Basic Topology

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1. Prove that the empty set is a subset of every set.

   Proof: For any element x of the empty set, x is also an element of every set since x does not exist. Hence, the empty set is a subset of every set.

2. A complex number z is said to be algebraic if there are integers $a_0, ..., a_n$, not all zero, such that

   $$a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n = 0.$$

   Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

   $$n + |a_0| + |a_1| + ... + |a_n| = N.$$

   Proof: For every positive integer N there are only finitely many equations with

   $$n + |a_0| + |a_1| + ... + |a_n| = N.$$

   (since $1 \leq n \leq N$ and $0 \leq |a_0| \leq N$). We collect those equations as C_N. Hence $\bigcup C_N$ is countable. For each algebraic number, we can form an equation and this equation lies in C_M for some M and thus the set of all algebraic numbers is countable.

3. Prove that there exist real numbers which are not algebraic.

   Proof: If not, $R^1 = \{$ all algebraic numbers $\}$ is countable, a contradiction.

4. Is the set of all irrational real numbers countable?

Solution: If $R - Q$ is countable, then $R^1 = (R - Q) \bigcup Q$ is countable, a contradiction. Thus $R - Q$ is uncountable.

5. Construct a bounded set of real numbers with exactly three limit points.

Solution: Put

$$A = \{1/n : n \in N\} \bigcup \{1 + 1/n : n \in N\} \bigcup \{2 + 1/n : n \in N\}.$$

A is bounded by 3, and A contains three limit points - 0, 1, 2.

6. Let E' be the set of all limit points of a set E. Prove that S' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \bigcup E'$.) Do E and E' always have the same limit points?

Proof: For any point p of $X - E'$, that is, p is not a limit point E, there exists a neighborhood of p such that q is not in E with $q \neq p$ for every q in that neighborhood.

Hence, p is an interior point of $X - E'$, that is, $X - E'$ is open, that is, E' is closed.

Next, if p is a limit point of E, then p is also a limit point of $\overline{E}$ since $\overline{E} = E \bigcup E'$. If p is a limit point of $\overline{E}$, then every neighborhood $N_r(p)$ of p contains a point $q \neq p$ such that $q \in \overline{E}$. If $q \in E$, we completed the proof. So we suppose that $q \in \overline{E} - E = E' - E$. Then q is a limit point of E. Hence,

$$N_{r'}(q)$$

where $r' = \frac{1}{2}\min(r - d(p, q), d(p, q))$ is a neighborhood of q and contains a point $x \neq q$ such that $x \in E$. Note that $N_{r'}(q)$ contains in $N_r(p) - \{p\}$. That is, $x \neq p$ and x is in $N_r(p)$. Hence, q also a limit point of E. Hence, E and $\overline{E}$ have the same limit points.

Last, the answer of the final sub-problem is no. Put

$$E = \{1/n : n \in N\},$$

and $E' = \{0\}$ and $(E')' = \phi$.

7. Let A_1, A_2, A_3, ... be subsets of a metric space. (a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$, for $n = 1, 2, 3, ...$ (b) If $B = \bigcup_{i=1}^\infty$, prove that $\overline{B} \supset \bigcup_{i=1}^\infty \overline{A_i}$. Show, by an example, that this inclusion can be proper.

   Proof of (a): (Method 1) $\overline{B_n}$ is the smallest closed subset of X that contains B_n. Note that $\bigcup \overline{A_i}$ is a closed subset of X that contains B_n, thus
$$\overline{B_n} \supset \bigcup_{i=1}^n \overline{A_i}.$$
   If $p \in \overline{B_n} - B_n$, then every neighborhood of p contained a point $q \neq p$ such that $q \in B_n$. If p is not in $\overline{A_i}$ for all i, then there exists some neighborhood $N_{r_i}(p)$ of p such that $(N_{r_i}(p) - p) \bigcap A_i = \phi$ for all i. Take $r = \min\{r_1, r_2, ..., r_n\}$, and we have $N_r(p) \bigcap B_n = \phi$, a contradiction. Thus $p \in \overline{A_i}$ for some i. Hence
$$\overline{B_n} \subset \bigcup_{i=1}^n \overline{A_i}.$$
   that is,
$$\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}.$$
   (Method 2) Since $\bigcup_{i=1}^n \overline{A_i}$ is closed and $B_n = \bigcup_{i=1}^n A_i \subset \bigcup_{i=1}^n \overline{A_i}$, $\overline{B_n} \subset \bigcup_{i=1}^n \overline{A_i}$.

   Proof of (b): Since $\overline{B}$ is closed and $\overline{B} \supset B \supset A_i$, $\overline{B} \supset \overline{A_i}$ for all i. Hence $\overline{B} \supset \bigcup \overline{A_i}$.

   Note: My example is $A_i = (1/i, \infty)$ for all i. Thus, $\overline{A_i} = [1/i, \infty)$, and $B = (0, \infty)$, $\overline{B} = [0, \infty)$. Note that 0 is not in $\overline{A_i}$ for all i. Thus this inclusion can be proper.

8. Is every point of every open set $E \subset R^2$ a limit point of E? Answer the same question for closed sets in R^2.

Solution: For the first part of this problem, the answer is yes.

