Numerical Sequences and Series

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1. Prove that the convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

   Solution: Since $\{s_n\}$ is convergent, for any $\epsilon > 0$, there exists N such that $|s_n - s| < \epsilon$ whenever $n \geq N$. By Exercise 1.13 I know that $||s_n| - |s|| \leq |s_n - s|$. Thus, $||s_n| - |s|| < \epsilon$, that is, $\{s_n\}$ is convergent.

   The converse is not true. Consider $s_n = (-1)^n$.

2. Calculate $\lim_{n\to\infty} (\sqrt{n^2 + n} - n)$.

   Solution:

$$\begin{aligned} \sqrt{n^2+n} - n &= \frac{n}{\sqrt{n^2+n}+n} \\ &= \frac{1}{\sqrt{1/n+1}+1} \\ &\to \frac{1}{2} \end{aligned}$$

   as $n \to \infty$.

3. If $s_n = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \ \ (n = 1, 2, 3, ...),$$

   prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, ...$

   Proof: First, I show that $\{s_n\}$ is strictly increasing. It is trivial that $s_2 = \sqrt{2 + \sqrt{s_1}} = \sqrt{2 + \sqrt{\sqrt{2}}} > \sqrt{2} = s_1$. Suppose $s_k > s_{k-1}$ when

$k < n$. By the induction hypothesis,

$$\begin{aligned} s_n &= \sqrt{2+\sqrt{s_{n-1}}} \\ &> \sqrt{2+\sqrt{s_{n-2}}} \\ &= s_{n-1} \end{aligned}$$

By the induction, $\{s_n\}$ is strictly increasing. Next, I show that $\{s_n\}$ is bounded by 2. Similarly, I apply the induction again. Hence $\{s_n\}$ is strictly increasing and bounded, that is, $\{s_n\}$ converges.

4.

5.

6.

7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n}$$

if $a_n \geq 0$.

Proof: By Cauchy's inequality,

$$\sum_{n=1}^{k} a_n \sum_{n=1}^{k} \frac{1}{n^2} \geq \sum_{n=1}^{k} a_n \frac{\sqrt{a_n}}{n}$$

for all $n \in N$. Also, both $\sum a_n$ and $\sum \frac{1}{n^2}$ are convergent; thus $\sum_{n=1}^{k} a_n \frac{\sqrt{a_n}}{n}$ is bounded. Besides, $\frac{\sqrt{a_n}}{n} \geq 0$ for all n. Hence $\sum \frac{\sqrt{a_n}}{n}$ is convergent.

8.

9. Find the radius of convergence of each of the following power series:

$$(a) \sum n^3 z^n, (b) \sum \frac{2^n}{n!} z^n, (c) \sum \frac{2^n}{n^2} z^n, (d) \sum \frac{n^3}{3^n} z^n.$$

Solution: (a) $\alpha_n = (n^3)^{1/n} \to 1$ as $n \to \infty$. Hence $R = 1/\alpha = 1$.
(b) $\alpha_n = (2^n/n!)^{1/n} = 2/(n!)^{1/n} \to 0$ as $n \to \infty$. Hence $R = +\infty$.
(c) $\alpha_n = (2^n/n^2)^{1/n} \to 2/1 = 2$ as $n \to \infty$. Hence $R = 1/\alpha = 1/2$.
(d) $\alpha_n = (n^3/3^n)^{1/n} \to 1/3$ as $n \to \infty$. Hence $R = 1/\alpha = 3$.

10.

