Continuity

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1. Suppose f is a real function define on R^1 which satisfies

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$$

for every $x \in R^1$. Does this imply that f is continuous?

Solution: No. Take $f(x) = 1$, if $x \in Z$; $f(x) = 0$. otherwise.

2. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Proof: If $f(\overline{E})$ is empty, the conclusion holds trivially. If $f(\overline{E})$ is non-empty, then we take an arbitrary point $y \in f(\overline{E})$. Thus, there exists $p \in \overline{E}$ such that $y = f(p)$. Thus $p \in E$ or $p \in E'$. Also, note that $\overline{f(E)} = f(E) \bigcup (f(E))'$. Now we consider the following two cases:

Case 1: If $p \in E$, then $y \in f(E) \subset \overline{f(E)}$. **Case 2:** Suppose $p \in E'$. Since f is continuous at $x = p$, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \epsilon$$

whenever $d_X(x, p) < \delta$ for all $x \in X$. Since p is a limit point of E, then for some $\delta > 0$ there exists $x \in E$. Thus $f(x) \in N_\epsilon(p)$ for some $f(x) \in f(E)$. Since ϵ is arbitrary, $f(p)$ is a limit point of $f(E)$ in Y. Thus $f(p) \in \overline{f(E)}$.

By case (1)(2), we proved that $f(\overline{E}) \subset \overline{f(E)}$.

Now we show that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$. Define

$$f(x) = \frac{1}{x}, X = (0, +\infty), Y = R^1, E = Z^+.$$

Thus

$$\begin{aligned} f(\overline{E}) &= f(E) = \{1/n : n \in Z^+\}, \\ \overline{f(E)} &= \overline{\{1/n : n \in Z^+\}} = \{0\} \bigcup \{1/n : n \in Z^+\}. \end{aligned}$$

3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

   Proof: Let $E = f(X) - Z(f)$, that is, the set of all $p \in X$ at which $f(p) \neq 0$. Take $p[-E$, and thus $f(p) \neq 0$. WLOG, we take $f(p) > 0$. Since f is continuous at $x = p$, thus for every $\epsilon > 0$ there exists a $\delta > 0$ such that

   $$|f(x) - f(p)| < \epsilon$$

   for all points $x \in X$ for which $d_X(x, p) < \delta$. Especially, we take $\epsilon = f(p)/2 > 0$. If $x \in N_\delta(p)$ for all x, then $f(x) > f(p)/2 > 0$, that is, $N_\delta(p) \subset E$, that is, p is an interior point of E. (If $f(p) < 0$, we take $\epsilon = -f(p)/2 > 0$). Since p is arbitrary, E is open. Thus, $Z(f)$ is closed.

4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

   Proof: First we need to show $f(E)$ is dense in $f(X)$, that is, every point of $f(X)$ is a limit point of $f(E)$, or a point of $f(E)$ (or both).

   Take any $y \in f(X)$, and then there exists a point $p \in X$ such that $y = f(p)$. Since E is dense in X, thus p is a limit point of E or $p \in E$.

If $p \in E$, then $y = f(p) \in f(E)$. Thus y is a point of $f(E)$, done. If p is a limit point of E and $p \notin E$. Since f is continuous on X, for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all points $x \in X$ for which $d_X(x, p) < \delta$. Since p is a limit point of E, there exists $q \in N_\delta(p)$ such that $q \neq p$ and $q \in E$. Hence

$$f(q) \in N_\epsilon(f(p)) = N_\epsilon(y).$$

and $f(q) \in f(E)$. Since $p \notin E$, $f(p) \notin f(E)$, and $f(q) \neq f(p)$. Hence $f(p)$ is a limit point of $f(E)$. Thus $f(E)$ is dense in $f(X)$.

Suppose $p \in X - E$. Since E is dense in X, p is a limit point of E and $p \notin E$. Hence we can take a sequence $\{q_n\} \to p$ such that $q_n \in E$ and $q_n \neq p$ for all n. (More precisely, since p is a limit point, every neighborhood $N_r(p)$ of p contains a point $q \neq p$ such that $q \in E$. Take $r = r_n = 1/n$, and thus $r_n \to 0$ as $n \to \infty$. At this time we can get $q = q_n \to p$ as $n \to \infty$.) Hence

$$\begin{aligned} g(p) &= g(\lim_{n\to\infty} q_n) \\ &= \lim_{n\to\infty} g(q_n) \\ &= \lim_{n\to\infty} f(q_n) \\ &= f(\lim_{n\to\infty} q_n) \\ &= f(p). \end{aligned}$$

Thus $g(p) = f(p)$ for all $p \in X$.