(Reason): For every point p of E, p is an interior point of E. That is, there is a neighborhood $N_r(p)$ of p such that $N_r(p)$ is a subset of E. Then for every real r', we can choose a point q such that $d(p,q) = 1/2\min(r, r')$. Note that $q \neq p, q \in N_{r'}(p)$, and $q \in N_r(p)$. Hence, every neighborhood $N_{r'}(p)$ contains a point $q \neq p$ such that $q \in N_r(p) \subset E$, that is, p is a limit points of E.

For the last part of this problem, the answer is no. Consider $A = \{(0,0)\}$. $A' = \phi$ and thus $(0,0)$ is not a limit point of E.

9. Let E^o denote the set of all interior points of a set E.

(a) Prove that E^o is always open.
(b) Prove that E is open if and only if $E^o = E$.
(c) If G is contained in E and G is open, prove that G is contained in E^o.
(d) Prove that the complement of E^o is the closure of the complement of E.
(e) Do E and $\overline{E}$ always have the same interiors?
(f) Do E and E^o always have the same closures?

Proof of (a): If E is non-empty, take $p \in E^o$. We need to show that $p \in (E^o)^o$. Since $p \in E^o$, there is a neighborhood N_r of p such that N_r is contained in E. For each $q \in N_r$, note that $N_s(q)$ is contained in $N_r(p)$, where $s = \min\{d(p,q), r - d(p,q)\}$. Hence q is also an interior point of E, that is, N_r is contained in E^o. Hence E^o is always open.

Proof of (b): ($\Rightarrow$) It is clear that E^o is contained in E. Since E is open, every point of E is an interior point of E, that is, E is contained in E^o. Therefore $E^o = E$.

($\Leftarrow$) Since every point of E is an interior point of E ($E^o(E) = E$), E is open.

Proof of (c): If $p \in G$, p is an interior point of G since G is open. Note that E contains G, and thus p is also an interior point of E. Hence $p \in E^o$. Therefore G is contained in E^o. (Thus E^o is the biggest open set contained in E. Similarly, $\overline{E}$ is the smallest closed set containing E.)

Proof of (d): Suppose $p \in X - E^o$. If $p \in X - E$, then $p \in \overline{X - E}$ clearly. If $p \in E$, then N is not contained in E for any neighborhood N of p. Thus N contains an point $q \in X - E$. Note that $q \neq p$, that is, p is a limit point of $X - E$. Hence $X - E^o$ is contained in $\overline{X - E}$.

Next, suppose $p \in \overline{X - E}$. If $p \in X - E$, then $p \in X - E^o$ clearly. If $p \in E$, then every neighborhood of p contains a point $q \neq p$ such that $q \in X - E$. Hence p is not an interior point of E. Hence $\overline{X - E}$ is contained in $X - E^o$. Therefore $X - E^o = \overline{X - E}$.

Solution of (e): No. Take $X = R^1$ and $E = Q$. Thus $E^o = \phi$ and $\overline{E}^o = (R^1)^o = R^1 \neq \phi$.

Solution of (f): No. Take $X = R^1$ and $E = Q$. Thus $\overline{E} = R^1$, and $\overline{E^o} = \overline{\phi} = \phi$.

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q) \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof: (a) $d(p,q) = 1 > 0$ if $p \neq q$; $d(p,p) = 0$. (b) $d(p,q) = d(q,p)$ since $p = q$ implies $q = p$ and $p \neq q$ implies $q \neq p$. (c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$ if $p = q$. If $p \neq q$, then either $r = p$ or

$r = q$, that is, $r \neq p$ or $r \neq q$. Thus, $d(p,q) = 1 \leq d(p,r) + d(r,q)$. By (a)-(c) we know that d is a metric.

Every subset of X is open and closed. We claim that for any $p \in X$, p is not a limit point. Since $d(p,q) = 1 > 1/2$ if $q \neq p$, there exists an neighborhood $N_{1/2}(p)$ of p contains no points of $q \neq p$ such that $q \in X$. Hence every subset of X contains no limit points and thus it is closed. Since $X - S$ is closed for every subset S of X, $S = X - (X - S)$ is open. Hence every subset of X is open.

Every finite subset of X is compact. Let $S = \{p_1, ..., p_n\}$ be finite. Consider an open cover $\{G_\alpha\}$ of S. Since S is covered by G_α, p_i is covered by G_{α_i}, thus $\{G_{\alpha_1}, ..., G_{\alpha_n}\}$ is finite subcover of S. Hence S is compact. Next, suppose S is infinite. Consider an open cover $\{G_p\}$ of S, where

$$G_p = N_{\frac{1}{2}}(p)$$

for every $p \in S$. Note that q is not in G_p if $q \neq p$. If S is compact, then S can be covered by finite subcover, say

$$G_{p_1}, ..., G_{p_n}.$$

Then there exists q such that $q \neq p_i$ for all i since S is infinite, a contradiction. Hence only every finite subset of X is compact.

11. For $x \in R^1$ and $y \in R^1$, define

$$\begin{aligned} d_1(x,y) &= (x,y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2 - y^2|, \\ d_4(x,y) &= |x - 2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution: (1) $d_1(x, y)$ is not a metric. Since $d_1(0, 2) = 4$, $d_1(0, 1) = 1$, and $d_1(1, 2) = 1$, $d_1(0, 2) > d_1(0, 1) + d_1(1, 2)$. Thus $d_1(x, y)$ is not a metric.
(2) $d_2(x, y)$ is a metric. (a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$. (b) $d(p, q) = \sqrt{|p - q|} = \sqrt{|q - p|} = d(q, p)$. (c) $|p - q| \leq |p - r| + |r - q|$, $\sqrt{|p - q|} \leq \sqrt{|p - r| + |r - q|} \leq \sqrt{|p - r|} + \sqrt{|r - q|}$. That is, $d(p, q) \leq d(p, r) + d(r, q)$. (3) $d_3(x, y)$ is not a metric since $d_3(1, -1) = 0$.
(4) $d_4(x, y)$ is not a metric since $d_4(1, 1) = 1 \neq 0$.
(5) $d_5(x, y)$ is a metric since $|x - y|$ is a metric.

