Differentiation

Written by *Men-Gen Tsai*

email: b89902089@ntu.edu.tw

1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \le (x - y)^2$$

for all real x and y. Prove that f is constant.

Proof: $|f(x)-f(y)| \le (x-y)^2$ for all real x and y. Fix y, $|\frac{f(x)-f(y)}{x-y}| \le |x-y|$. Let $x \to y$, therefore,

$$0 \le \lim_{x \to y} \frac{f(x) - f(y)}{x - y} \le \lim_{x \to y} |x - y| = 0$$

It implies that $(f(x) - f(y))/(x - y) \to 0$ as $x \to y$. Hence $f'(y) = 0$, $f = const.$

2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentible, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Proof: For every pair $x > y$ in (a, b), $f(x) - f(y) = f'(c)(x - y)$ where $y < c < x$ by Mean-Value Theorem. Note that $c \in (a, b)$ and $f'(x) > 0$ in (a, b), hence $f'(c) > 0$. $f(x) - f(y) > 0$, $f(x) > f(y)$ if $x > y$, f is strictly increasing in (a, b).

Let $\Delta g = g(x_0 + h) - g(x_0)$. Note that $x_0 = f(g(x_0))$, and thus,

$$(x_0 + h) - x_0 = f(g(x_0 + h)) - f(g(x_0)),$$

$$h = f(g(x_0) + \Delta g) - f(g(x_0)) = f(g + \Delta g) - f(g).$$

Thus we apply the fundamental lemma of differentiation,

$$h = [f'(g) + \eta(\Delta g)]\Delta g,$$

$$\frac{1}{f'(g) + \eta(\Delta g)} = \frac{\Delta g}{h}$$

Note that $f'(g(x)) > 0$ for all $x \in (a, b)$ and $\eta(\Delta g) \to 0$ as $h \to 0$, thus,

$$\lim_{h \to 0} \Delta g / h = \lim_{h \to 0} \frac{1}{f'(g) + \eta(\Delta g)} = \frac{1}{f'(g(x))}.$$

Thus $g'(x) = \frac{1}{f'(g(x))}$, $g'(f(x)) = \frac{1}{f'(x)}$.

3. Suppose g is a real function on R^1, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissible values of ϵ can be determined which depends only on M.)

Proof: For every $x < y$, and $x, y \in R$, we will show that $f(x) \neq f(y)$. By using Mean-Value Theorem:

$$g(x) - g(y) = g'(c)(x - y) \ \ where \ \ x < c < y,$$

$$(x - y) + \epsilon((x) - g(y)) = (\epsilon g'(c) + 1)(x - y),$$

that is,

$$f(x) - f(y) = (\epsilon g'(c) + 1)(x - y). \ \ (*)$$

Since $|g'(x)| \leq M$, $-M \leq g'(x) \leq M$ for all $x \in R$. Thus $1 - \epsilon M \leq \epsilon g'(c) + 1 \leq 1 + \epsilon M$, where $x < c < y$. Take $c = \frac{1}{2M}$, and $\epsilon g'(c) + 1 > 0$ where $x < c < y$ for all x, y. Take into equation (*), and $f(x) - f(y) < 0$ since $x - y < 0$, that is, $f(x) \neq f(y)$, that is, f is one-to-one (injective).

4. If
$$C_0 + \frac{C_1}{2} + ... + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$
where C_0, ..., C_n are real constants, prove that the equation
$$C_0 + C_1 x + ... + C_{n-1}x^{n-1} + C_n x^n = 0$$
has at least one real root between 0 and 1.

Proof: Let $f(x) = C_0 x + ... + \frac{C_n}{n+1}x^{n+1}$. f is differentiable in R^1 and $f(0) = f(1) = 0$. Thus, $f(1) - f(0) = f'(c)$ where $c \in (0, 1)$ by Mean-Value Theorem. Note that
$$f'(x) = C_0 + C_1 x + ... + C_{n-1}x^{n-1} + C_n x^n.$$
Thus, $c \in (0, 1)$ is one real root between 0 and 1 of that equation.