5. If f is a real continuous function defined on a closed set $E \subset R^1$, prove that there exist continuous real function g on R^1 such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to R^1.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector valued functions. *Hint:* Let the graph of g be a straight line on each of the segments which constitute

the complement of E (compare Exercise 29, Chap. 2). The result remains true if R^1 is replaced by any metric space, but the proof is not so simple.

Proof: Note that the following fact:

> Every open set of real numbers is the union of a countable collection of disjoint open intervals.

Thus, consider $E^c = \bigcup(a_i, b_i)$, where $i \in Z$, and $a_i < b_i < a_{i+1} < b_{i+1}$. We extend g on (a_i, b_i) as following:

$$g(x) = f(a_i) + (x - a_i)\frac{f(b_i) - f(a_i)}{b_i - a_i}$$

($g(x) = f(x)$ for $x \in E$). Thus g is well-defined on R^1, and g is continuous on R^1 clearly.

Next, consider $f(x) = 1/x$ on a open set $E = R - 0$. f is continuous on E, but we cannot redefine $f(0) =$ any real number to make new $f(x)$ continue at $x = 0$.

Next, consider a vector valued function

$$f(x) = (f_1(x), ..., f_n(x)),$$

where $f_i(x)$ is a real valued function. Since f is continuous on E, each component of f, f_i, is also continuous on E, thus we can extend f_i, say g_i, for each $i = 1, ..., n$. Thus,

$$g(x) = (g_1(x), ..., g_n(x))$$

is a extension of $f(x)$ since each component of g, g_i, is continuous on R^1 implies g is continuous on R^n.

Note: The above fact only holds in R^1. If we change R^1 into any metric spaces, we have no the previous fact.

6. If f is defined on E, the *graph* of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plain.

   Suppose E is compact, and prove that that f is continuous on E if and only if its graph is compact.

   Proof: ($\Rightarrow$) Let $G = \{(x, f(x)) : x \in E\}$. Since f is a continuous mapping of a compact set E into $f(E)$, by Theorem 4.14 $f(E)$ is also compact. We claim that the product of finitely many compact sets is compact. Thus $G = E \times f(E)$ is also compact.

   ($\Leftarrow$) (Due to Shin-Yi Lee) Define

$$g(x) = (x, f(x))$$

   from E to G for $x \in E$. We claim that $g(x)$ is continuous on E. Consider $h(x, f(x)) = x$ from G to E. Thus h is injective, continuous on a compact set G. Hence its inverse function $g(x)$ is injective and continuous on a compact set E.

   Since $g(x)$ is continuous on E, the component of $g(x)$, say $f(x)$, is continuous on a compact E.

   Proof of Claim: We prove that the product of two compact spaces is compact; the claim follows by induction for any finite product.

   Step 1. Suppose that we are given sets X and Y, with Y is compact. Suppose that $x_0 \in X$, and N is an open set of $X \times Y$ containing the "slice" $x_0 \times Y$ of $X \times Y$. We prove the following:

   > There is a neighborhood W of x_0 in X such that N contains the entire set $W \times Y$.

The set $W \times Y$ is often called a **tube** about $x_0 \times Y$.

First let us cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$) lying in N. The set $x_0 \times Y$ is compact, being homeomorphic to Y. Therefore, we can cover $x_0 \times Y$ by finitely many such basis elements

$$U_1 \times V_1, ..., U_n \times V_n.$$

(We assume that each of the basis elements $U_i \times V_i$ actually intersects $x_0 \times Y$, since otherwise that basis element would be superfluous; we could discard it from the finite collection and still have a covering of $x_0 \times Y$.) Define

$$W = U_1 \bigcap ... \bigcap U_n.$$

The set W is open, and it contains x_0 because each set $U_i \times V_i$ intersects $x_0 \times Y$.

