The Riemann-Stieltjes Integral

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1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

   Proof: Note that $L(P, f, \alpha) = 0$ for all partition P of $[a, b]$. Thus

   $$\underline{\int}_b^a f d\alpha = 0.$$

   Take a partition P such that

   $$P = \{a, a + \frac{1}{n}(b-a), ..., a + \frac{k}{n}(b-a), ..., a + \frac{n-1}{n}(b-a), b\}$$

   for all $N \ni n > 1$. Thus

   $$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \leq \frac{2(b-a)}{n}$$

   for all $N \ni n > 1$. Thus

   $$0 \leq \inf U(P, f, \alpha) \leq \frac{2(b-a)}{n}$$

   for all $n \in N$. Thus $\inf U(P, f, \alpha) = 0$. Hence $\overline{\int}_b^a f d\alpha = 0$; thus, $\int_b^a f d\alpha = 0$.

2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

   Proof: Suppose not, then there is $p \in [a, b]$ such that $f(p) > 0$. Since f is continuous at $x = p$, for $\epsilon = f(p)/2$, there exist $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ whenever $x \in (x - \delta, x + \delta) \bigcap [a, b]$, that is,

   $$0 < \frac{1}{2}f(p) < f(x) < \frac{3}{2}f(p)$$

for $x \in B_r(p) \subset [a,b]$ where r is small enough. Next, consider a partition P of $[a,b]$ such that

$$P = \{a, p - \frac{r}{2}, p + \frac{r}{2}, b\}.$$

Thus

$$L(P,f) \geq r \cdot \frac{1}{2} f(p) = \frac{rf(p)}{2}.$$

Thus

$$\sup L(P,f) \geq L(P,f) \geq \frac{rf(p)}{2} > 0,$$

a contradition since $\int_a^b f(x)dx = \sup L(P,f) = 0$. Hence $f = 0$ for all $x \in [a,b]$.

Note: The above conclusion holds under the condition that f is continuous. If f is not necessary continuous, then we cannot get this conclusion. (A counter-example is shown in Exercise 6.1).

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4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin R$ on $[a,b]$ for any $a < b$.

Proof: Take any partition P of $[a,b]$, say

$$a = x_0 \leq x_1 \leq .. \leq x_{n-1} \leq x_n = b.$$

By P we can construct the new partition P' without repeated points, and $U(P,f) = U(P'f)$, $L(P,f) = L(P',f)$. Say P'

$$a = y_0 < y_1 < ... < y_{m-1} < y_m = b.$$

Note that Q is dense in R, and $R - Q$ is also dense in R. Hence

$$\begin{aligned} M_i &= \sup_{y_{i-1} \leq x \leq y_i} f(x) = 1, \\ m_i &= \inf_{y_{i-1} \leq x \leq y_i} f(x) = 0. \end{aligned}$$

Hence

$$\begin{aligned} U(P,f) &= U(P',f) = \sum_{i=1}^{m} M_i \Delta y_i = b-a, \\ L(P,f) &= L(P',f) = \sum_{i=1}^{m} m_i \Delta y_i = 0. \end{aligned}$$

for any partition of $[a,b]$. Hence

$$\begin{aligned} \overline{\int_a^b} f dx &= \inf U(P,f) = b-a, \\ \underline{\int_a^b} f dx &= \sup L(P,f) = 0. \end{aligned}$$

Thus $f \notin R[a,b]$.

5. Suppose f is bounded real function on $[a,b]$, and $f^2 \in R$ on $[a,b]$. Does it follow that $f \in R$? Does the answer change if we assume that $f^3 \in R$?

   Proof: The first answer is **NO**. Define $f(x) = -1$ for all irrational $x \in [a,b]$, $f(x) = 1$ for all rational $x \in [a,b]$. Similarly, by Exercise 6.4, $f \notin R$.

   However, if we assume that $f^3 \in R$, the answer is **YES**. Let $\Phi = x^{1/3}$, we apply Theorem 6.11 and get the conclusion that $f \in R$.

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7. Suppose f is a real funtion on $(0,1]$ and $f \in R$ on $[c,1]$ for every $c > 0$. Define
   $$\int_0^1 f(x)dx = \lim_{c \to 0} \int_c^1 f(x)dx$$
   if this limit exists (and is finite).

   (a) If $f \in R$ on $[0,1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Proof of (a): Take any $c > 0$, and note that f is bounded, say $|f| \leq M$ for some M. Hence

$$\begin{aligned}
\int_0^c -M dx \leq & \quad \int_0^c f(x)dx & \leq \int_0^c M dx \\
-cM \leq & \quad \int_0^c f(x)dx & \leq cM \\
-cM \leq & \int_c^1 f(x)dx - \int_0^1 f(x)dx & \leq cM.
\end{aligned}$$

Letting $c \to 0$ and thus $\int_c^1 f(x)dx - \int_0^1 f(x)dx \to 0$, that is, $\int_c^1 f(x)dx \to \int_0^1 f(x)dx$ as $c \to 0$. Thus this definition of the integral agrees with the old one.