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof: $f(x + 1) - f(x) = f'(c)(x + 1 - x)$ where $x < c < x + 1$ by Mean-Value Theorem. Thus, $g(x) = f'(c)$ where $x < c < x + 1$, that is,
$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} f'(c) = \lim_{c \to +\infty} f'(c) = 0.$$

6. Suppose
(a) f is continuous for $x \geq 0$,
(b) $f'(x)$ exists for $x > 0$,
(c) $f(0) = 0$,

(d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Proof: Our goal is to show $g'(x) > 0$ for all $x > 0$
$\Leftrightarrow g'(x) = \frac{xf'(x) - f(x)}{x^2} > 0 \Leftrightarrow f'(x) > \frac{f(x)}{x}$.
Since $f'(x)$ exists, $f(x) - f(0) = f'(c)(x - 0)$ where $0 < c < x$ by Mean-Value Theorem. $\Rightarrow f'(c) = \frac{f(x)}{x}$ where $0 < c < x$. Since f' is monotonically increasing, $f'(x) > f'(c)$, that is, $f'(x) > \frac{f(x)}{x}$ for all $x > 0$.

7. Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

Proof:

$$\frac{f'(t)}{g'(t)} = \frac{\lim_{t \to x} \frac{f(t) - f(x)}{t - x}}{\lim_{t \to x} \frac{g(t) - g(x)}{t - x}} = \frac{\lim_{t \to x} f(t)}{\lim_{t \to x} g(t)} = \lim_{t \to x} \frac{f(t)}{g(t)}$$

Surely, this holds also for complex functions.

8. Suppose $f'(x)$ is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

Proof: Since $f'(x)$ is continuous on a compact set $[a, b]$, $f'(x)$ is uniformly continuous on $[a, b]$. Hence, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f'(t) - f'(x)| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. Thus, $f'(c) = \frac{f(t)-f(x)}{t-x}$ where c between t and x by Mean-Value Theorem. Note that $0 < |c - x| < \delta$ and thus $|f'(c) - f'(x)| < \epsilon$, thus,

$$|\frac{f(t) - f(x)}{t - x} - f'(x)| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$.

Note: It does not hold for vector-valued functions. If not, take

$$f(x) = (\cos x, \sin x),$$

$[a, b] = [0, 2\pi]$, and $x = 0$. Hence $f'(x) = (-\sin x, \cos x)$. Take any $1 > \epsilon > 0$, there exists $\delta > 0$ such that

$$|\frac{f(t) - f(0)}{t - 0} - f'(0)| < \epsilon$$

whenever $0 < |t| < \delta$ by our hypothesis. With calculating,

$$|(\frac{\cos t - 1}{t}, \frac{\sin t}{t}) - (0, 1)| < \epsilon$$

$$|(\frac{\cos t - 1}{t}, \frac{\sin t}{t} - 1)| < \epsilon$$

$$(\frac{\cos t - 1}{t})^2 + (\frac{\sin t}{t} - 1)^2 < \epsilon^2 < \epsilon$$

$$\frac{2}{t^2} + 1 - \frac{2(\cos t + \sin t)}{t} < \epsilon$$

since $1 > \epsilon > 0$. Note that

$$\frac{2}{t^2} + 1 - \frac{4}{t} < \frac{2}{t^2} + 1 - \frac{2(\cos t + \sin t)}{t}$$

But $\frac{2}{t^2} + 1 - \frac{4}{t} \to +\infty$ as $t \to 0$. It contradicts.

9. Let f be a continuous real function on R^1, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 0$ as $x \to 0$. Dose it follow that $f'(0)$ exists?

Note: We prove a more general exercise as following.
Suppose that f is continuous on an open interval I containing x_0, suppose that f' is defined on I except possibly at x_0, and suppose that $f'(x) \to L$ as $x \to x_0$. Prove that $f'(x_0) = L$.

Proof of the Note: Using L'Hospital's rule:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} f'(x_0 + h)$$

By our hypothesis: $f'(x) \to L$ as $x \to x_0$. Thus,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = L,$$

Thus $f'(x_0)$ exists and

$$f'(x_0) = L.$$

10. Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Compare with Example 5.18. *Hint:*

$$\frac{f(x)}{g(x)} = (\frac{f(x)}{x} - A)\frac{x}{g(x)} + A\frac{x}{g(x)}.$$

Apply Theorem 5.13 to the real and imaginary parts of $\frac{f(x)}{x}$ and $\frac{g(x)}{x}$.

