Supplement on $\limsup$ and $\liminf$

Introduction

In order to make us understand the information more on approaches of a given real sequence $\{a_n\}_{n=1}^{\infty}$, we give two definitions, thier names are upper limit and lower limit. It is fundamental but important tools in analysis. We do **NOT** give them proofs. The reader can see the book, **Infinite Series by Chao Wen-Min, pp 84-103. (Chinese Version)**

Definition of limit sup and limit inf

Definition Given a real sequence $\{a_n\}_{n=1}^{\infty}$, we define

$$b_n = \sup\{a_m : m \geq n\}$$

and

$$c_n = \inf\{a_m : m \geq n\}.$$

Example $\{1 + (-1)^n\}_{n=1}^{\infty} = \{0, 2, 0, 2, \ldots\}$, so we have

$$b_n = 2 \text{ and } c_n = 0 \text{ for all } n.$$

Example $\{(-1)^n n\}_{n=1}^{\infty} = \{-1, 2, -3, 4, \ldots\}$, so we have

$$b_n = +\infty \text{ and } c_n = -\infty \text{ for all } n.$$

Example $\{-n\}_{n=1}^{\infty} = \{-1, -2, -3, \ldots\}$, so we have

$$b_n = -n \text{ and } c_n = -\infty \text{ for all } n.$$

Proposition Given a real sequence $\{a_n\}_{n=1}^{\infty}$, and thus define b_n and c_n as the same as before.

1 $b_n \neq -\infty$, and $c_n \neq \infty$ $\forall n \in N$.
2 If there is a positive integer p such that $b_p = +\infty$, then $b_n = +\infty$ $\forall n \in N$.
If there is a positive integer q such that $c_q = -\infty$, then $c_n = -\infty$ $\forall n \in N$.
3 $\{b_n\}$ is decreasing and $\{c_n\}$ is increasing.

By property 3, we can give definitions on the upper limit and the lower limit of a given sequence as follows.

Definition Given a real sequence $\{a_n\}$ and let b_n and c_n as the same as before.

(1) If every $b_n \in R$, then

$$\inf\{b_n : n \in N\}$$

is called the upper limit of $\{a_n\}$, denoted by

$$\lim_{n\to\infty} \sup a_n.$$

That is,

$$\lim_{n\to\infty} \sup a_n = \inf_n b_n.$$

If every $b_n = +\infty$, then we define

$$\lim_{n\to\infty} \sup a_n = +\infty.$$

(2) If every $c_n \in R$, then

$$\sup\{c_n : n \in N\}$$

is called the lower limit of $\{a_n\}$, denoted by

$$\lim_{n\to\infty} \inf a_n.$$

That is,

$$\lim_{n\to\infty} \inf a_n = \sup_n c_n.$$

If every $c_n = -\infty,$ then we define

$$\lim_{n\to\infty} \inf a_n = -\infty.$$

Remark The concept of lower limit and upper limit first appear in the book (**Analyse Alge'brique**) written by **Cauchy** in 1821. But until 1882, **Paul du Bois-Reymond** gave explanations on them, it becomes well-known.

Example $\{1 + (-1)^n\}_{n=1}^{\infty} = \{0, 2, 0, 2, \dots\}$, so we have

$$b_n = 2 \text{ and } c_n = 0 \text{ for all } n$$

which implies that

$$\lim \sup a_n = 2 \text{ and } \lim \inf a_n = 0.$$

Example $\{(-1)^n n\}_{n=1}^{\infty} = \{-1, 2, -3, 4, \dots\},$ so we have

$$b_n = +\infty \text{ and } c_n = -\infty \text{ for all } n$$

which implies that

$$\lim \sup a_n = +\infty \text{ and } \lim \inf a_n = -\infty.$$

Example $\{-n\}_{n=1}^{\infty} = \{-1, -2, -3, \dots\},$ so we have

$$b_n = -n \text{ and } c_n = -\infty \text{ for all } n$$

which implies that

$$\lim \sup a_n = -\infty \text{ and } \lim \inf a_n = -\infty.$$

Relations with convergence and divergence for upper (lower) limit

Theorem Let $\{a_n\}$ be a real sequence, then $\{a_n\}$ converges if, and only if, the upper limit and the lower limit are real with

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} a_n.$$

Theorem Let $\{a_n\}$ be a real sequence, then we have

(1) $\lim_{n\to\infty} \sup a_n = +\infty \Leftrightarrow \{a_n\}$ has no upper bound.