Claim: $d(x, y)$ is a metric, then

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric.

Proof of Claim: (a) $d'(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$. (b) $d'(p, q) = d'(q, p)$. (c) Let $x = d(p, q)$, $y = d(p, r)$, and $z = d(r, q)$. Then $x \leq y+z$.

$$\begin{aligned}
& d'(p, q) \leq d'(p, r) + d'(r, q) \\
\Leftrightarrow\quad & \frac{x}{1 + x} \leq \frac{y}{1 + y} + \frac{z}{1 + z} \\
\Leftrightarrow\quad & x(1 + y)(1 + z) \leq y(1 + z)(1 + x) + z(1 + x)(1 + y) \\
\Leftrightarrow\quad & x + xy + xz + xyz \leq (y + xy + yz + xyz) + (z + xz + yz + xyz) \\
\Leftrightarrow\quad & x \leq y + z + 2yz + xyz \\
\Leftarrow\quad & x \leq y + z
\end{aligned}$$

Thus, d' is also a metric.

12. Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, ...$ Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Proof: Suppose that $\{O_\alpha\}$ is an arbitrary open covering of K. Let $E \in \{O_\alpha\}$ consists 0. Since E is open and $0 \in E$, 0 is an interior point of E. Thus there is a neighborhood $N = N_r(0)$ of 0 such that $N \subset E$. Thus N contains

$$\frac{1}{[1/r]+1}, \frac{1}{[1/r]+2}, \ldots$$

Next, we take finitely many open sets $E_n \in \{O_\alpha\}$ such that $1/n \in E_n$ for $n = 1, 2, ..., [1/r]$. Hence $\{E, E_1, ..., E_{[1/r]}$ is a finite subcover of K. Therefore, K is compact.

Note: The unique limit point of K is 0. Suppose $p \neq 0$ is a limit point of K. Clearly, $0 < p < 1$. (p cannot be 1). Thus there exists $n \in Z^+$ such that

$$\frac{1}{n+1} < p < \frac{1}{n}.$$

Hence $N_r(p)$ where $r = \min\{\frac{1}{n} - p, p - \frac{1}{n+1}\}$ contains no points of K, a contradiction.

13. Construct a compact set of real numbers whose limit points form a countable set.

    Solution: Let K be consist of 0 and the numbers $1/n$ for $n = 1, 2, 3, ...$ Let $xK = \{xk : k \in K\}$ and $x + K = \{x + k : k \in K\}$ for $x \in R^1$. I take

$$\begin{aligned} S_n &= (1 - \frac{1}{2^n}) + \frac{K}{2^{n+1}}, \\ S &= \bigcup_{n=1}^{\infty} S_n \bigcup \{1\}. \end{aligned}$$

    Claim: S is compact and the set of all limit points of S is $K \bigcup \{1\}$. Clearly, S lies in $[0, 1]$, that is, S is bounded in R^1. Note that $S_n \subset [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$. By Exercise 12 and its note, I have that all limit points of $S \bigcap [0, 1)$ is

$$0, \frac{1}{2}, ..., \frac{1}{2^n}, \ldots$$

Clearly, 1 is also a limit point of S. Therefore, the set of all limit points of S is $K \bigcup\{1\}$. Note that $K \bigcup\{1\} \subset S$, that is, K is compact. I completed the proof of my claim.

14. Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.

    Solution: Take $\{O_n\} = \{(1/n, 1)\}$ for $n = 1, 2, 3, ...$ The following is my proof. For every $x \in (0,1)$,

    $$x \in (\frac{1}{[1/x]+1}, 0) \in \{O_n\}$$

    Hence $\{O_n\}$ is an open covering of $(0,1)$. Suppose there exists a finite subcovering

    $$(\frac{1}{n_1}, 1), ..., (\frac{1}{n_k}, 1)$$

    where $n_1 < n_2 < ... < n_k$, respectively. Clearly $\frac{1}{2n_p} \in (0,1)$ is not in any elements of that subcover, a contradiction.

    Note: By the above we know that $(0,1)$ is not compact.

15. Show that Theorem 2.36 and its Corollary become false (in R^1, for example) if the word "compact" is replaced by "closed" or by "bounded."

    Theorem 2.36: If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

    Corollary: If $\{K_n\}$ is a sequence of nonempty compact sets such that K_n contains K_{n+1} $(n = 1, 2, 3, ...)$, then $\bigcap K_n$ is not empty.

    Solution: For closed: $[n, \infty)$. For bounded: $(-1/n, 1/n) - \{0\}$.

16. Regard Q, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in Q?

    Proof: Let $S = (\sqrt{2}, \sqrt{3}) \bigcup (-\sqrt{3}, -\sqrt{2})$. Then $E = \{p \in Q : p \in S\}$. Clearly, E is bounded in Q. Since Q is dense in R, every limit point of Q is in Q. (I regard Q as a metric space). Hence, E is closed in Q.

