mathbb{R}$ is Riemann integrable and $g : [0, 1] \to \mathbb{R}$ satisfies $|g(x)-g(y)| \le |f(x) - f(y)|$ for all $x, y \in [0, 1]$. Prove that g is also Riemann integrable.

Lemma. *Let I be a subset of $[0, 1]$. Then $\sup_I g - \inf_I g \le \sup_I f - \inf_I f$.*

Proof of the Lemma. Let $x, y \in I$. By definition of upper and lower bounds, we have $f(x) - f(y) \le \sup_I f - \inf_I f$. Reversing the roles of x, y, we in fact have $|f(x) - f(y)| \le \sup_I f - \inf_I f$. Putting this together with the hypothesis, we get $g(x) - g(y) \le \sup_I f - \inf_I f$. Holding y fixed. we see that $g(y) + \sup_I f - \inf_I f$ is an upper bound for the set $\{g(x) : x \in I\}$. By definition of *least* upper bound, this yields $\sup_I g \le g(y) + \sup_I f - \inf_I f$. Transposing, freeing y, and applying the definition of *greatest* lower bound then completes the proof of the lemma.

Proof of the Problem. Let $\epsilon > 0$. Apply integrability of f to get a partition P of $[0, 1]$ satisfying $U(f, P) - L(f, P) < \epsilon$. But then the Lemma tells us that $U(g, P) - L(g, P) \le U(f, P) - L(f, P) < \epsilon$, and g meets the "convenient criterion" for integrability.

9 (16 points). For each $n \in J$, let $f_n, g_n : \mathbb{R} \to \mathbb{R}$ with $0 \le f_n(x) \le g_n(x)$ for all $x \in \mathbb{R}$. Prove that if the series $\sum g_n$ converges uniformly, then so does the series $\sum f_n$.

Proof. (This is a generalization of the Weierstrass-M test, but that result cannot be used to prove this one because the *converse* of the WM test is not valid.)

Write (s_n) and (t_n) for the sequences of partial sums of given series. Uniform convergence of $\sum g_n$ tells us that the sequence (t_n) is uniformly Cauchy. But for all $x \in \mathbb{R}$ and $m, n \in J$, we have $|s_m(x) - s_n(x)| \le |t_m(x) - t_n(x)|$, so the sequence (s_n) is also uniformly Cauchy. It follows that (s_n) is also uniformly convergent, which is what it means for the series $\sum f_n$ to converge uniformly.

10 (16 points). Suppose $\mathcal{F}$ is a uniformly bounded, equicontinuous family of functions in $\mathcal{C}[0, 1]$, and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that the family of composites $\mathcal{H} := \{g \circ f : f \in \mathcal{F}\}$ is also equicontinuous.

Proof. Choose a positive real number M so that $|f(x)| \le M$ for all $f \in \mathcal{F}$ and all $x \in \mathbb{R}$.

Let $\epsilon > 0$. Since g is *uniformly* continuous on the compact interval $[-M, M]$, we can find an $\eta > 0$ so that $|g(z) - g(w)| < \epsilon$ whenever $z, w \in [-M, M]$ with $|z - w| < \eta$. Apply equicontinuity of $\mathcal{F}$ to find $\delta > 0$ so that $|f(x) - f(y)| < \eta$ whenever $f \in \mathcal{F}$ and $|x - y| < \delta$.

Now suppose $h \in \mathcal{H}$ and $|x-y| < \delta$. Then $h = g \circ f$ for some $f \in \mathcal{F}$. We then have $|f(x)-f(y)| < \eta$, whence $|h(x) - h(y)| = |g(f(x)) - g(f(y))| < \epsilon$, as desired.

Bonus (10 points). Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at 0 but discontinuous at each $a \ne 0 \in \mathbb{R}$.

Solution. Take $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$.