11. Suppose $a_n > 0$, $s_n = a_1 + ... + a_n$, and $\sum a_n$ diverges.
(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.
(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.
(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.
(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \quad \textit{and} \quad \sum \frac{a_n}{1+n^2 a_n}?$$

Proof of (a): Note that

$$\begin{aligned} \frac{a_n}{1+a_n} \to 0 \quad &\Leftrightarrow \quad \frac{1}{\frac{1}{a_n}+1} \to 0 \\ &\Leftrightarrow \quad \frac{1}{a_n} \to \infty \\ &\Leftrightarrow \quad a_n \to 0 \end{aligned}$$

as $n \to \infty$. If $\sum \frac{a_n}{1+a_n}$ converges, then $a_n \to 0$ as $n \to \infty$. Thus for some $\epsilon' = 1$ there is an N_1 such that $a_n < 1$ whenever $n \geq N_1$. Since $\sum \frac{a_n}{1+a_n}$ converges, for any $\epsilon > 0$ there is an N_2 such that

$$\frac{a_m}{1+a_m} + ... + \frac{a_n}{1+a_n} < \epsilon$$

all $n > m \geq N_2$. Take $N = \max(N_1, N_2)$. Thus

$$\begin{aligned}\epsilon &> \frac{a_m}{1+a_m} + ... + \frac{a_n}{1+a_n} \\ &> \frac{a_m}{1+1} + ... + \frac{a_n}{1+1} \\ &= \frac{a_m + ... + a_n}{2}\end{aligned}$$

for all $n > m \geq N$. Thus

$$a_m + ... + a_n < 2\epsilon$$

for all $n > m \geq N$. It is a contradiction. Hence $\sum \frac{a_n}{1+a_n}$ diverges.
Proof of (b):

$$\begin{aligned}\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + ... + \frac{a_{N+k}}{s_{N+k}} \\ &= \frac{a_{N+1} + ... + a_{N+k}}{s_{N+k}} \\ &= \frac{s_{N+k} - s_N}{s_{N+k}} \\ &= 1 - \frac{s_N}{s_{N+k}}\end{aligned}$$

If $\sum \frac{a_n}{s_n}$ converges, for any $\epsilon > 0$ there exists N such that

$$\frac{a_m}{s_m} + ... + \frac{a_n}{s_n} < \epsilon$$

for all m, n whenever $n > m \geq N$. Fix $m = N$ and let $n = N + k$. Thus

$$\begin{aligned}\epsilon &> \frac{a_m}{s_m} + ... + \frac{a_n}{s_n} \\ &= \frac{a_N}{s_N} + ... + \frac{a_{N+k}}{s_{N+k}} \\ &\geq 1 - \frac{s_N}{s_{N+k}}\end{aligned}$$

for all $k \in N$. But $s_{N+k} \to \infty$ as $k \to \infty$ since $\sum a_n$ diverges and $a_n > 0$. Take $\epsilon = 1/2$ and we obtain a contradiction. Hence $\sum \frac{a_n}{s_n}$ diverges.

Proof of (c):

$$
\begin{aligned}
s_{n-1} \leq s_n \quad &\Leftrightarrow \quad \frac{1}{s_n^2} \leq \frac{1}{s_n s_{n-1}} \\
&\Leftrightarrow \quad \frac{a_n}{s_n^2} \leq \frac{a_n}{s_n s_{n-1}} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} \\
&\Leftrightarrow \quad \frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}
\end{aligned}
$$

for all n.

Hence

$$
\begin{aligned}
\sum_{n=2}^{k} \frac{a_n}{s_n^2} \quad &\leq \quad \sum_{n=2}^{k} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n}\right) \\
&= \quad \frac{1}{s_1} - \frac{1}{s_n}.
\end{aligned}
$$

Note that $\frac{1}{s_n} \to 0$ as $n \to \infty$ since $\sum a_n$ diverges. Hence $\sum \frac{a_n}{s_n^2}$ converges.

Proof of (d): $\sum \frac{a_n}{1+na_n}$ may converge or diverge, and $\sum \frac{a_n}{1+n^2 a_n}$ converges. To see this, we put $a_n = 1/n$. $\frac{a_n}{1+na_n} = \frac{1}{2n}$, that is, $\sum \frac{a_n}{1+na_n} = 2 \sum 1/n$ diverges. Besides, if we put

$$
a_n = \frac{1}{n(\log n)^p}
$$

where $p > 1$ and $n \geq 2$, then

$$
\begin{aligned}
\frac{a_n}{1+na_n} \quad &= \quad \frac{1}{n(\log n)^{2p}((\log n)^p + 1)} \\
&< \quad \frac{1}{2n(\log n)^{3p}}
\end{aligned}
$$