    To prove that E is not compact, we form a open covering of E as follows:

    $$\{G_\alpha\} = \{N_r(p) : p \in E \ \ and \ \ (p-r, p+r) \subset S\}$$

    Surely, $\{G_\alpha\}$ is a open covering of E. If E is compact, then there are finitely many indices $\alpha_1, ..., \alpha_n$ such that

    $$E \subset G_{\alpha_1} \bigcup ... \bigcup G_{\alpha_n}.$$

    For every $G_{\alpha_i} = N_{r_i}(p_i)$, take $p = \max_{1 \leq i \leq n} p_i$. Thus, p is the nearest point to $\sqrt{3}$. But $N_r(p)$ lies in E, thus $[p+r, \sqrt{3})$ cannot be covered since Q is dense in R, a contradiction. Hence E is not compact.

    Finally, the answer is yes. Take any $p \in Q$, then there exists a neighborhood $N(p)$ of p contained in E. (Take r small enough where $N_r(p) = N(p)$, and Q is dense in R.) Thus every point in $N(p)$ is also in Q. Hence E is also open.

17. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

    Solution:

    $$E = \Big\{ \sum_{n=1}^{\infty} \frac{a_n}{10^n} : a_n = 4 \ \ or \ \ a_n = 7 \Big\}$$

Claim: E is uncountable.
Proof of Claim: If not, we list E as follows:

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}...a_{1n}... \\ x_2 &= 0.a_{21}a_{22}...a_{2n}... \\ ... & \quad ... \\ x_k &= 0.a_{k1}a_{k2}...a_{kn}... \\ ... & \quad ... \end{aligned}$$

(Prevent ending with all digits 9) Let $x = 0.x_1x_2...x_n...$ where

$$x_n = \begin{cases} 4 & \text{if } a_{nn} = 7 \\ 7 & \text{if } a_{nn} = 4 \end{cases}$$

By my construction, $x \notin E$, a contradiction. Thus E is uncountable.

Claim: E is not dense in $[0, 1]$.
Proof of Claim: Note that $E \bigcap (0.4\overline{7}, 0.7\overline{4}) = \phi$. Hence E is not dense in $[0, 1]$.

Claim: E is compact.
Proof of Claim: Clearly, E is bounded. For every limit point p of E, I show that $p \in E$. If not, write the decimal expansion of p as follows

$$p = 0.p_1p_2...p_n...$$

Since $p \notin E$, there exists the smallest k such that $p_k \neq 4$ and $p_k \neq 7$. When $p_k = 0, 1, 2, 3$, select the smallest l such that $p_l = 7$ if possible. (If l does not exist, then $p < 0.\overline{4}$. Thus there is a neighborhood of p such that contains no points of E, a contradiction.) Thus

$$0.p_1...p_{l-1}4p_{l+1}...p_{k-1}\overline{7} < p < 0.p_1...p_{k-1}\overline{4}.$$

Thus there is a neighborhood of p such that contains no points of E, a contradiction.

When $p_k = 5, 6$,

$$0.p_1...p_{k-1}4\overline{7} < p < 0.p_1...p_{k-1}7\overline{4}.$$

Thus there is a neighborhood of p such that contains no points of E, a contradiction.

When $p_k = 8, 9$, it is similar. Hence E is closed. Therefore E is compact.

Claim: E is perfect.
Proof of Claim: Take any $p \in E$, and I claim that p is a limit point of E. Write $p = 0.p_1p_2...p_n...$ Let

$$x_k = 0.y_1y_2...y_n...$$

where

$$y_n = \begin{cases} p_k & \text{if } k \neq n \\ 4 & \text{if } p_n = 7 \\ 7 & \text{if } p_n = 4 \end{cases}$$

Thus, $|x_k - p| \to 0$ as $k \to \infty$. Also, $x_k \neq p$ for all k. Hence p is a limit point of E. Therefore E is perfect.

18. Is there a nonempty perfect set in R^1 which contains no rational number?

    Solution: Yes. The following claim will show the reason.

    Claim: Given a measure zero set S, we have a perfect set P contains no elements in S.

    Proof of Claim: (due to SYLee). Since S has measure zero, there exists a collection of open intervals $\{I_n\}$ such that

$$S \subset \bigcup I_n \textit{ and } \sum |I_n| < 1.$$

Consider $E = R^1 - \bigcup I_n$. E is nonempty since E has positive measure. Thus E is uncountable and E is closed. Therefore there exists a nonempty perfect set P contained in E by Exercise 28. $P \bigcap S = \phi$. Thus P is our required perfect set.

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

    (b) Prove the same for disjoint open sets.

    (c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

    (d) Prove that every connected metric space with at least two points is uncountable. Hint: Use (c).

    Proof of (a): Recall the definition of **separated**: A and B are separated if $A \bigcap \overline{B}$ and $\overline{A} \bigcap B$ are empty. Since A and B are closed sets, $A = \overline{A}$ and $B = \overline{B}$. Hence $A \bigcap \overline{B} = \overline{A} \bigcap B = A \bigcap B = \phi$. Hence A and B are separated.