Proof: Write $f(x) = f_1(x) + if_2(x)$, where $f_1(x)$, $f_2(x)$ are real-valued functions. Thus,

$$\frac{df(x)}{dx} = \frac{df_1(x)}{dx} + i\frac{df_2(x)}{dx},$$

Apply L'Hospital's rule to $\frac{f_1(x)}{x}$ and $\frac{f_2(x)}{x}$, we have

$$\lim_{x\to 0} \frac{f_1(x)}{x} = \lim_{x\to 0} f_1'(x)$$

$$\lim_{x\to 0} \frac{f_2(x)}{x} = \lim_{x\to 0} f_2'(x)$$

Combine $f_1(x)$ and $f_2(x)$, we have

$$\lim_{x\to 0} \frac{f_1(x)}{x} + i \lim_{x\to 0} \frac{f_2(x)}{x} = \lim_{x\to 0} \frac{f_1(x)}{x} + i\frac{f_2(x)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

or

$$\lim_{x\to 0} \frac{f_1(x)}{x} + i \lim_{x\to 0} \frac{f_2(x)}{x} = \lim_{x\to 0} f_1'(x) + i \lim_{x\to 0} f_2'(x) = \lim_{x\to 0} f'(x)$$

Thus, $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} f'(x)$. Similarly, $\lim_{x\to 0} \frac{g(x)}{x} = \lim_{x\to 0} g'(x)$. Note that $B \neq 0$. Thus,

$$\begin{aligned} \lim_{x\to 0} \frac{f(x)}{g(x)} &= \lim_{x\to 0} (\frac{f(x)}{x} - A)\frac{x}{g(x)} + A\frac{x}{g(x)} \\ &= (A - A)\frac{1}{B} + \frac{A}{B} = \frac{A}{B}. \end{aligned}$$

Note: In Theorem 5.13, we know $g(x) \to +\infty$ as $x \to 0$. ($f(x) = x$, and $g(x) = x + x^2 e^{\frac{i}{x^2}}$).

11. Suppose f is defined in a neighborhood of x, and suppose f"(x) exists. Show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f"(x)$ dose not. Hint: Use Theorem 5.13.

Proof: By using L'Hospital's rule: (respect to h.)

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

Note that

$$\begin{aligned} f''(x) &= \frac{1}{2}(f''(x)+f''(x)) \\ &= \frac{1}{2}(\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}+\lim_{h\to 0}\frac{f'(x-h)-f'(x)}{-h}) \\ &= \frac{1}{2}\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{h} \\ &= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} \end{aligned}$$

Thus,

$$\frac{f(x+h)+f(x-h)-2f(x)}{h^2}\to f''(x)$$

as $h\to 0$. Counter-example: $f(x)=x|x|$ for all real x.

12. If $f(x)=|x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

Proof: $f'(x)=3|x|^2$ if $x\neq 0$. Consider

$$\frac{f(h)-f(0)}{h}=\frac{|h|^3}{h}$$

Note that $|h|/h$ is bounded and $|h|^2\to 0$ as $h\to 0$. Thus,

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=0.$$

Hence, $f'(x)=3|x|^2$ for all x. Similarly,

$$f''(x)=6|x|.$$

Thus,

$$\frac{f''(h) - f(0)}{h} = 6\frac{|h|}{h}$$

Since $\frac{|h|}{h} = 1$ if $h > 0$ and $= -1$ if $h < 0$, $f'''(0)$ does not exist.

13. Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & (\text{if } x \neq 0), \\ 0 & (\text{if } x = 0). \end{cases}$$

Prove the following statements:

(a) f is continuous if and only if $a > 0$.
(b) $f'(0)$ exists if and only if $a > 1$.
(c) f' is bounded if and only if $a \geq 1 + c$.
(d) f' is continuous if and only if $a > 1 + c$.
(e) $f''(0)$ exists if and only if $a > 2 + c$.
(f) f'' is bounded if and only if $a \geq 2 + 2c$.
(g) f'' is continuous if and only if $a > 2 + 2c$.

Proof: For (a): ($\Rightarrow$) f is continuous iff for any sequence $\{x_n\} \to 0$ with $x_n \neq 0$, $x_n^a \sin x_n^{-c} \to 0$ as $n \to \infty$. In particular, take

$$x_n = (\frac{1}{2n\pi + \pi/2})^{\frac{1}{c}} > 0$$

and thus $x_n^a \to 0$ as $n \to \infty$. Hence $a > 0$. (If not, then $a = 0$ or $a < 0$. When $a = 0$, $x_n^a = 1$. When $a < 0$, $x_n^a = 1/x_n^{-a} \to \infty$ as $n \to \infty$. It contradicts.)