for large enough n. By Theorem 3.25 and Theorem 3.29, $\sum \frac{a_n}{1+na_n}$ converges. Next,

$$
\sum \frac{a_n}{1+n^2 a_n} \quad = \quad \sum \frac{1}{1/a_n + n^2}
$$

$$< \sum \frac{1}{n^2}.$$

for all a_n. Note that $\sum \frac{1}{n^2}$ converges, and thus $\sum \frac{a_n}{1+n^2 a_n}$ converges.

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Proof of (a):

$$\begin{aligned} \frac{a_m}{r_m} + ... + \frac{a_n}{r_n} &> \frac{a_m + ... + a_n}{r_m} \\ &= \frac{r_m - r_n}{r_m} \\ &= 1 - \frac{r_n}{r_m} \end{aligned}$$

if $m < n$. If $\sum \frac{a_n}{r_n}$ converges, for any $\epsilon > 0$ there exists N such that

$$\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} < \epsilon$$

for all m, n whenever $n > m \geq N$. Fix $m = N$. Thus

$$\begin{aligned} \frac{a_m}{r_m} + ... + \frac{a_n}{r_n} &> 1 - \frac{r_n}{r_m} \\ &= 1 - \frac{r_n}{r_N} \end{aligned}$$

for all $n > N$. But $r_n \to 0$ as $n \to \infty$; thus $\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} \to 1$ as $n \to \infty$. If we take $\epsilon = 1/2$, we will get a contradiction.

Proof of (b): Note that

$$\begin{aligned}
r_{n+1} < r_n &\Leftrightarrow \sqrt{r_{n+1}} < \sqrt{r_n} \\
&\Leftrightarrow \sqrt{r_n} + \sqrt{r_{n+1}} < 2\sqrt{r_n} \\
&\Leftrightarrow \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}} < 2 \\
&\Leftrightarrow (\sqrt{r_n} - \sqrt{r_{n+1}})\frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\
&\Leftrightarrow \frac{r_n - r_{n+1}}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\
&\Leftrightarrow \frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})
\end{aligned}$$

since $a_n > 0$ for all n.

Hence,

$$\begin{aligned}
\sum_{n=1}^{k} \frac{a_n}{\sqrt{r_n}} &< \sum_{n=1}^{k} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \\
&= 2(\sqrt{r_1} - \sqrt{r_{k+1}})
\end{aligned}$$

Note that $r_n \to 0$ as $n \to \infty$. Thus $\sum \frac{a_n}{\sqrt{r_n}}$ is bounded. Hence $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Note: If we say $\sum a_n$ converges faster than $\sum b_n$, it means that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

According the above exercise, we can construct a faster convergent series from a known convergent one easily. It implies that there is no **perfect** tests to test all convergences of the series from a known convergent one.

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Note: Given $\sum a_n$ and $\sum b_n$, we put $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ and call $\sum c_n$ the Cauchy product of the two given series.

Proof: Put $A_n = \sum_{k=0}^{n} |a_k|$, $B_n = \sum_{k=0}^{n} |b_k|$, $C_n = \sum_{k=0}^{n} |c_k|$. Then

$$\begin{aligned}
C_n &= |a_0b_0| + |a_0b_1 + a_1b_0| + ... + |a_0b_n + a_1b_{n-1} + ... + a_nb_0| \\
&\le |a_0||b_0| + (|a_0||b_1| + |a_1||b_0|) + ... \\
&\quad +(|a_0||b_n| + |a_1||b_{n-1}| + ... + |a_n||b_0|) \\
&= |a_0|B_n + |a_1|B_{n-1} + ... + |a_n|B_0 \\
&\le |a_0|B_n + |a_1|B_n + ... + |a_n|B_n \\
&= (|a_0| + |a_1| + ... + |a_n|)B_n = A_nB_n \le AB
\end{aligned}$$

where $A = \lim A_n$ and $B = \lim B_n$. Hence $\{C_n\}$ is bounded. Note that $\{C_n\}$ is increasing, and thus C_n is a convergent sequence, that is, the Cauchy product of two absolutely convergent series converges absolutely.