(2) $\lim_{n\to\infty} \sup a_n = -\infty \Leftrightarrow$ for any $M > 0,$ there is a positive integer n_0 such that as $n \geq n_0,$ we have

$$a_n \leq -M.$$

(3) $\lim_{n\to\infty} \sup a_n = a$ if, and only if, (a) given any $\varepsilon > 0,$ there are infinite many numbers n such that

$$a - \varepsilon < a_n$$

and (b) given any $\varepsilon > 0,$ there is a positive integer n_0 such that as $n \geq n_0,$ we have

$$a_n < a + \varepsilon.$$

Similarly, we also have

Theorem Let $\{a_n\}$ be a real sequence, then we have

(1) $\lim_{n\to\infty} \inf a_n = -\infty \Leftrightarrow \{a_n\}$ has no lower bound.

(2) $\lim_{n\to\infty} \inf a_n = +\infty \Leftrightarrow$ for any $M > 0,$ there is a positive integer n_0 such that as $n \geq n_0,$ we have

$$a_n \geq M.$$

(3) $\lim_{n\to\infty} \inf a_n = a$ if, and only if, (a) given any $\varepsilon > 0,$ there are infinite many numbers n such that

$$a + \varepsilon > a_n$$

and (b) given any $\varepsilon > 0,$ there is a positive integer n_0 such that as $n \geq n_0,$ we have

$$a_n > a - \varepsilon.$$

From Theorem 2 an Theorem 3, the sequence is divergent, we give the following definitios.

Definition Let $\{a_n\}$ be a real sequence, then we have

(1) If $\lim_{n\to\infty} \sup a_n = -\infty,$ then we call the sequence $\{a_n\}$ diverges to $-\infty,$ denoted by

$$\lim_{n\to\infty} a_n = -\infty.$$

(2) If $\lim_{n\to\infty} \inf a_n = +\infty,$ then we call the sequence $\{a_n\}$ diverges to $+\infty,$ denoted by

$$\lim_{n\to\infty} a_n = +\infty.$$

Theorem Let $\{a_n\}$ be a real sequence. If a is a limit point of $\{a_n\},$ then we have

$$\lim_{n\to\infty} \inf a_n \leq a \leq \lim_{n\to\infty} \sup a_n.$$

Some useful results

Theorem Let $\{a_n\}$ be a real sequence, then

(1) $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n.$
(2) $\lim_{n\to\infty} \inf(-a_n) = -\lim_{n\to\infty} \sup a_n$ and $\lim_{n\to\infty} \sup(-a_n) = -\lim_{n\to\infty} \inf a_n$
(3) If every $a_n > 0,$ and $0 < \lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n < +\infty,$ then we have

$$\lim_{n\to\infty} \sup \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} \inf a_n} \text{ and } \lim_{n\to\infty} \inf \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} \sup a_n}.$$

Theorem Let $\{a_n\}$ and $\{b_n\}$be two real sequences.

(1) If there is a positive integer n_0 such that $a_n \leq b_n,$ then we have

$$\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \inf b_n \text{ and } \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup b_n.$$

(2) Suppose that $-\infty < \lim_{n\to\infty} \inf a_n,$ $\lim_{n\to\infty} \inf b_n,$ $\lim_{n\to\infty} \sup a_n,$ $\lim_{n\to\infty} \sup b_n < +\infty,$ then

$$\begin{aligned}
&\lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \inf b_n \\
&\leq \lim_{n\to\infty} \inf(a_n + b_n) \\
&\leq \lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \sup b_n \text{ (or } \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \inf b_n \text{)} \\
&\leq \lim_{n\to\infty} \sup(a_n + b_n) \\
&\leq \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n.
\end{aligned}$$

In particular, if $\{a_n\}$ converges, we have

$$\lim_{n\to\infty} \sup(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} \sup b_n$$

and

$$\lim_{n\to\infty} \inf(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} \inf b_n.$$

(3) Suppose that $-\infty < \lim_{n\to\infty} \inf a_n,\ \lim_{n\to\infty} \inf b_n,\ \lim_{n\to\infty} \sup a_n,\ \lim_{n\to\infty} \sup b_n < +\infty$, and $a_n > 0,\ b_n > 0\ \forall n,$ then

$$\begin{aligned}
&\left(\lim_{n\to\infty} \inf a_n\right)\left(\lim_{n\to\infty} \inf b_n\right)\\
&\le \lim_{n\to\infty} \inf(a_n b_n)\\
&\le \left(\lim_{n\to\infty} \inf a_n\right)\left(\lim_{n\to\infty} \sup b_n\right) \text{ (or } \left(\lim_{n\to\infty} \inf b_n\right)\left(\lim_{n\to\infty} \sup a_n\right))\\
&\le \lim_{n\to\infty} \sup(a_n b_n)\\
&\le \left(\lim_{n\to\infty} \sup a_n\right)\left(\lim_{n\to\infty} \sup b_n\right).
\end{aligned}$$