($\Leftarrow$) f is continuous on $[-1, 1] - \{0\}$ clearly. Note that

$$-|x^a| \leq x^a \sin(x^{-c}) \leq |x^a|,$$

and $|x^a| \to 0$ as $x \to 0$ since $a > 0$. Thus f is continuous at $x = 0$. Hence f is continuous.

For (b): $f'(0)$ exists iff $x^{a-1}\sin(x^{-c}) \to 0$ as $x \to 0$. In the previous proof we know that $f'(0)$ exists if and only if $a-1>0$. Also, $f'(0)=0$.

14. Let f be a differentiable real function defined in (a,b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exist for every $x \in (a,b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a,b)$.

15. Suppose $a \in R^1$, f is a twice-differentiable real function on (a,∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a,∞). Prove that

$$M_1^2 \leq 4M_0M_2.$$

Hint: If $h>0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x+2h)-f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a=-1$, define

$$f(x) = \begin{cases} 2x^2-1 & (-1<x<0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty), \end{cases}$$

and show that $M_0=1, M_1=4, M_2=4$. Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?

Proof: Suppose $h>0$. By using Taylor's theorem:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(\xi)$$

for some $x < \xi < x + 2h$. Thus

$$h|f'(x)| \leq |f(x+h)| + |f(x)| + \frac{h^2}{2}|f''(\xi)|$$

$$h|f'(x)| \leq 2M_0 + \frac{h^2}{2}M_2.$$

$$h^2M_2 - 2h|f'(x)| + 4M_0 \geq 0 \;\; (*)$$

Since equation (*) holds for all $h > 0$, its determinant must be non-positive.

$$4|f'(x)|^2 - 4M_2(4M_0) \leq 0$$

$$|f'(x)|^2 \leq 4M_0M_2$$

$$(M_1)^2 \leq 2M_0M_2$$

Note: There is a similar exercise:
Suppose $f(x)(-\infty < x < +\infty)$ is a twice-differentiable real function, and

$$M_k = \sup_{-\infty<x<+\infty} |f^{(k)}(x)| < +\infty \;\; (k = 0, 1, 2).$$

Prove that $M_1^2 \leq 2M_0M_2$.

Proof of Note:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi_1)}{2}h^2$$

$(x < \xi_1 < x + h$ or $x > \xi_1 > x + h)$ .... (*)

$$f(x-h) = f(x) - f'(x)h + \frac{f''(\xi_2)}{2}h^2$$

$(x - h < \xi_2 < x$ or $x - h > \xi_2 > x)$ .... (**)
(*) minus (**):

$$f(x+h) - f(x-h) = 2f'(x)h + \frac{h^2}{2}(f''(\xi_1) - f''(\xi_2)).$$

$$2h|f'(x)| \leq |2hf'(x)|$$

$$2h|f'(x)| \leq |f(x+h)| + |f(x-h)| + \frac{h^2}{2}(|f''(\xi_1)| + |f''(\xi_2)|)$$

$$2h|f'(x)| \leq 2M_0 + h^2 M_2$$

$$M_2 h^2 - 2|f'(x)|h + 2M_0 \geq 0$$

Since this equation holds for all h, its determinant must be non-positive:

$$4|f'(x)|^2 - 4M_2(2M_0) \leq 0,$$

$$|f'(x)|^2 \leq 2M_0 M_2$$

Thus

$$M_1^2 \leq 2M_0 M_2$$

16. Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$. Hint: Let $a \to \infty$ in Exercise 15.

    Proof: Suppose $a \in (0, \infty)$, and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$ on (a, ∞). Hence, $M_1^2 \leq 4M_0M_2$. Let $a \to \infty$, $M_0 = \sup|f(x)| \to 0$. Since M_2 is bounded, therefore $M_1^2 \to 0$ as $a \to \infty$. It implies that $\sup|f'(x)| \to 0$ as $x \to \infty$.

17. Suppose f is a real, three times differentiable function on $[-1, 1]$, such that

    $$f(-1) = 0, f(0) = 0, f(1) = 1, f'(0) = 0.$$

    Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.
    Note that equality holds for $\frac{1}{2}(x^3 + x^2)$.