14. If $\{a_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + ... + s_n}{n+1} (n = 0, 1, 2, ...).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i).$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\epsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1.$$

Then $(m+1)/(n-m) \leq 1/\epsilon$ and $|s_n - s_i| < M\epsilon$. Hence

$$\limsup_{n\to\infty} |s_n - \sigma| \leq M\epsilon.$$

Since ϵ was arbitrary, $\lim s_n = \sigma$.

Proof of (a): The proof is straightforward. Let $t_n = s_n - s$, $\tau_n = \sigma_n - s$. (Or you may suppose that $s = 0$.) Then

$$\tau_n = \frac{t_0 + t_1 + ... + t_n}{n+1}.$$

Choose $M > 0$ such that $|t_n| \leq M$ for all n. Given $\epsilon > 0$, choose N so that $n > N$ implies $|t_n| < \epsilon$. Taking $n > N$ in $\tau_n = (t_0 + t_1 + ... + t_n)/(n+1)$, and then

$$\begin{aligned} |\tau_n| &\leq \frac{|t_0| + ... + |t_N|}{n+1} + \frac{|t_{N+1} + ... + |t_n}{n+1} \\ &< \frac{(N+1)M}{n+1} + \epsilon. \end{aligned}$$

Hence, $\limsup_{n\to\infty} |\tau_n| \leq \epsilon$. Since ϵ is arbitrary, it follows that $\lim_{n\to\infty} |\tau_n| = 0$, that is, $\lim \sigma_n = s$.

Proof of (b): Let $s_n = (-1)^n$. Hence $|\sigma_n| \leq 1/(n+1)$, that is, $\lim \sigma_n = 0$. However, $\lim s_n$ does not exists.

Proof of (c): Let

$$s_n = \begin{cases} 1 & , n = 0, \\ n^{1/4} + n^{-1} & , n = k^2 \text{ for some integer } k, \\ n^{-1} & , \text{otherwise.} \end{cases}$$

It is obvious that $s_n > 0$ and $\limsup s_n = \infty$. Also,

$$s_0 + ... + s_n = 1 + nn^{-1} + \left\lfloor \sqrt{n} \right\rfloor n^{1/4} = 2 + \left\lfloor \sqrt{n} \right\rfloor n^{1/4}.$$

That is,

$$\sigma_n = \frac{2}{n+1} + \frac{\lfloor \sqrt{n} \rfloor n^{1/4}}{n+1}$$

The first term $2/(n+1) \to 0$ as $n \to \infty$. Note that

$$0 \leq \frac{\lfloor \sqrt{n} \rfloor n^{1/4}}{n+1} < n^{1/2}n^{1/4}n^{-1} = n^{-1/4}.$$

It implies that the last term $\to 0$. Hence, $\lim \sigma_n = 0$.

Proof of (d):

$$\begin{aligned}
\sum_{k=1}^{n} ka_k &= \sum_{k=1}^{n} k(s_k - s_{k-1}) = \sum_{k=1}^{n} ks_k - \sum_{k=1}^{n} ks_{k-1} \\
&= \sum_{k=1}^{n} ks_k - \sum_{k=0}^{n-1}(k+1)s_k \\
&= ns_n + \sum_{k=1}^{n-1} ks_k - \sum_{k=1}^{n-1}(k+1)s_k - s_0 \\
&= ns_n - \sum_{k=1}^{n-1} s_k - s_0 = (n+1)s_n - \sum_{k=0}^{n} s_k \\
&= (n+1)(s_n - \sigma_n).
\end{aligned}$$

That is,

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} ka_k.$$

Note that $\{na_n\}$ is a complex sequence. By (a),

$$\lim_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=1}^{n} ka_k\right) = \lim_{n\to\infty} na_n = 0.$$

Also, $\lim \sigma_n = \sigma$. Hence by the previous equation, $\lim s = \sigma$.