    Proof of (b): Suppose $A \bigcap \overline{B}$ is not empty. Thus there exists p such that $p \in A$ and $p \in \overline{B}$. For $p \in A$, there exists a neighborhood $N_r(p)$ of p contained in A since A is open. For $p \in \overline{B} = B \bigcup B'$, if $p \in B$, then $p \in A \bigcap B$. Note that A and B are disjoint, and it's a contradiction. If $p \in B'$, then p is a limit point of B. Thus every neighborhood of p contains a point $q \neq p$ such that $q \in B$. Take an neighborhood $N_r(p)$of p containing a point $q \neq p$ such that $q \in B$. Note that $N_r(p) \subset A$, thus $q \in A$. With A and B are disjoint, we get a contradiction. Hence $A \bigcap \bar{(}B)$ is empty.

    Similarly, $\overline{A} \bigcap B$ is also empty. Thus A and B are separated.

    Proof of (c): Suppose $A \bigcap \overline{B}$ is not empty. Thus there exists x such that $x \in A$ and $x \in \overline{B}$. Since $x \in A$, $d(p, x) < \delta$. $x \in \overline{B} = B \bigcup B'$, thus if $x \in B$, then $d(p, x) > \delta$, a contradiction. The only possible is x is a limit point of B. Hence we take a neighborhood $N_r(x)$ of x contains y with $y \in B$ where $r = \frac{\delta - d(x,p)}{2}$. Clearly, $d(y, p) > \delta$. But,

$$
\begin{aligned}
d(y, p) &\leq d(y, x) + d(x, p) \\
&< r + d(x, p) \\
&= \frac{\delta - d(x, p)}{2} + d(x, p) \\
&= \frac{\delta + d(x, p)}{2} \\
&< \frac{\delta + \delta}{2} = \delta.
\end{aligned}
$$

A contradiction. Hence $A \bigcap \overline{B}$ is empty. Similarly, $\overline{A} \bigcap B$ is also empty. Thus A and B are separated.

Note: Take care of $\delta > 0$. Think a while and you can prove the next sub-exercise.

Proof of (d): Let X be a connected metric space. Take $p \in X$, $q \in X$ with $p \neq q$, thus $d(p, q) > 0$ is fixed. Let

$$A = \{x \in X : d(x, p) < \delta\}; B = \{x \in X : d(x, p) > \delta\}.$$

Take $\delta = \delta_t = td(p, q)$ where $t \in (0, 1)$. Thus $0 < \delta < d(p, q)$. $p \in A$ since $d(p, p) = 0 < \delta$, and $q \in B$ since $d(p, q) > \delta$. Thus A and B are non-empty.

By (c), A and B are separated. If $X = A \bigcup B$, then X is not connected, a contradiction. Thus there exists $y_t \in X$ such that $y \notin A \bigcup B$. Let

$$E = E_t = \{x \in X : d(x, p) = \delta_t\} \ni y_t.$$

For any real $t \in (0, 1)$, E_t is non-empty. Next, E_t and E_s are disjoint if $t \neq s$ (since a metric is well-defined). Thus X contains a uncountable set $\{y_t : t \in (0,1)\}$ since $(0,1)$ is uncountable. Therefore, X is uncountable.

Note: It is a good exercise. If that metric space contains only one point, then it must be separated.

Similar Exercise Given by SYLee: (a) Let $A = \{x : d(p,x) < r\}$ and $B = \{x : d(p,x) > r\}$ for some p in a metric space X. Show that A, B are separated.
(b) Show that a connected metric space with at least two points must be uncountable. [Hint: Use (a)]

Proof of (a): By definition of separated sets, we want to show $\overline{A} \bigcap B = \phi$, and $\overline{B} \bigcap A = \phi$. In order to do these, it is sufficient to show $\overline{A} \bigcap B = \phi$. Let $x \in \overline{A} \bigcap B = \phi$, then we have:

$$(1)\ \ x \in \overline{A} \Rightarrow d(x,p) \leq r (2)\ \ x \in B \Rightarrow d(x,p) > r$$

It is impossible. So, $\overline{A} \bigcap B = \phi$.

Proof of (b): Suppose that C is countable, say $C = a, b, x_3, ...$ We want to show C is disconnected. So, if C is a connected metric space with at least two points, it must be uncountable. Consider the set $S = \{d(a, x_i) : x_i \in C\}$, and thus let $r \in R - S$ and $\inf S < r < \sup S$. And construct A and B as in (a), we have $C = A \bigcup B$, where A and B are separated. That is C is disconnected.

Another Proof of (b): Let $a \in C$, $b \in C$, consider the continuous function f from C into R defined by $f(x) = d(x,a)$. So, $f(C)$ is connected and $f(a) = 0$, $f(b) > 0$. That is, $f(C)$ is an interval. Therefore, C is uncountable.

20. Are closures and interiors of connected sets always connected? (Look at subsets of R^2.)

Solution: Closures of connected sets is always connected, but interiors of those is not. The counterexample is

$$S = N_1(2,0)\bigcup N_1(-2,0)\bigcup\{x-axis\} \subset R^2.$$

Since S is path-connected, S is connect. But $S^o = N_1(2)\bigcup N_1(-2)$ is disconnected clearly.

Claim: If S is a connected subset of a metric space, then $\overline{S}$ is connected.