Hint: Use Theorem 5.15, with $\alpha = 0$ and $\beta = 1, -1$, to show that there exist $s \in (0,1)$ and $t \in (-1,0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.$$

Proof: By Theorem 5.15, we take $\alpha = 0, \beta = 1$,

$$f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6}$$

where $s \in (0,1)$. Take $\alpha = 0, and \beta = -1$,

$$f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6}$$

where $t \in (-1,0)$. Thus

$$1 = \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6}, s \in (0,1) \ \ (*)$$

$$0 = \frac{f''(0)}{2} - \frac{f^{(3)}(s)}{6}, s \in (-1,0) \ \ (**)$$

Equation (*) - equation (**):

$$\frac{f^{(3)}(s)}{6} + \frac{f^{(3)}(t)}{6}, s \in (0,1), t \in (-1,0).$$

$$f^{(3)}(s) + f^{(3)}(t) = 6, \ \ s, t \in (-1,1).$$

$$f^{(3)}(x) \geq 3 \ \ for \ \ some \ \ x \in (-1,1).$$

Theorem 5.15: Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P be as in Taylor's theorem (5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

19. Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

Prove the following statements:

(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.
(b) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.
(c) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that

$\lim D_n$ exists but is different from $f'(0)$.

Proof: For (a):

$$D_n = \frac{f(\beta_n) - f(0)}{\beta_n}\frac{\beta_n}{\beta_n - \alpha_n} + \frac{f(\alpha_n) - f(0)}{\alpha_n}\frac{-\alpha_n}{\beta_n - \alpha_n}$$

Note that

$$f'(0) = \lim_{n\to\infty}\frac{f(\alpha_n) - f(0)}{\alpha_n} = \lim_{n\to\infty}\frac{f(\beta_n) - f(0)}{\beta_n}$$

Thus for any $\epsilon > 0$, there exists N such that

$$L - \epsilon < \frac{f(\alpha_n) - f(0)}{\alpha_n} < L + \epsilon,$$

$$L - \epsilon < \frac{f(\beta_n) - f(0)}{\beta_n} < L + \epsilon,$$

whenever $n > N$ where $L = f'(0)$ respectively. Note that $\beta_n/(\beta_n - \alpha_n)$ and $-\alpha_n/(\beta_n - \alpha_n)$ are positive. Hence,

$$\frac{\beta_n}{\beta_n - \alpha_n}(L - \epsilon) < \frac{f(\beta_n) - f(0)}{\beta_n}\frac{\beta_n}{\beta_n - \alpha_n} < \frac{\beta_n}{\beta_n - \alpha_n}(L + \epsilon)$$

$$\frac{-\alpha_n}{\beta_n - \alpha_n}(L - \epsilon) < \frac{f(\alpha_n) - f(0)}{\alpha_n}\frac{-\alpha_n}{\beta_n - \alpha_n} < \frac{-\alpha_n}{\beta_n - \alpha_n}(L + \epsilon)$$

Combine two inequations,

$$L - \epsilon < D_n < L + \epsilon$$

Hence, $\lim D_n = L = f'(0)$.

For (b): We process as above prove, but note that $-\alpha_n/(\beta_n - \alpha_n) < 0$. Thus we only have the following inequations:

$$\frac{\beta_n}{\beta_n - \alpha_n}(L - \epsilon) < \frac{f(\beta_n) - f(0)}{\beta_n}\frac{\beta_n}{\beta_n - \alpha_n} < \frac{\beta_n}{\beta_n - \alpha_n}(L + \epsilon)$$

$$\frac{-\alpha_n}{\beta_n - \alpha_n}(L+\epsilon) < \frac{f(\alpha_n) - f(0)}{\alpha_n}\frac{-\alpha_n}{\beta_n - \alpha_n} < \frac{-\alpha_n}{\beta_n - \alpha_n}(L-\epsilon)$$

Combine them:

$$L - \frac{\beta_n + \alpha_n}{\beta_n - \alpha_n}\epsilon < D_n < L + \frac{\beta_n + \alpha_n}{\beta_n - \alpha_n}\epsilon$$

Note that $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, ie,

$$|\frac{\beta_n}{\beta_n - \alpha_n}| \leq M$$

for some constant M. Thus

$$|\frac{\beta_n + \alpha_n}{\beta_n - \alpha_n}| = |\frac{2\beta_n}{\beta_n - \alpha_n} - 1| \leq 2M + 1$$

Hence,

$$L - (2M+1)\epsilon < D_n < L + (2M+1)\epsilon$$

Hence, $\lim D_n = L = f'(0)$.