Proof of (e): If $m < n$, then

$$\begin{aligned}
&\sum_{i=m+1}^{n}(s_n - s_i) + (m+1)(\sigma_n - \sigma_m) \\
= \;& (n-m)s_n - \sum_{i=m+1}^{n} s_i + (m+1)(\sigma_n - \sigma_m) \\
= \;& (n-m)s_n - \left(\sum_{i=0}^{n} s_i - \sum_{i=0}^{m} s_i\right) + (m+1)(\sigma_n - \sigma_m) \\
= \;& (n-m)s_n - (n+1)\sigma_n + (m+1)\sigma_m + (m+1)(\sigma_n - \sigma_m) \\
= \;& (n-m)s_n - (n-m)\sigma_n.
\end{aligned}$$

Hence,

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i).$$

For these i, recall $a_n = s_n - s_{n-1}$ and $|na_n| \leq M$ for all n,

$$\begin{aligned} |s_n - s_i| &= \Big|\sum_{k=i+1}^{n} a_k\Big| \leq \sum_{k=i+1}^{n} |a_k| \leq \sum_{k=i+1}^{n} \frac{M}{i+1} = \frac{(n-i)M}{i+1} \\ &\leq \frac{(n-(m+1))M}{(m+1)+1} = \frac{(n-m-1)M}{m+2}. \end{aligned}$$

Fix $\epsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1.$$

Thus

$$\frac{n-m}{m+1} \geq \epsilon \quad \text{and} \quad \frac{n-m-1}{m+2} < \epsilon,$$

or

$$\frac{m+1}{n-m} \leq \frac{1}{\epsilon} \quad \text{and} \quad |s_n - s_i| < M\epsilon.$$

Hence,

$$|s_n - \sigma| \leq |\sigma_n - \sigma| + \frac{1}{\epsilon}(|\sigma_n - \sigma| + |\sigma_m - \sigma|) + M\epsilon.$$

Let $n \to \infty$ and thus $m \to \infty$ too, and thus

$$\limsup_{n\to\infty} |s_n - \sigma| \leq M\epsilon.$$

Since ϵ was arbitrary, $\lim s_n = \sigma$.

15.

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define x_2, x_3, x_4, ..., by the recursion formula

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.
(b) Put $\epsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\epsilon_{n+1} < \beta(\frac{\epsilon_1}{\beta})^{2^n} \quad (n = 1, 2, 3, ...).$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon_1/\beta < \frac{1}{10}$ and therefore

$$\epsilon_5 < 4 \cdot 10^{-16}, \epsilon_6 < 4 \cdot 10^{-32}.$$

Proof of (a):

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}(x_n + \frac{\alpha}{x_n}) \\ &= \frac{1}{2}(x_n - \frac{\alpha}{x_n}) \\ &= \frac{1}{2}(\frac{x_n^2 - \alpha}{x_n}) \\ &> 0 \end{aligned}$$

since $x_n > \alpha$. Hence $\{x_n\}$ decreases monotonically. Also, $\{x_n\}$ is bounded by 0; thus $\{x_n\}$ converges. Let $\lim x_n = x$. Hence

$$\begin{aligned} \lim x_{n+1} = \lim \frac{1}{2}(x_n + \frac{\alpha}{x_n}) \quad &\Leftrightarrow \quad x = \frac{1}{2}(x + \frac{\alpha}{x}) \\ &\Leftrightarrow \quad x^2 = \alpha. \end{aligned}$$

Note that $x_n > 0$ for all n. Thus $x = \sqrt{\alpha}$. $\lim x_n = \sqrt{\alpha}$.

Proof of (b):

$$\begin{aligned} & x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}) \\ \Rightarrow \quad & x_{n+1} - \sqrt{\alpha} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}) - \sqrt{\alpha} \\ \Rightarrow \quad & x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{x_n} \\ \Rightarrow \quad & x_{n+1} - \sqrt{\alpha} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} \\ \Rightarrow \quad & \epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}. \end{aligned}$$

Hence

$$\epsilon_{n+1} < \beta(\frac{\epsilon_1}{\beta})^{2^n}$$

where $\beta = 2\sqrt{\alpha}$ by induction.