We assert that the sets $U_i \times V_i$, which were chosen to cover the slice $x_0 \times Y$, actually cover the tube $W \times Y$. Let $x \times y \in W \times Y$. Consider the point $x_0 \times y$ of the slice $x_0 \times Y$ having the same y-coordinate as this point. Now $x_0 \times y \in U_i \times V_i$ for some i, so that $y \in V_i$. But $x \in U_j$ for every j (because $x \in W$). Therefore, we have $x \times y \in U_i \times V_i$, as desired.

Since all the sets $U_i \times V_i \subset N$, and since they cover $W \times Y$, the tube $W \times Y \subset N$ also.

Step 2. Now we prove the claim. Let X and Y be compact sets. Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements $A_1, ..., A_m$ of $\mathcal{A}$. Their union $N = A_1 \bigcup ... \bigcup A_m$ is an open set containing $x_0 \times Y$; by Step 1, the open set N contains a tube $W \times Y$ about $x_0 \times Y$, where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, ..., A_m$ of $\mathcal{A}$.

Thus, for each $x \in X$, we can choose a neighborhood W_x of x such that the tube $W_x \times Y$ can be covered by finitely many element of $\mathcal{A}$. The collection of all the neighborhoods W_x is an open covering of X; therefore by compactness of X, there exists a finite subcollection

$$\{W_1, ..., W_k\}$$

covering X. The union of the tubes

$$W_1 \times Y, ..., W_k \times Y$$

is all of $X \times Y$; since each may be covered by finitely many elements of $\mathcal{A}$, so may $X \times Y$ be covered.

7. If $E \subset X$ and if f is a function defined on X, the restriction of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on R^2 by: $f(0,0) = g(0,0) = 0$, $f(x,y) = xy^2/(x^2+y^4)$, $g(x,y) = xy^2/(x^2+y^6)$ if $(x,y) \neq (0,0)$. Prove that f is bounded on R^2, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restriction of both f and g to every straight line in R^2 are continuous!

   Proof: Since $x^2 + y^4 \geq 2xy^2$, $f(x,y) \leq 2$ for all $(x,y) \in R^2$. That is, f is bounded (by 2). Next, select

   $$(x_n, y_n) = (\frac{1}{n^3}, \frac{1}{n}).$$

   $(x_n, y_n) \to (0,0)$ as $n \to \infty$, and $g(x_n, y_n) = n/2 \to \infty$ as $n \to \infty$, that is, $g(x,y)$ is unbounded in every neighborhood of $(0,0)$ by choosing large enough n.

   Next, select

   $$(x_n, y_n) = (\frac{1}{n^2}, \frac{1}{n}).$$

$(x_n, y_n) \to (0,0)$ as $n \to \infty$, and $f(x_n, y_n) = 1/2$ for all n. Thus,

$$\lim_{n\to\infty} f(x_n, y_n) = \frac{1}{2} \neq 0 = f(0,0).$$

for some sequence $\{(x_n, y_n)\}$ in R^2. Thus, f is not continuous at $(0,0)$.

Finally, we consider two cases of straight lines in R^2: (1) $x = c$ and (2) $y = ax + b$. (equation of straight lines).

(1) $x = c$: If $c \neq 0$, $f(x,y) = cy^2/(c^2+y^4)$ and $g(x,y) = cy^2/(c^2+y^6)$ are continuous since cy^2, c^2+y^4, and c^2+y^6 are continuous on R^1 respect to y, and c^2+y^4, c^2+y^6 are nonzero. If $c = 0$, then $f(x,y) = g(x,y) = 0$, and it is continuous trivially.

(2) $y = ax + b$: If $b \neq 0$, then this line dose not pass $(0,0)$. Then $f(x,y) = x(ax+b)^2/(x^2+(ax+b)^4)$ and $g(x,y) = x(ax+b)^2/(x^2+(ax+b)^6)$. By previous method we conclude that $f(x,y)$ and $g(x,y)$ are continuous. If $b = 0$, then $f(x,y) = 0$ if $(x,y) = (0,0)$; $f(x,y) = a^2x/(1+a^4x^2)$, and $g(x,y) = 0$ if $(x,y) = (0,0)$; $g(x,y) = a^2x/(1+a^6x^4)$. Thus, $f(x,y) \to 0/1 = 0 = f(0,0)$ and $g(x,y) \to 0/1 = 0 = f(0,0)$ as $x \to 0$. Thus, f and g are continuous.

