The Real and Complex Number Systems

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6. Fix $b > 1$.

   (a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that
   $$(b^m)^{1/n} = (b^p)^{1/q}.$$
   Hence it makes sense to define $b^r = (b^m)^{1/n}$.

   (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

   (c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that
   $$b^r = \sup B(r)$$
   where r is rational. Hence it makes sense to define
   $$b^x = \sup B(x)$$
   for every real x.

   (d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof: For (a): $mq = np$ since $m/n = p/q$. Thus $b^{mq} = b^{np}$. By Theorem 1.21 we know that $(b^{mq})^{1/(mn)} = (b^{np})^{1/(mn)}$, that is, $(b^m)^{1/n} = (b^p)^{1/q}$, that is, b^r is well-defined.

For (b): Let $r = m/n$ and $s = p/q$ where m, n, p, q are integers, and $n > 0, q > 0$. Hence $(b^{r+s})^{nq} = (b^{m/n+p/q})^{nq} = (b^{(mq+np)/(nq)})^{nq} = b^{mq+np} = b^{mq}b^{np} = (b^{m/n})^{nq}(b^{p/q})^{nq} = (b^{m/n}b^{p/q})^{nq}$. By Theorem 1.21 we know that $((b^{r+s})^{nq})^{1/(nq)} = ((b^{m/n}b^{p/q})^{nq})^{1/(nq)}$, that is $b^{r+s} = b^{m/n}b^{p/q} = b^r b^s$.

For (c): Note that $b^r \in B(r)$. For all $b^t \in B(r)$ where t is rational and $t \leq r$. Hence, $b^r = b^t b^{r-t} \geq b^t 1^{r-t}$ since $b > 1$ and $r - t \geq 0$. Hence b^r is an upper bound of $B(r)$. Hence $b^r = \sup B(r)$.

For (d): $b^x b^y = \sup B(x) \sup B(y) \geq b^{t_x} b^{t_y} = b^{t_x+t_y}$ for all rational $t_x \leq x$ and $t_y \leq y$. Note that $t_x + t_y \leq x + y$ and $t_x + t_y$ is rational. Therefore, $\sup B(x) \sup B(y)$ is a upper bound of $B(x + y)$, that is, $b^x b^y \geq \sup B(x + y) = b^(x + y)$.

Conversely, we claim that $b^x b^r = b^{x+r}$ if $x \in R^1$ and $r \in Q$. The following is my proof.

$$\begin{aligned}
b^{x+r} &= \sup B(x+r) = \sup\{b^s : s \leq x + r, s \in Q\} \\
&= \sup\{b^{s-r}b^r : s - r \leq x, s - r \in Q\} \\
&= b^r \sup\{b^{s-r} : s - r \leq x, s - r \in Q\} \\
&= b^r \sup B(x) \\
&= b^r b^x.
\end{aligned}$$

And we also claim that $b^{x+y} \geq b^x$ if $y \geq 0$. The following is my proof:

$(r \in Q)$

$$B(x) = \{b^r : r \leq x\} \subset \{b^r : r \leq x + y\} = B(x + y),$$

Therefore, $\sup B(x + y) \geq \sup B(x)$, that is, $b^{x+y} \geq b^x$.

Hence,

$$\begin{aligned}
b^{x+y} &= \sup B(x + y) \\
&= \sup\{b^r : r \leq x + y, r \in Q\} \\
&= \sup\{b^s b^{r-s} : r \leq x + y, s \leq x, r \in Q, s \in Q\} \\
&\geq \sup\{\sup B(x) b^{r-s} : r \leq x + y, s \leq x, r \in Q, s \in Q\} \\
&= \sup B(x) \sup\{b^{r-s} : r \leq x + y, s \leq x, r \in Q, s \in Q\} \\
&= \sup B(x) \sup\{b^{r-s} : r - s \leq x + y - s, s \leq x, r - s \in Q\} \\
&= \sup B(x) \sup B(x + y - s) \\
&\geq \sup B(x) \sup B(y) \\
&= b^x b^y
\end{aligned}$$

Therefore, $b^{x+y} = b^x b^y$.

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