For (c): By using Mean-Value Theorem,

$$D_n = f'(t_n)$$

where t_n is between α_n and β_n. Note that

$$\min\{\alpha_n, \beta_n\} < t_n < \max\{\alpha_n, \beta_n\}$$

and

$$\max\{\alpha_n, \beta_n\} = \frac{1}{2}(\alpha_n + \beta_n + |\alpha_n - \beta_n|)$$
$$\min\{\alpha_n, \beta_n\} = \frac{1}{2}(\alpha_n + \beta_n - |\alpha_n - \beta_n|)$$

Thus, $\max\{\alpha_n, \beta_n\} \to 0$ and $\min\{\alpha_n, \beta_n\} \to 0$ as $\alpha_n \to 0$ and $\beta_n \to 0$. By squeezing principle for limits, $t_n \to 0$. With the continuity of f', we have

$$\lim D_n = \lim f'(t_n) = f'(\lim t_n) = f'(0).$$

Example: Let f be defined by

$$f(x) = \begin{cases} x^2 \sin(1/x) & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Thus $f'(x)$ is not continuous at $x = 0$, and $f'(0) = 0$. Take $\alpha_n = \frac{1}{\pi/2 + 2n\pi}$ and $\beta_n = \frac{1}{2n\pi}$. It is clear that $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Also,

$$D_n = \frac{-4n\pi}{\pi(\pi/2 + 2n\pi)} \to -\frac{2}{\pi}$$

as $n \to \infty$. Thus, $\lim D_n = -2/\pi$ exists and is different from $0 = f'(0)$.

20.

21.

22. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, ...$

(d) Show that the process describe in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to ....$$

Proof: For (a): If not, then there exists two distinct fixed points, say x and y, of f. Thus $f(x) = x$ and $f(y) = y$. Since f is differeniable, by applying Mean-Value Theorem we know that

$$f(x) - f(y) = f'(t)(x - y)$$

where t is between x and y. Since $x \neq y$, $f'(t) = 1$. It contradicts.

For (b): We show that $0 < f'(t) < 1$ for all real t first:

$$f'(t) = 1 + (-1)(1 + e^t)^{-2}e^t = 1 - \frac{e^t}{(1 + e^t)^2}$$

Since

$$e^t > 0$$

$$(1 + e^t)^2 = (1 + e^t)(1 + e^t) > 1(1 + e^t) = 1 + e^t > e^t > 0$$

for all real t, thus

$$(1 + e^t)^{-2}e^t > 0$$

$$(1 + e^t)^{-2}e^t < 1$$

for all real t. Hence $0 < f'(t) < 1$ for all real t.

Next, since $f(t) - t = (1 - e^t)^{-1} > 0$ for all real t, $f(t)$ has no fixed point.

For (c): Suppose $x_{n+1} \neq x_n$ for all n. (If $x_{n+1} = x_n$, then $x_n = x_{n+1} =$ ... and x_n is a fixed point of f).

By Mean-Value Theorem,

$$f(x_{n+1}) - f(x_n) = f'(t_n)(x_{n+1} - x_n)$$

where t_n is betweem x_n and x_{n+1}. Thus,

$$|f(x_{n+1}) - f(x_n)| = |f'(t_n)||(x_{n+1} - x_n)|$$

Note that $|f'(t_n)|$ is bounded by $A < 1$, $f(x_n) = x_{n+1}$, and $f(x_{n+1}) = x_{n+2}$. Thus

$$|x_{n+2} - x_{n+1}| \leq A|x_{n+1} - x_n|$$

$$|x_{n+1} - x_n| \leq CA^{n-1}$$

where $C = |x_2 - x_1|$. For two positive integers $p > q$,

$$\begin{aligned} |x_p - x_q| &\leq |x_p - x_{p-1}| + ... + |x_{q+1} - x_q| \\ &= C(A^{q-1} + A^q + ... + A^{p-2}) \\ &\leq \frac{CA^{q-1}}{1-A}. \end{aligned}$$

Hence

$$|x_p - x_q| \leq \frac{CA^{q-1}}{1-A}.$$

Hence, for any $\epsilon > 0$, there exists $N = [\log_A \frac{\epsilon(1-A)}{C}] + 2$ such that $|x_p - x_q| < \epsilon$ whenever $p > q \geq N$. By Cauchy criterion we know that $\{x_n\}$ converges to x. Thus,

$$\lim_{n\to\infty} x_{n+1} = f(\lim_{n\to\infty} x_n)$$

since f is continuous. Thus,

$$x = f(x).$$

x is a fixed point of f.

For (d): Since $x_{n+1} = f(x_n)$, it is trivial.

23.

24.

25. Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$.

Complete the details in the following outline of *Newton's method* for computing ξ.