Both of two cases implies that the restriction of both f and g to every straight line in R^2 are continuous.

8. Let f be a real uniformly continuous function on the bounded set E in R^1. Prove that f is bounded on E.

   Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

   Proof: Let E is bounded by $M > 0$, that is, $|x| \leq M$ for all $x \in E$. Since f is uniformly continuous, take $\epsilon = 1$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $|x - y| < \delta$ where $x, y \in E$. For every $x \in E$, there exists an integer $n = n_x$ such that

$$n\delta \le x < (n+1)\delta.$$

Since E is bounded, the collection of $S = \{n_x : x \in E\}$ is finite. Suppose $x \in E$ and x is the only one element satisfying $n\delta \le x < (n+1)\delta$ for some n. Let $x = x_n$, and thus

$$|f(x)| \le |f(x_n)|$$

for all $x \in E \bigcap [n\delta, (n+1)\delta)$. If there are more than two or equal to two element satisfying that condition, then take some one as x_n. Since

$$|x - x_n| < \delta$$

for all $x \in E \bigcap [n\delta, (n+1)\delta)$. Thus

$$|f(x) - f(x_n)| < 1$$

for all $x \in E \bigcap [n\delta, (n+1)\delta)$, that is,

$$|f(x)| < 1 + |f(x_n)|.$$

Hence

$$|f(x)| < \max_{n \in S} (1 + f(x_n)).$$

(Since S is finite, that maximum is meaningful). Thus $f(x)$ is bounded.

Note: If boundedness of E is omitted from the hypothesis, define $f(x) = x$ for $x \in E = R^1$. Hence f is uniformly continuous on E, but $f(E) = R^1$ is unbounded.

9. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\epsilon > 0$ there exists a $\delta > 0$ such that $\text{diam} f(E) < \epsilon$ for all $E \subset X$ with $\text{diam} E < \delta$.

Proof: Recall the original definition of uniformly continuity:

for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all p and q in X for which $d_X(p, q) < \delta$.

($\Leftarrow$) Given $\epsilon > 0$. $\forall p, q \in X$ for which $d_X(p, q) < \delta$. Take

$$E = \{p, q\},$$

and thus $\text{diam}E = \sup_{p,q \in E} d(p, q) = d(p, q) < \delta$. Hence $\text{diam}f(E) < \epsilon$. Note that $\text{diam}f(E) \geq d(f(p), f(q))$ since $p, q \in E$. Hence $d(f(p), f(q)) < \epsilon$. Thus for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all p and q in X for which $d_X(p, q) < \delta$.

($\Rightarrow$) $\forall E \subset X$ with $\text{diam}E < \delta$. $\forall p, q \in R$, $d(p, q) \leq \text{diam}E < \delta$. Thus we have

$$d(f(p), f(q)) < \frac{\epsilon}{2}$$

for all $p, q \in E$. Hence $\text{diam}f(E) \leq \epsilon/2 < \epsilon$. Thus to every $\epsilon > 0$ there exists a $\delta > 0$ such that $\text{diam}f(E) < \epsilon$ for all $E \subset X$ with $\text{diam}E < \delta$.

10. Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\epsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$. Use Theorem 2.37 to obtain a contradiction.

11. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.

    Proof: Let $\{x_n\}$ be a Cauchy sequence in X. $\forall \epsilon' > 0$, $\exists N$ such that $d_X(x_n, x_m) < \epsilon'$ whenever $m, n \geq N$. Since f is a uniformly continuous, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$. Take $\epsilon' = \delta$. Thus

$$d_Y(f(x_n), f(x_m)) < \epsilon$$

whenever $m, n \geq N$ for some N; that is, $\{f(x_n)\}$ is a Cauchy sequence.

12. A uniformly continuous function of a uniformly continuous function is uniformly continuous.

    State this more precisely and prove it.

13. Let E be a dense subset of a metric space X, and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X (see Exercise 5 for terminology). (Uniqueness follows from Exercise 4.) *Hint:* For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < 1/n$. Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p))$, $f(V_2(p))$, ..., consists of a single point, say $g(p)$, of R^1. Prove that the function g so define on X is the desired extension of f.