Pf of Claim: If not, then $\overline{S}$ is a union of two nonempty separated set A and B. Thus $A\bigcap\overline{B} = \overline{A}\bigcap B = \phi$. Note that

$$\begin{aligned} S &= \overline{S} - T \\ &= A\bigcup B - T \\ &= (A\bigcup B)\bigcap T^c \\ &= (A\bigcap T^c)\bigcup(B\bigcap T^c) \end{aligned}$$

where $T = \overline{S} - S$. Thus

$$\begin{aligned} (A\bigcap T^c)\bigcap\overline{B\bigcap T^c} &\subset (A\bigcap T^c)\bigcap\overline{B}\bigcap\overline{T^c} \\ &\subset A\bigcap\overline{B} \\ &= \phi. \end{aligned}$$

Hence $(A\bigcap T^c)\bigcap\overline{B\bigcap T^c} = \phi$. Similarly, $\overline{A\bigcap T^c}\bigcap(B\bigcap T^c) = \phi$.

Now we claim that both $A\bigcap T^c$ and $B\bigcap T^c$ are nonempty. Suppose that $B\bigcap T^c = \phi$. Thus

$$\begin{aligned} A\bigcap T^c = S &\Leftrightarrow A\bigcap(\overline{S}-S)^c = S \\ &\Leftrightarrow A\bigcap(A\bigcup B - S)^c = S \\ &\Leftrightarrow A\bigcap((A\bigcup B)\bigcap S^c)^c = S \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \quad A \bigcap ((A^c \bigcap B^c) \bigcup S) = S \\
&\Leftrightarrow \quad (A \bigcap S) \bigcup (A \bigcap A^c \bigcap B^c) = S \\
&\Leftrightarrow \quad A \bigcap S = S.
\end{aligned}$$

Thus B is empty, a contradiction. Thus $B \bigcap T^c$ is nonempty. Similarly, $A \bigcap T^c$ nonempty. Therefore S is a union of two nonempty separated sets, a contradiction. Hence $\overline{S}$ is connected.

21. Let A and B be separated subsets of some R^k, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(a) Prove that A_0 and B_0 are separated subsets of R^1.

(b) Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \bigcup B$.

(c) Prove that every convex subset of R^k is connected.

Proof of (a): I claim that $A_0 \bigcap \overline{B_0}$ is empty. ($B_0 \bigcap \overline{A_0}$ is similar). If not, take $x \in A_0 \bigcap \overline{B_0}$. $x \in A_0$ and $x \in \overline{B_0}$. $x \in B_0$ or x is a limit point of B_0. $x \in B_0$ will make $x \in A_0 \bigcap B_0$, that is, $\mathbf{p}(x) \in A \bigcap B$, a contradiction since A and B are separated.

Claim: x is a limit point of $B_0 \Rightarrow \mathbf{p}(x)$ is a limit point of B. Take any neighborhood N_r of $\mathbf{p}(x)$, and $\mathbf{p}(t)$ lies in B for small enough t. More precisely,

$$x - \frac{r}{|\mathbf{b}-\mathbf{a}|} < t < x + \frac{r}{|\mathbf{b}-\mathbf{a}|}.$$

Since x is a limit point of B_0, and $(x - r/|\mathbf{b}-\mathbf{a}|, x + r/|\mathbf{b}-\mathbf{a}|)$ is a neighborhood N of x, thus N contains a point $y \neq x$ such that $y \in B_0$, that is, $\mathbf{p}(y) \in B$. Also, $\mathbf{p}(y) \in N_r$. Therefore, $\mathbf{p}(x)$ is a limit point of B. Hence $\mathbf{p}(x) \in A \bigcap \overline{B}$, a contradiction since A and B are separated.

Hence $A_0 \bigcap \overline{B_0}$ is empty, that is, A_0 and B_0 are separated subsets of R^1.

Proof of (b): Suppose not. For every $t_0 \in (0,1)$, neither $\mathbf{p}(t_0) \in A$ nor $\mathbf{p}(t_0) \in B$ (since A and B are separated). Also, $\mathbf{p}(t_0) \in A \bigcup B$ for all $t_0 \in (0,1)$. Hence $(0,1) = A_0 \bigcup B_0$, a contradiction since $(0,1)$ is connected. I completed the proof.

Proof of (c): Let S be a convex subset of R^k. If S is not connected, then S is a union of two nonempty separated sets A and B. By (b), there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \bigcup B$. But S is convex, $\mathbf{p}(t_0)$ must lie in $A \bigcup B$, a contradiction. Hence S is connected.

22. A metric space is called separable if it contains a countable dense subset. Show that R^k is separable. *Hint:* Consider the set of points which have only rational coordinates.

    Proof: Consider $S =$ the set of points which have only rational coordinates. For any point $x = (x_1, x_2, ..., x_k) \in R^k$, we can find a rational sequence $\{r_{i_j}\} \to x_j$ for $j = 1, ..., k$ since Q is dense in R^1. Thus,

    $$r_i = (r_{i_1}, r_{i_2}, ..., r_{i_k}) \to x$$

    and $r_i \in S$ for all i. Hence S is dense in R^k. Also, S is countable, that is, S is a countable dense subset in R^k, R^k is separable.

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

    Prove that every separable metric space has a countable base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

Proof: Let X be a separable metric space, and S be a countable dense subset of X. Let a collection $\{V_\alpha\} = \{$ all neighborhoods with rational radius and center in S $\}$. We claim that $\{V_\alpha\}$ is a base for X.

