1 (16 points). Compute:

a) $F'(x)$ if $F(x) = \int_{2x}^{x^2} \frac{1}{\sqrt{1+t^4}} dt$,

b) $\int_0^1 x^3 d\alpha(x)$ where $\alpha(x) = \begin{cases} x^2, & x \le \frac{1}{2} \\ 3 + x^2, & x > \frac{1}{2}. \end{cases}$.

Solution. For (a), let G be an antiderviative of the integrand function, i.e., $G'(x) = \frac{1}{\sqrt{1+x^4}}$. By the Fundamental Theorem of Calculus, we have $F(x) = G(x^2) - G(2x)$. Applying the chain rule, we then get

$$F'(x) = 2xG'(x^2) - 2G'(2x) = \frac{2x}{\sqrt{1+x^8}} - \frac{2}{\sqrt{1+16x^4}}.$$

For (b), take $\alpha_1(x) = x^2$ and $\alpha_2(x) = \begin{cases} 0, & x \le \frac{1}{2} \\ 1, & x > \frac{1}{2}. \end{cases}$. Then

$$\int_0^1 x^3 d\alpha(x) = \int_0^1 x^3 d\alpha_1(x) + 3\int_0^1 x^3 d\alpha_2(x) = \frac{2}{5} + \frac{3}{8} = \frac{31}{40}.$$

2 (24 points). Give examples of the following. No proofs are required.

a) a function $f : [0,1] \to \mathbb{R}$ which is not Riemann integrable,
b) monotone functions $f, \alpha : [0,1] \to \mathbb{R}$ such that $f \notin \mathcal{R}(\alpha)$ on $[0,1]$,
c) a sequence of functions $(f_n : \mathbb{R} \to \mathbb{R})$ which converges pointwise, but not uniformly,
d) a family of continuous functions from $[0,1]$ to $\mathbb{R}$ which is not equicontinuous.

Solution.

a) Take $f(x) = 1$ for x rational and 0 otherwise.
b) The easiest course is to take $f = \alpha$, with α discontinuous, e.g., $f(0) = \alpha(0) = 0$, while $f(x) = \alpha(x) = 1$ for $0 < x \le 1$.
c) The simplest example is $f_n(x) = \frac{x}{n}$ for $n \in J$ and $x \in \mathbb{R}$.
d) The usual example is $f_n(x) = x^n$ for $n \in J$ and $x \in [0,1]$.

3 (20 points). Suppose $f : [0,1] \to [0,\infty)$ is continuous at $\frac{1}{2}$. Prove that if the lower integral $\underline{\int_0^1} f = 0$, then $f(\frac{1}{2}) = 0$.

Proof. Assume for purposes of contradiction that $f(\frac{1}{2}) > 0$. Apply continuity to find $\delta > 0$ so that $|x - \frac{1}{2}| < \delta$ implies $|f(x) - f(\frac{1}{2})| < \frac{f(\frac{1}{2})}{2}$. Consider the partition $P : 0 < \frac{1}{2} < \frac{1}{2} + \frac{\delta}{2} < 1$. Since $f(x) > \frac{f(\frac{1}{2})}{2}$ for all x in the second subinterval of this partition, we reach the contrdiction $\underline{\int_0^1} f \ge L(f,P) \ge \frac{\delta f(\frac{1}{2})}{4} > 0$.

4 (20 points). For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n(1-x)$. Prove that the sequence (f_n) is uniformly convergent.

Proofs. Clearly the sequence (f_n) converges to 0 pointwise.

First) Since $f_{n+1}(x) \leq f_n(x)$ for each $n \in J$ and $x \in [0,1]$, Dini's Theorem (7.13) tells us that the convergence is uniform.

Second) f_n attains its absolute maximum at its critical point $\frac{n}{n+1}$. Hence

$$||f_n|| = f_n\left(\frac{n}{n+1}\right) = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n < \frac{1}{n+1},$$

whence $\lim_{n\to\infty} ||f_n|| = 0$ as desired.

Third) Let $\epsilon > 0$. Then $\lim_{n\to\infty}(1-\epsilon)^n = 0$, so we can choose $N \in J$ so that $(1-\epsilon)^N < \epsilon$.

Now suppose $n \geq N$ and $x \in [0,1]$. If $x \leq 1-\epsilon$, we have $|f_n(x)| \leq (1-\epsilon)^N < \epsilon$. Otherwise, $|f_n(x)| \leq 1 - x < \epsilon$ anyway, and we are done.

5 (20 points). Suppose (f_n) is a uniformly convergent sequence of continuous functions mapping a compact metric space into $\mathbb{R}$. Prove that $\mathcal{F} := \{f_n : n \in J\}$ is an equicontinuous family of functions.

Proof. By compactness of the underlying metric space K, we know each f_n is uniformly continuous. Thus for each $n \in J$, there is a number $\delta_n > 0$ such that $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ whenever $d_K(x,y) < \delta_n$. Apply uniform convergence to get $N \in J$ so that $||f_n - f_N|| < \frac{\epsilon}{3}$ for all $n \geq N$. Finally set $\delta := \min\{\delta_1, \ldots \delta_N\}$.

It remains to show that this δ works. So suppose $n \in J$, while $x, y \in K$ with $d_K(x,y) < \delta$. If $n < N$, we have the desired inequality $|f_n(x) - f_n(y)| < \epsilon$ since $\delta \leq \delta_n$ by construction. On the other hand, when $n \geq N$, the triangle inequality yields

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\frac{\epsilon}{3} = \epsilon,$$

and all cases are covered.

Bonus (10 points). For each $n \in \mathbb{N}$, let $g_n : [0,1] \to \mathbb{R}$ by $g_n(x) = nx^n(1-x)$. Prove (in contrast to Problem 4) that the sequence (g_n) is not uniformly convergent.

Proof. The ratio test shows that the series $\sum g_n(x)$ converges for each $x \in (0,1)$, so the (g_n) converge to 0 pointwise on $[0,1]$. Since g_n is a scalar multiple of the function f_n of Problem 4, it attains its maximum at the same point $\frac{n}{n+1}$. In particular, for each $n \in J$, we have $||g_n|| = g_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^{n+1}$, whence

$$\lim_{n\to\infty} ||g_n|| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1} = \frac{1}{e} > 0,$$

so (g_n) cannot converge to 0 uniformly.