(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n->oo} x_n = \xi.$$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) If $A = \frac{M}{2\delta}$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercise 16 and 18, Chap. 3)

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

(f) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Proof: For (a): You can see the picture in the following URL: *http://archives.math.utk.edu/visual.calculus/3/newton.5/1.html.*

For (b): We show that $x_n \geq x_{n+1} \geq \xi$. (induction). By Mean-Value Theorem, $f(x_n) - f(\xi) = f'(c_n)(x_n - \xi)$ where $c_n \in (\xi, x_n)$. Since $f'' \geq 0$, f' is increasing and thus

$$\frac{f(x_n)}{x_n - \xi} = f'(c_n) \leq f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

$$f(x_n)(x_n - \xi) \leq f(x_n)(x_n - x_{n+1})$$

Note that $f(x_n) > f(\xi) = 0$ since $f' \geq \delta > 0$ and f is strictly increasing. Thus,

$$x_n - \xi \leq x_n - x_{n+1}$$

$$\xi \leq x_{n+1}$$

Note that $f(x_n) > 0$ and $f'(x_n) > 0$. Thus $x_{n+1} < x_n$. Hence,

$$x_n > x_{n+1} \geq \xi.$$

Thus, $\{x_n\}$ converges to a real number ζ. Suppose $\zeta \neq \xi$, then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note that $\frac{f(x_n)}{f'(x_n)} > \frac{f(\zeta)}{\delta}$. Let $\alpha = \frac{f(\zeta)}{\delta} > 0$, be a constant. Thus,

$$x_{n+1} < x_n - \alpha$$

for all n. Thus, $x_n < x_1 - (n-1)\alpha$, that is, $x_n \to -\infty$ as $n \to \infty$. It contradicts. Thus, $\{x_n\}$ converges to ξ.

For (c): By using Taylor's theorem,

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2(x_n - \xi)^2}$$

$$0 = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2(x_n - \xi)^2}$$

$$0 = \frac{f(x_n)}{f'(x_n)} - x_n + \xi + \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$
$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

where $t_n \in (\xi, x_n)$.

For (d): By (b) we know that $0 \leq x_{n+1} - \xi$ for all n. Next by (c) we know that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

Note that $f'' \leq M$ and $f' \geq \delta > 0$. Thus

$$x_{n+1} - \xi \leq A(x_n - \xi)^2 \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}$$

by the induction. Thus,

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

For (e): If x_0 is a fixed point of $g(x)$, then $g(x_0) = x_0$, that is,

$$x_0 - \frac{f(x_0)}{f'(x_0)} = x_0$$
$$f(x_0) = 0.$$

It implies that $x_0 = \xi$ and x_0 is unique since f is strictly increasing. Thus, we choose $x_1 \in (\xi, b)$ and apply Newton's method, we can find out ξ. Hence we can find out x_0.

Next, by calculating

$$g'(x) = \frac{f(x)f''(x)}{f'(x)^2}$$
$$0 \leq g'(x) \leq f(x)\frac{M}{\delta^2}.$$

As x near ξ from right hand side, $g'(x)$ near $f(\xi) = 0$.

For (f): $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n$ by calculating. Thus,

$$x_n = (-2)^{n-1}x_1$$

for all n, thus $\{x_n\}$ does not converges for any choice of x_1, and we cannot find ξ such that $f(\xi) = 0$ in this case.

26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. *Hint:* Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.

Proof: Suppose $A > 0$. (If not, then $f = 0$ on $[a, b]$ clearly.) Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$f(x) - f(a) = f'(c)(x - a)$$

where c is between x and a by using Mean-Value Theorem. Thus

$$|f(x)| \leq M_1(x - a) \leq M_1(x_0 - a) \leq A(x_0 - a)M_0$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$ by taking $x_0 = a + \frac{1}{2A}$. Repeat the above argument by replacing a with x_0, and note that $\frac{1}{2A}$ is a constant. Hence, $f = 0$ on $[a, b]$.

27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \le x \le b$, $\alpha \le y \le \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c \quad (\alpha \le c \le \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \le f(x) \le \beta$, and

$$f'(x) = \phi(x, f(x)) \quad (a \le x \le b)$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \le A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Hint: Apply Exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.