Proof of (c):

$$\frac{\epsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{2\sqrt{3}(2+\sqrt{3})} = \frac{1}{6+4\sqrt{3}} < \frac{1}{10}.$$

Thus

$$\begin{aligned} \epsilon_5 &< \beta(\frac{\epsilon_1}{\beta})^{2^4} < 2\sqrt{3} \cdot 10^{-16} < 4 \cdot 10^{-16}, \\ \epsilon_6 &< \beta(\frac{\epsilon_1}{\beta})^{2^5} < 2\sqrt{3} \cdot 10^{-32} < 4 \cdot 10^{-32}. \end{aligned}$$

Note: It is an application of Newton's method. Let $f(x) = x^2 - \alpha$ in Exercise 5.25.

17.

18.

19.

20.

21.

22. Suppose X is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.) *Hint:* Find a shrinking sequence of neighborhoods E_n such that $\overline{E_n} \subset G_n$, and apply Exercise 21.

    Proof: I've proved it in Chapter 2 Exercise 30.

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint:* For any m, n,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

Proof: For any $\epsilon > 0$, there exists N such that $d(p_n, p_m) < \epsilon$ and $d(q_m, q_n) < \epsilon$ whenever $m, n \geq N$. Note that

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

It follows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < 2\epsilon.$$

Thus $\{d(p_n, q_n)\}$ is a Cauchy sequence in X. Hence $\{d(p_n, q_n)\}$ converges.

24. Let X be a metric space. (a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X equivalent if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\triangle(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\triangle(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that $\triangle$ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

Proof of (a): Suppose there are three Cauchy sequences $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$. First, $d(p_n, p_n) = 0$ for all n. Hence, $d(p_n, p_n) = 0$ as $n \to \infty$. Thus it is reflexive. Next, $d(q_n, p_n) = d(p_n, q_n) \to 0$ as $n \to \infty$. Thus it is symmetric. Finally, if $d(p_n, q_n) \to 0$ as $n \to \infty$ and if $d(q_n, r_n) \to 0$ as $n \to \infty$, $d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n) \to 0 + 0 = 0$ as $n \to \infty$. Thus it is transitive. Hence this is an equivalence relation.

Proof of (b):

Proof of (c): Let $\{P_n\}$ be a Cauchy sequence in $(X^*, \triangle)$. We wish to show that there is a point $P \in X^*$ such that $\triangle(P_n, P) \to 0$ as $n \to \infty$. For each P_n, there is a Cauchy sequence in X, denoted $\{Q_k n\}$, such that $\triangle(P_n, Q_k n) \to 0$ as $k \to \infty$. Let $\epsilon_n > 0$ be a sequence tending to 0 as $n \to \infty$. From the double sequence $\{Q_k n\}$ we can extract a subsequence Q'_n such that $\triangle(P_n, Q'_n) < \epsilon_n$ for all n. From the triangle inequality, it follows that

$$\triangle(Q'_n, Q'_m) \leq \triangle(Q'_n, P_n) + \triangle(P_n, P_m) + \triangle(P_m, Q'_m). \tag{1}$$

Since $\{P_n\}$ is a Cauchy sequence, given $\epsilon > 0$, there is an $N > 0$ such that $\triangle(P_n, P_m) < \epsilon$ for $m, n > N$. We choose m and n so large that $\epsilon_m < \epsilon$, $\epsilon_n < \epsilon$. Thus (1) shows that $\{Q'_n\}$ is a Cauchy sequence in X. Let P be the corresponding equivalence class in S. Since

$$\triangle(P, P_n) \leq \triangle(P, Q'_n) + \triangle(Q'_n, P_n) < 2\epsilon$$

for $n > N$, we conclude that $P_n \to P$ as $n \to \infty$. That is, X^* is complete.

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