14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

    Proof: Let $g(x) = f(x) - x$. If $g(1) = 0$ or $g(0) = 0$, then the conclusion holds trivially. Now suppose $g(1) \neq 0$ and $g(0) \neq 0$. Since f is from I to I, $0 \neq f(x) \neq 1$. Thus,

$$\begin{aligned} g(1) &= f(1) - 1 < 0, \\ g(0) &= f(0) - 0 > 0. \end{aligned}$$

    Since g is continuous on $[0, 1]$, by Intermediate Value Theorem (Theorem 4.23)

$$g(c) = 0$$

    for some $c \in (0, 1)$. Hence $f(c) = c$ for some $c \in (0, 1)$.

15. Call a mapping of X into Y *open* if $f(V)$ is an open set Y whenever V is an open set X.

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Proof: Suppose not, there exist three points $x_1 < x_2 < x_3 \in R^1$ such that

$$f(x_2) > f(x_1), f(x_2) > f(x_3)$$

or

$$f(x_2) < f(x_1), f(x_2) < f(x_3).$$

WLOG, we only consider the case that $f(x_2) > f(x_1)$, $f(x_2) > f(x_3)$ for some $x_1 < x_2 < x_3$. Since f is continuous on R^1, for

$$\epsilon = \frac{f(x_2) - f(x_1)}{2} > 0$$

there exists $\delta_1 > 0$ such that

$$|f(x) - f(x_1)| < \epsilon$$

whenever $|x - x_1| < \delta_1$. That is,

$$f(x) < \frac{f(x_1) + f(x_2)}{2} < f(x_2)$$

whenever $x < x_1 + \delta_1$. Note that $\delta_1 < x_2 - x_1$. Hence we can take $y_1 \in (x_1, x_1 + \delta_1)$. Similarly, for

$$\epsilon = \frac{f(x_2) - f(x_3)}{2} > 0$$

there exists $\delta_2 > 0$ such that

$$|f(x) - f(x_3)| < \epsilon$$

whenever $|x - x_3| < \delta_2$. That is,

$$f(x) < \frac{f(x_2) + f(x_3)}{2} < f(x_2)$$

whenever $x > x_3 - \delta_2$. Note that $\delta_2 < x_3 - x_2$. Hence we can take $y_2 \in (x_3 - \delta_2, x_3)$. Note that $y_2 > y_1$. Since f is continuous on a closed set $[y_1, y_2]$, f take a maximum value at $p \in [y_1, y_2]$. Note that

$$\sup_{x \in (x_1, x_3)} f(x) \leq \sup_{x \in [y_1, y_2]} f(x)$$

by previous inequations. Also,

$$\sup_{x \in (x_1, x_3)} f(x) \geq \sup_{x \in [y_1, y_2]} f(x)$$

Hence $\sup_{x \in (x_1, x_3)} f(x) = \sup_{x \in [y_1, y_2]} f(x)$. Since (x_1, x_3) is an open set, $f((x_1, x_3))$ is also open. Note that $f(p) \in f((x_1, x_3))$ but $f(p)$ is not an interior point of $f((x_1, x_3))$. (otherwise $f(p) + \epsilon \in f((x_1, x_3))$ for some $\epsilon > 0$. That is, $f(p) + \epsilon > f(p)$, a contradiction with the maximum of $f(p)$).

16. Let $[x]$ denote the largest integer contained in x, this is, $[x]$ is a integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the function $[x]$ and (x) have?

17. Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable. *Hint:* Let E be the set on which $f(x-) < f(x+)$. With each point x of E, associate a triple (p, q, r) of rational numbers such that

    (a) $f(x-) < p < f(x+)$,
    (b) $a < q < t < x$ implies $f(t) < p$,
    (c) $x < t < r < b$ implies $f(t) > p$.

    The set of all such triples is countable. Show that each triple is associated with at most one point of E. Deal similarly with the other possible types of simple discontinuities.

18. Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \frac{1}{n} & (x = \frac{m}{n}). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

19. Suppose f is a real function with domain R^1 which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b.

Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

Hint: If $x_n \to x_0$, but $f(x_n) > r > f(x_0)$ for some r and all n, then $f(t_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Find a contradiction. (N. J. Fine, Amer. Math. Monthly, vol. 73, 1966, p. 782.)