Solution of (b): (Due to Shin-Yi Lee) Define

$$f(x) = n(-1)^n, \quad where \ \frac{1}{n+1} < x \leq \frac{1}{n}.$$

(Due to Meng-Gen Tsai). Define

$$f(x) = \frac{\sin(1/x)}{x}.$$

8. Suppose $f \in R$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

Proof: Since f is nonnegative and decreases monotonically on $[1, \infty)$, we have for $n \geq 2$,

$$\sum_{n=2}^{k} f(n) \leq \int_1^k f(x)dx \leq \sum_{n=1}^{k-1} f(n).$$

We define

$$g(x) = \int_1^y f(t)dt.$$

If this integral converges, then $g(x)$ is a nondecreasing function which tends to a limit, and so g(k) is a bounded nondecreasing sequence. Thus denote $s_k = \sum_{n=2}^{k} f(n)$, we see that $s_k \leq g(k)$, and s_k tends to a limit. The fact that $\sum_{n=1}^{\infty} f(n)$ converges if $\int_1^{\infty} f(x)dx$ converges is now established. If the integral diverges, then $g(x) \to +\infty$ as $x \to +\infty$. Therefore $g(k) \to +\infty$. Since $g(k) \leq \sum_{n=1}^{k-1} f(n)$, we conclude that the series diverges. Hence $\int_1^{\infty} f(x)dx$ converges if $\sum_{n=1}^{\infty} f(n)$. Thus, $\int_1^{\infty} f(x)dx$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges.

9. Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercise 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^{\infty} \frac{\cos x}{1+x} dx = \int_0^{\infty} \frac{\sin x}{(1+x)^2} dx.$$

Show that one of these integrals converges *absolutely*, but that the other does not.

Proof: Recall Theorem 6.22 (integration by parts):

> Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in R$, and $G' = g \in R$. Then
>
> $$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

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13. Define

$$f(x) = \int_x^{x+1} \sin(t^2)dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.

Hint: Put $t^2 = u$ and integrate by parts, to show that $f(x)$ is equal to

$$\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du.$$

Replace $\cos u$ by -1.

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < c/x$ and c is a constant.

(c) Find the upper and lower bound limit of $xf(x)$, as $x \to \infty$.

(d) Does $\int_0^\infty \sin(t^2)$ dt converges?

Proof of (a): Put $t^2 = u$,

$$\begin{aligned} f(x) &= \int_x^{x+1} \sin(t^2)dt \\ &= \int_{x^2}^{(x+1)^2} \frac{\sin u}{2u^{1/2}}du \\ &= \frac{-\cos u}{2u^{1/2}}\Big|_{u=x^2}^{u=(x+1)^2} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du \\ &= \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du. \end{aligned}$$

To get a bound of $\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du$, we replace $\cos u$ by -1:

$$\begin{aligned} \left|\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du\right| &< \left|\int_{x^2}^{(x+1)^2} \frac{-1}{4u^{3/2}} du\right| \\ &= \frac{1}{2x(x+1)} \end{aligned}$$

for $x > 0$. Hence

$$\begin{aligned} |f(x)| &\leq \left|\frac{\cos(x^2)}{2x}\right| + \left|\frac{\cos[(x+1)^2]}{2(x+1)}\right| + \left|\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du\right| \\ &< \frac{1}{2x} + \frac{1}{2(x+1)} + \frac{1}{2x(x+1)} \\ &= \frac{1}{x} \end{aligned}$$

for $x > 0$.

Proof of (b): By (a),

$$f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du.$$

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + \frac{1}{x+1}\cos[(x+1)^2] - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du.$$

Let

$$r(x) = \frac{1}{x+1}\cos[(x+1)^2] - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du.$$

Note that

$$\left|\frac{1}{x+1}\cos[(x+1)^2]\right| \leq \frac{1}{x+1} < \frac{1}{x},$$

$$2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du < 2x\frac{1}{2x(x+1)} < \frac{1}{x}$$

if $x > 0$. Thus $|r(x)| < 1/x + 1/x = c/x$ and $c = 2$ is a constant.

Note: Here, $x > 0$.

Proof of (c):

Proof of (d):

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16. For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

(a) $\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s-1}} dx$

and that

(b) $\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x-[x]}{x^{s-1}} dx$,

where $[x]$ denotes the greatest integer $\leq x$.
Prove that the integral in (b) converges for all $s > 0$.
Hint: To prove (a), compute the difference between the integral over $[1, N]$ and the Nth partial sum of the series that defines $\zeta(s)$.

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