In particular, if $\{a_n\}$ converges, we have

$$\lim_{n\to\infty} \sup(a_n b_n) = \left(\lim_{n\to\infty} a_n\right) \lim_{n\to\infty} \sup b_n$$

and

$$\lim_{n\to\infty} \inf(a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) \lim_{n\to\infty} \inf b_n.$$

Theorem Let $\{a_n\}$ be a **positive** real sequence, then

$$\lim_{n\to\infty} \inf \frac{a_{n+1}}{a_n} \le \lim_{n\to\infty} \inf(a_n)^{1/n} \le \lim_{n\to\infty} \sup(a_n)^{1/n} \le \lim_{n\to\infty} \sup \frac{a_{n+1}}{a_n}.$$

Remark We can use the inequalities to show

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = 1/e.$$

Theorem Let $\{a_n\}$ be a real sequence, then

$$\lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \inf \frac{a_1 + \ldots + a_n}{n} \le \lim_{n\to\infty} \sup \frac{a_1 + \ldots + a_n}{n} \le \lim_{n\to\infty} \sup a_n.$$

Exercise Let $f : [a,d] \to R$ be a continuous function, and $\{a_n\}$ is a real sequence. If f is increasing and for every n, $\lim_{n\to\infty} \inf a_n,\ \lim_{n\to\infty} \sup a_n \in [a,d],$ then

$$\lim_{n\to\infty} \sup f(a_n) = f\left(\lim_{n\to\infty} \sup a_n\right) \text{ and } \lim_{n\to\infty} \inf f(a_n) = f\left(\lim_{n\to\infty} \inf a_n\right).$$

Remark: (1) The condition that f is increasing cannot be removed. For example,

$$f(x) = |x|,$$

and

$$a_k = \begin{cases} 1/k \text{ if } k \text{ is even} \\ -1 - 1/k \text{ if } k \text{ is odd.} \end{cases}$$

(2) The proof is easy if we list the definition of limit sup and limit inf. So, we omit it.

Exercise Let $\{a_n\}$ be a real sequence satisfying $a_{n+p} \le a_n + a_p$ for all $n,\ p.$ Show that $\{\frac{a_n}{n}\}$ converges.

Hint: Consider its limit inf.

Remark: The exercise is useful in the theory of **Topological Entorpy**.

Infinite Series And Infinite Products

Sequences

8.1 (a) Given a real-valed sequence $\{a_n\}$ bounded above, let $u_n = \sup\{a_k : k \geq n\}$. Then $u_n \searrow$ and hence $U = \lim_{n\to\infty} u_n$ is either finite or $-\infty$. Prove that

$$U = \lim_{n\to\infty} \sup a_n = \lim_{n\to\infty}(\sup\{a_k : k \geq n\}).$$

Proof: It is clear that $u_n \searrow$ and hence $U = \lim_{n\to\infty} u_n$ is either finite or $-\infty$.

If $U = -\infty$, then given any $M > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$u_n \leq -M$$

which implies that, as $n \geq N$, $a_n \leq -M$. So, $\lim_{n\to\infty} a_n = -\infty$. That is, $\{a_n\}$ is not bounded below. In addition, if $\{a_n\}$ has a finite limit supreior, say a. Then given $\varepsilon > 0$, and given $m > 0$, there exists an integer $n > m$ such that

$$a_n > a - \varepsilon$$

which contradicts to $\lim_{n\to\infty} a_n = -\infty$. From above results, we obtain

$$U = \lim_{n\to\infty} \sup a_n$$

in the case of $U = -\infty$.

If U is finite, then given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$U \leq u_n < U + \varepsilon.$$

So, as $n \geq N$, $u_n < U + \varepsilon$ which implies that, as $n \geq N$, $a_n < U + \varepsilon$. In addition, given $\varepsilon' > 0$, and $m > 0$, there exists an integer $n > m$,

$$U - \varepsilon' < a_n$$

by $U \leq u_n = \sup\{a_k : k \geq n\}$ if $n \geq N$. From above results, we obtain

$$U = \lim_{n\to\infty} \sup a_n$$

in the case of U is finite.