Proof: Let $S = \{x : f(x) = r\}$. If $x_n \to x_0$, but $f(x_n) > r > f(x_0)$ for some r and all n since Q is dense in R^1, then $f(t_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Hence x_0 is a limit point of S. Since S is closed, $f(x_0) = r$, a contradiction. Hence, $\limsup f(x_n) \leq f(x_0)$. Similarly, $\liminf f(x_n) \geq f(x_0)$. Hence, $\lim f(x_n) = f(x_0)$, and f is continuous at x_0.

Note: Original problem is stated as follows:

> Let f be a function from the reals to the reals, differentiable at every point. Suppose that, for every r, the set of points

x, where $f'(x) = r$, is closed. Prove that f' is continuous.

If we replace Q into any dense subsets of R^1, the conclusion also holds.

20. If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by
$$\rho_E(x) = \inf_{z \in E} d(x, z).$$
(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
(b) Prove that $\rho_E(x)$ is a uniformly continuous function on X, by showing that
$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$
for all $x \in X$, $y \in X$. *Hint:* $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$, so that
$$\rho_E(x) \leq d(x, y) + \rho_E(y).$$

Proof of (a): ($\Leftarrow$) If $x \in E \subset \overline{E}$, then
$$\inf_{z \in E} d(x, z) \leq d(x, x) = 0$$
since we take $z = x \in E$. Hence $\rho_E(x) = 0$ if $x \in E$. Suppose $x \in \overline{E} - E$, that is, x is a limit point of E. Thus for every neighborhood of x contains a point $y \neq x$ such that $q \in E$. It implies that $d(x, y) \to 0$ for some $y \in E$, that is, $\rho_E(x) = \inf_{z \in E} d(x, z) = 0$ exactly.

($\Rightarrow$) Suppose $\rho_E(x) = \inf_{z \in E} d(x, z) = 0$. Fixed some $x \in X$. If $d(x, z) = 0$ for some $z \in E$, then $x = z$, that is $x \in E \subset \overline{E}$. If $d(x, z) > 0$ for all $z \in E$, then by $\inf_{z \in E} d(x, z) = 0$, for any $\epsilon > 0$ there exists $z \in E$ such that
$$d(x, z) < \epsilon,$$
that is,
$$z \in N_\epsilon(x).$$
Since ϵ is arbitrary and $z \in E$, x is a limit point of E. Thus $x \in E' \subset \overline{E}$.

Proof of (b): For all $x \in X$, $y \in X$, $z \in E$,

$$\rho_E(x) \leq d(x,z) \leq d(x,y) + d(y,z).$$

Take infimum on both sides, and we get that

$$\rho_E(x) \leq d(x,y) + \rho_E(y).$$

Similarly, we also have

$$\rho_E(y) \leq d(x,y) + \rho_E(x).$$

Hence

$$|\rho_E(x) - \rho_E(y)| \leq d(x,y)$$

for all $x \in X$, $y \in X$. Thus ρ_E is a uniformly continuous function on X.

Exercise 1: (Due to Shin-Yi Lee) In a metric space (S,d), let A be a nonempty subset of S. Define a function $f_A(x) : S \to R$ by the equation

$$f_A(x) = \inf\{d(x,y) : y \in A\}$$

for every $x \in S$. The value $f_A(x)$ is called the distance from x to A.

(a) Prove that f_A is uniformly continuous on S.
(b) Prove $\overline{A} = \{x \in S : f_A(x) = 0\}$.

Exercise 2: (Due to Shin-Yi Lee) In a metric space (S,d). Let A and B be two disjoint closed subsets of S. Prove that there are two open subset of S, say U and V such that $A \subset U$ and $B \subset V$ with $U \bigcap V = \phi$.

It will be shown in Exercise 4.22.

21. Suppose K and F are disjoint sets in a metric space X. K is compact. F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$. *Hint:* ρ_F is a continuous positive function on K.

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Proof: Let

$$\rho_F(x) = \inf_{z \in F} d(x, z)$$

for all $x \in K$. By Exercise 4.20(a), we know that

$$\rho_F(x) = 0 \Leftrightarrow x \in \overline{F} = F$$

(since F is closed). That is, $\rho_F(x) = 0$ if and only if $x \in F$. Since K and F are disjoint, $\rho_F(x)$ is a positive function. Also, by Exercise 4.20(b) $\rho_F(x)$ is continuous. Thus $\rho_F(x)$ is a continuous positive function.