Proof: Suppose y_1 and y_2 are solutions of that problem. Since

$$|\phi(x, y_2) - \phi(x, y_1)| \le A|y_2 - y_1|,$$

$y(a) = c$, $y_1' = \phi(x, y_1)$, and $y_2' = \phi(x, y_2)$, by Exercise 26 we know that $y_1 - y_2 = 0$, $y_1 = y_2$. Hence, such a problem has at most one solution.

Note: Suppose there is initial-value problem

$$y' = y^{1/2}, \quad y(0) = 0.$$

If $y^{1/2} \neq 0$, then $y^{1/2}dy = dx$. By integrating each side and noting that $y(0) = 0$, we know that $f(x) = x^2/4$. With $y^{1/2} = 0$, or $y = 0$. All solutions of that problem are

$$f(x) = 0 \ \textit{ and } \ f(x) = x^2/4$$

Why the uniqueness theorem does not hold for this problem? One reason is that there does not exist a constant A satisfying

$$|y_1' - y_2'| \leq A|y_1 - y_2|$$

if y_1 and y_2 are solutions of that problem. (since $2/x \to \infty$ as $x \to 0$ and thus A does not exist).

28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi_j(x, y_1, ..., y_k), \ \ y_j(a) = c_j \ \ (j = 1, ..., k)$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \phi(x, y), \ \ \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, ..., y_k)$ ranges over a k-cell, ϕ is the mapping of a $(k+1)$-cell into the Euclidean k-space whose components are the function $\phi_1, ..., \phi_k$, and $\mathbf{c}$ is the vector $(c_1, ..., c_k)$. Use Exercise 26, for vector-valued functions.

Theorem: Let $\phi_j (j = 1, ..., k)$ be real functions defined on a rectangle R_j in the plane given by $a \leq x \leq b$, $\alpha_j \leq y_j \leq \beta_j$.

A *solution* of the initial-value problem

$$y_j' = \phi(x, y_j), \ \ y_j(a) = c_j \ \ (\alpha_j \leq c_j \leq \beta_j)$$

is, by definition, a differentiable function f_j on $[a, b]$ such that $f_j(a) = c_j$, $\alpha_j \leq f_j(x) \leq \beta_j$, and

$$f_j'(x) = \phi_j(x, f_j(x)) \quad (a \leq x \leq b)$$

Then this problem has at most one solution if there is a constant A such that

$$|\phi_j(x, y_{j_2}) - \phi_j(x, y_{j_1})| \leq A|y_{j_2} - y_{j_1}|$$

whenever $(x, y_{j_1}) \in R_j$ and $(x, y_{j_2}) \in R_j$.

Proof: Suppose $\mathbf{y_1}$ and $\mathbf{y_2}$ are solutions of that problem. For each components of $\mathbf{y_1}$ and $\mathbf{y_2}$, say y_{1j} and y_{2j} respectively, $y_{1j} = y_{2j}$ by using Exercise 26. Thus, $\mathbf{y_1} = \mathbf{y_2}$

29. Specialize Exercise 28 by considering the system

$$y_j' = y_{j+1} \quad (j = 1, ..., k-1),$$

$$y_k' = f(x) - \sum_{j=1}^{k} g_j(x) y_j$$

where $f, g_1, ..., g_k$ are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x) y^{(k-1)} + ... + g_2(x) y' + g_1(x) y = f(x),$$

subject to initial conditions

$$y(a) = c_1, y'(a) = c_1, ..., y^{(k-1)}(a) = c_k.$$

Theorem: Let R_j be a rectangle in the plain, given by $a \leq x \leq b$, $\min y_j \leq y_j \leq \max y_j$. (since y_j is continuous on the compact set, say

$[a, b]$, we know that y_j attains minimal and maximal.) If there is a constant A such that

$$\begin{cases} |y_{j+1,1} - y_{j+1,2}| \leq A|y_{j,1} - y_{j,2}| & (j < k) \\ |\sum_{j=1}^{k} g_j(x)(y_{j,1} - y_{j,2})| \leq A|y_{k,1} - y_{k,2}| \end{cases}$$

whenever $(x, y_{j,1}) \in R_j$ and $(x, y_{j,2}) \in R_j$.

Proof: Since the system $y_1', ..., y_k'$ with initial conditions satisfies a fact that there is a constant A such that $|\mathbf{y_1'} - \mathbf{y_2'}| \leq A|\mathbf{y_1} - \mathbf{y_2}|$, that system has at most one solution. Hence,

$$y^{(k)} + g_k(x)y^{(k-1)} + ... + g_2(x)y' + g_1(x)y = f(x),$$

with initial conditions has at most one solution.