For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, there exists a neighborhood $N_r(p)$ of p such that $N_r(p) \subset G$ since x is an interior point of G. Since S is dense in X, there exists $\{s_n\} \to x$. Take a rational number r_n such that $r_n < \frac{r}{2}$, and $\{V_\alpha\} \ni N_{r_n}(s_n) \subset N_r(p)$ for enough large n. Hence we have $x \in V_\alpha \subset G$ for some α. Hence $\{V_\alpha\}$ is a base for X.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, ..., x_j \in X$, choose x_{j+1}, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, ..., j$. Show that this process must stop after finite number of steps, and that X can therefore be covered by finite many neighborhoods of radius δ. Take $\delta = 1/n (n = 1, 2, 3, ...)$, and consider the centers of the corresponding neighborhoods.

    Proof: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, ..., x_j \in X$, choose x_{j+1}, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, ..., j$. If this process cannot stop, then consider the set $A = \{x_1, x_2, ..., x_k\}$. If p is a limit point of A, then a neighborhood $N_{\delta/3}(p)$ of p contains a point $q \neq p$ such that $q \in A$. $q = x_k$ for only one $k \in N$. If not, $d(x_i, x_j) \leq d(x_i, p) + d(x_j, p) \leq \delta/3 + \delta/3 < \delta$, and it contradicts the fact that $d(x_i, x_j) \geq \delta$ for $i \neq j$. Hence, this process must stop after finite number of steps.

    Suppose this process stop after k steps, and X is covered by $N_\delta(x_1)$, $N_\delta(x_2)$, ..., $N_\delta(x_k)$, that is, X can therefore be covered by finite many neighborhoods of radius δ.

    Take $\delta = 1/n (n = 1, 2, 3, ...)$, and consider the set A of the centers of the corresponding neighborhoods.

Fix $p \in X$. Suppose that p is not in A, and every neighborhood $N_r(p)$. Note that $N_{r/2}(p)$ can be covered by finite many neighborhoods $N_s(x_1), ..., N_s(x_k)$ of radius $s = 1/n$ where $n = [2/r] + 1$ and $x_i \in A$ for $i = 1, ..., k$. Hence, $d(x_1, p) \leq d(x_1, q) + d(q, p) \leq r/2 + s < r$ where $q \in N_{r/2}(p) \bigcap N_s(x_1)$. Therefore, $x_1 \in N_r(p)$ and $x_1 \neq p$ since p is not in A. Hence, p is a limit point of A if p is not in A, that is, A is a countable dense subset, that is, X is separable.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhood of radius $1/n$ whose union covers K.

Proof: For every positive integer n, there are finitely many neighborhood of radius $1/n$ whose union covers K (since K is compact). Collect all of them, say $\{V_\alpha\}$, and it forms a countable collection. We claim $\{V_\alpha\}$ is a base.

For every $x \in X$ and every open set $G \subset X$, there exists $N_r(x)$ such that $N_r(x) \subset G$ since x is an interior point of G. Hence $x \in N_m(p) \in \{V_\alpha\}$ for some p where $m = [2/r] + 1$. For every $y \in N_m(p)$, we have

$$d(y, x) \leq d(y, p) + d(p, x) < m + m = 2m < r.$$

Hence $N_m(p) \subset G$, that is, $V_\alpha \subset G$ for some α, and therefore $\{V_\alpha\}$ is a countable base of K. Next, collect all of the center of V_α, say D, and we claim D is dense in K (D is countable since V_α is countable). For all $p \in K$ and any $\epsilon > 0$ we can find $N_n(x_n) \in \{V_\alpha\}$ where $n = [1/\epsilon] + 1$. Note that $x_n \in D$ for all n and $d(p, x_n) \to 0$ as $n \to \infty$. Hence D is dense in K.

26. Let X be a metric space in which every infinite subsets has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a countable subcover $\{G_n\}$, $n = 1, 2, 3, ...$ If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \bigcup ... \bigcup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

Proof: By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a countable subcover $\{G_n\}$, $n = 1, 2, 3, ...$ If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \bigcup ... \bigcup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set contains a point from each F_n, consider a limit point of E.

Note that $F_k \supset F_{k+1} \supset ...$ and F_n is closed for all n, thus p lies in F_k for all k. Hence p lies in $\bigcap F_n$, but $\bigcap F_n$ is empty, a contradiction.

27. Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.

Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \bigcap E$ is at most countable. *Hint:* Let $\{V_n\}$ be a countable base of R^k, let W be the union of those V_n for which $E \bigcap V_n$ is at most countable, and show that $P = W^c$.

Proof: Let $\{V_n\}$ be a countable base of R^k, let W be the union of those V_n for which $E \bigcap V_n$ is at most countable, and we will show that $P = W^c$. Suppose $x \in P$. (x is a condensation point of E). If $x \in V_n$ for some n, then $E \bigcap V_n$ is uncountable since V_n is open. Thus $x \in W^c$. (If $x \in W$, then there exists V_n such that $x \in V_n$ and $E \bigcap V_n$

is uncountable, a contradiction). Therefore $P \subset W^c$.

Conversely, suppose $x \in W^c$. $x \notin V_n$ for any n such that $E \bigcap V_n$ is countable. Take any neighborhood $N(x)$ of x. Take $x \in V_n \subset N(x)$, and $E \bigcap V_n$ is uncountable. Thus $E \bigcap N(x)$ is also uncountable, x is a condensation point of E. Thus $W^c \subset P$. Therefore $P = W^c$. Note that W is countable, and thus $W \subset W \bigcap E = P^c \bigcap E$ is at most countable.