Since K is compact, $\rho_F(x)$ takes minimum $m > 0$ for some $x_0 \in K$. Take $\delta = m/2 > 0$ as desired.

Next, let $X = R^1$,

$$\begin{aligned} A &= Z^+ - \{2\}, \\ B &= \{n + 1/n : n \in Z^+\}. \end{aligned}$$

Hence A and B are disjoint, and they are not compact. Suppose there exists such $\delta > 0$. Take

$$x = [\frac{1}{\delta}] + 1 \in A, y = x + \frac{1}{x} \in B.$$

However,

$$d(x, y) = \frac{1}{x} < \frac{1}{1/\delta} = \delta,$$

a contradiction.

22. Let A and B be disjoint nonempty closed sets in a metric space X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}, \quad (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of Exercise 3: Every closed set A contained in X is $Z(f)$ for some continuous real f on X. Setting

$$V = f^{-1}([0, 1/2)), W = f^{-1}((1/2, 1]),$$

show that V and W are open and disjoint, and that A is contained in V, B is contained in W. (Thus pairs of disjoint closed set in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called normality.)

Proof: Note that $\rho_A(p)$ and $\rho_B(p)$ are (uniformly) continuous on X, and $\rho_A(p) + \rho_B(p) > 0$. (Clearly, $\rho_A(p) + \rho_B(p) \geq 0$ by the definition. If $\rho_A(p) + \rho_B(p) = 0$, then $p \in A \bigcap B$ by Exercise 20, a contradiction). Thus $f(p) = \rho_A(p)/(\rho_A(p)+\rho_B(p))$ is continuous on X. Next, $f(p) \geq 0$, and $f(p) \leq 1$ since $\rho_A(p) \leq \rho_A(p) + \rho_B(p)$. Thus $f(X)$ lies in $[0, 1]$.

Next, $f(p) = 0 \Leftrightarrow \rho_A(p) = 0 \Leftrightarrow p \in A$ precisely, and $f(p) = 1 \Leftrightarrow \rho_B(p) = 0 \Leftrightarrow p \in B$ precisely by Exercise 20.

Now we prove a converse of Exercise 3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X. If $Z(f) = \phi$, then $f(x) = 1$ for all $x \in X$ satisfies our requirement. If $Z(f) \neq \phi$, we consider two possible cases: (1) $Z(f) = X$; (2) $Z(f) \neq X$. If $Z(f) = X$, then $f(x) = 0$ for all $x \in X$. If $Z(f) \neq X$, we can choose $p \in X$ such that $f(p) \neq 0$. Note that $Z(f)$ and $\{p\}$ are one pair of disjoint closed sets. Hence we let

$$f(x) = \frac{\rho_{Z(f)}(x)}{\rho_{Z(f)}(x) + \rho_{\{p\}}(x)}.$$

By the previous result, we know that $f(x)$ satisfies our requirement. Hence we complete the whole proof.

Note that $[0, 1/2)$ and $(1/2, 1]$ are two open sets of $f(X)$. Since f is continuous, $V = f^{-1}([0, 1/2))$ and $W = f^{-1}((1/2, 1])$ are two open sets. $f^{-1}(\{0\}) \subset f^{-1}([0, 1/2))$, and $f^{-1}(\{1\}) \subset f^{-1}((1/2, 1])$. Thus, $A \subset V$ and $B \subset W$. Thus a metric space X is normal.

23. A real-valued function f defined in (a, b) is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.)

If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

24. Assume that f is a continuous real function defined in (a, b) such that

$$f(\frac{x + y}{2}) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex.

25. If $A \subset R^k$ and $B \subset R^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.

(a) If K is compact and C is closed in R^k, prove that $K + C$ is closed. *Hint:* Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$.

Then K and F are disjoint. Choose δ as in Exercise 21. Show that the open ball with center $\mathbf{z}$ and radius δ does not intersect $K + C$.

(b) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of R^1 whose sum $C_1 + C_2$ is *not* closed, by showing that $C_1 + C_2$ is a countable dense subset of R^1.

26. Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.

    Prove that f is uniformly continuous if h is uniformly continuous. *Hint:* g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.

    Prove also that f is continuous if h is continuous.

    Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypothese, even when X and Z are compact.