To show that P is perfect, it is enough to show that P contains no isolated point. (since P is closed). If p is an isolated point of P, then there exists a neighborhood N of p such that $N \bigcap E = \phi$. p is not a condensation point of E, a contradiction. Therefore P is perfect.

28. Prove that every closed set in a separable metric space is the union of a (possible empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in R^k has isolated points.) *Hint:* Use Exercise 27.

    Proof: Let X be a separable metric space, let E be a closed set on X. Suppose E is uncountable. (If E is countable, there is nothing to prove.) Let P be the set of all condensation points of E. Since X has a countable base, P is perfect, and $P^c \bigcap E$ is at most countable by Exercise 27. Since E is closed, $P \subset E$. Also, $P^c \bigcap E = E - P$. Hence $E = P \bigcup (E - P)$.

    For corollary: if there is no isolated point in E, then E is perfect. Thus E is uncountable, a contradiction.

    Note: It's also called **Cauchy-Bendixon Theorem**.

29. Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.

Proof: (due to H.L.Royden, Real Analysis) Since O is open, for each x in O, there is a $y > x$ such that $(x, y) \subset O$. Let $b = \sup\{y : (x, y) \subset O\}$. Let $a = \inf\{z : (z, x) \subset O\}$. Then $a < x < b$, and $I_x = (a, b)$ is an open interval containing x.

Now $I_x \subset O$, for if $w \in I_x$, say $x < w < b$, we have by the definition of b a number $y > w$ such that $(x, y) \subset O$, and so $w \in O$).

Moreover, $b \notin O$, for if $b \in O$, then for some $\epsilon > 0$ we have $(b-\epsilon, b+\epsilon) \subset O$, whence $(x, b+\epsilon) \subset O$, contradicting the definition of b. Similarly, $a \notin O$.

Consider the collection of open intervals $\{I_x\}$, $x \in O$. Since each $x \in O$ is contained in I_x, and each $I_x \subset O$, we have $O = \bigcup I_x$.

Let (a, b) and (c, d) be two intervals in this collection with a point in common. Then we must have $c < b$ and $a < d$. Since $c \notin O$, it does not belong to (a, b) and we have $c \leq a$. Since $a \notin O$ and hence not to (c, d), we have $a \leq c$. Thus $a = c$. Similarly, $b = d$, and $(a, b) = (c, d)$. Thus two different intervals in the collection $\{I_x\}$ must be disjoint. Thus O is the union of the disjoint collection $\{I_x\}$ of open intervals, and it remains only to show that this collection is countable. But each open interval contains a rational number since Q is dense in R. Since we have a collection of disjoint open intervals, each open interval contains a different rational number, and the collection can be put in one-to-one correspondence with a subset of the rationals. Thus it is a countable collection.

30. Imitate the proof of Theorem 2.43 to obtain the following result:

    If $R^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of R^k, then at least one F_n has a nonempty interior.

    Equivalent statement: If G_n is a dense open subset of R^k, for

$n = 1, 2, 3, ...$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in R^k).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

Proof: I prove Baire's theorem directly. Let G_n be a dense open subset of R^k for $n = 1, 2, 3, ...$ I need to prove that $\bigcap_1^\infty G_n$ intersects any nonempty open subset of R^k is not empty.

Let G_0 is a nonempty open subset of R^k. Since G_1 is dense and G_0 is nonempty, $G_0 \bigcap G_1 \neq \phi$. Suppose $x_1 \in G_0 \bigcap G_1$. Since G_0 and G_1 are open, $G_0 \bigcap G_1$ is also open, that is, there exist a neighborhood V_1 such that $\overline{V_1} \subset G_0 \bigcap G_1$. Next, since G_2 is a dense open set and V_1 is a nonempty open set, $V_1 \bigcap G_2 \neq \phi$. Thus, I can find a nonempty open set V_2 such that $\overline{V_2} \subset V_1 \bigcap G_2$. Suppose I have get n nonempty open sets $V_1, V_2, ..., V_n$ such that $\overline{V_1} \subset G_0 \bigcap G_1$ and $\overline{V_{i+1}} \subset V_i \bigcap G_{n+1}$ for all $i = 1, 2, ..., n-1$. Since G_{n+1} is a dense open set and V_n is a nonempty open set, $V_n \bigcap G_{n+1}$ is a nonempty open set. Thus I can find a nonempty open set V_{n+1} such that $\overline{V_{n+1}} \subset V_n \bigcap G_{n+1}$. By induction, I can form a sequence of open sets $\{V_n : n \in Z^+\}$ such that $\overline{V_1} \subset G_0 \bigcap G_1$ and $\overline{V_{i+1}} \subset V_i \bigcap G_{i+1}$ for all $n \in Z^+$. Since $\overline{V_1}$ is bounded and $\overline{V_1} \supset \overline{V_2} \supset ... \supset \overline{V_n} \supset ...$, by Theorem 2.39 I know that

$$\bigcap_{n=1}^{\infty} \overline{V_n} \neq \phi.$$

Since $\overline{V_1} \subset G_0 \bigcap G_1$ and $\overline{V_{n+1}} \subset G_{n+1}$, $G_0 \bigcap (\bigcap_{n=1}^\infty G_n) \neq \phi$. Proved.

Note: By Baire's theorem, I've proved the equivalent statement. Next, F_n has a empty interior if and only if $G_n = R^k - F_n$ is dense in R^k. Hence we completed all proof.