(b)Similarly, if $\{a_n\}$ is bounded below, prove that

$$V = \lim_{n\to\infty} \inf a_n = \lim_{n\to\infty}(\inf\{a_k : k \geq n\}).$$

Proof: Since the proof is similar to (a), we omit it.

If U and V are finite, show that:

(c) There exists a subsequence of $\{a_n\}$ which converges to U and a subsequence which converges to V.

Proof: Since $U = \lim\sup_{n\to\infty} a_n$ by (a), then

(i) Given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$a_n < U + \varepsilon.$$

(ii) Given $\varepsilon > 0$, and $m > 0$, there exists an integer $P(m) > m$,

$$U - \varepsilon < a_{P(m)}.$$

Hence, $\{a_{P(m)}\}$ is a convergent subsequence of $\{a_n\}$ with limit U.

Similarly for the case of V.

(d) If $U = V,$ every subsequnce of $\{a_n\}$ converges to $U.$

Proof: By (a) and (b), given $\varepsilon > 0,$ then there exists a positive integer N_1 such that as $n \geq N_1,$ we have

$$a_n < U + \varepsilon$$

and there exists a positive integer N_2 such that as $n \geq N_2,$ we have

$$U - \varepsilon < a_n.$$

Hence, as $n \geq \max(N_1, N_2),$ we have

$$U - \varepsilon < a_n < U + \varepsilon.$$

That is, $\{a_n\}$ is a convergent sequence with limit $U.$ So, every subsequnce of $\{a_n\}$ converges to $U.$

8.2 Given two real-valed sequence $\{a_n\}$ and $\{b_n\}$ bounded below. Prove hat

(a) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$

Proof: Note that $\{a_n\}$ and $\{b_n\}$ bounded below, we have $\limsup_{n\to\infty} a_n = +\infty$ or is finite. And $\limsup_{n\to\infty} b_n = +\infty$ or is finite. It is clear if one of these limit superior is $+\infty$, so we may assume that both are finite. Let $a = \limsup_{n\to\infty} a_n$ and $b = \limsup_{n\to\infty} b_n.$ Then given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$a_n + b_n < a + b + \varepsilon/2. \qquad *$$

In addition, let $c = \limsup_{n\to\infty}(a_n + b_n),$ where $c < +\infty$ by (*). So, for the same $\varepsilon > 0,$ and given $m = N$ there exists a positive integer K such that as $K \geq N,$ we have

$$c - \varepsilon/2 < a_K + b_K. \qquad **$$

By (*) and (**), we obtain that

$$c - \varepsilon/2 < a_K + b_K < a + b + \varepsilon/2$$

which implies that

$$c \leq a + b$$

since ε is arbitrary. So,

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Remark: (1) The equality may **NOT** hold. For example,

$$a_n = (-1)^n \text{ and } b_n = (-1)^{n+1}.$$

(2) The reader should noted that the finitely many terms does **NOT** change the relation of order. The fact is based on process of proof.

(b) $\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$ if $a_n > 0,$ $b_n > 0$ for all $n,$ and if both $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ are finite or both are infinite.

Proof: Let $\limsup_{n\to\infty} a_n = a$ and $\limsup_{n\to\infty} b_n = b.$ It is clear that we may assume that a and b are finite. Given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$a_n b_n < (a + \varepsilon)(b + \varepsilon) = ab + \varepsilon(a + b + \varepsilon). \qquad *$$

In addition, let $c = \limsup_{n\to\infty}(a_n b_n),$ where $c < +\infty$ by (*). So, for the same $\varepsilon > 0,$ and given $m = N$ there exists a positive integer K such that as $K \geq N,$ we have

$$c - \varepsilon < a_K + b_K. \qquad **$$

By (*) and (**), we obtain that

$$c - \varepsilon < a_K + b_K < a + b + \varepsilon(a + b + \varepsilon)$$

which implies that

$$c \leq a + b$$

since ε is arbitrary. So,

$$\limsup_{n\to\infty}(a_n b_n) \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right).$$

Remark: (1) The equality may **NOT** hold. For example,

$$a_n = 1/n \text{ if } n \text{ is odd and } a_n = 1 \text{ if } n \text{ is even.}$$

and

$$b_n = 1 \text{ if } n \text{ is odd and } b_n = 1/n \text{ if } n \text{ is even.}$$

(2) The reader should noted that the finitely many terms does **NOT** change the relation of order. The fact is based on the process of the proof.

(3) The reader should be noted that if letting $A_n = \log a_n$ and $B_n = \log b_n$, then by (a) and $\log x$ is an increasing function on $(0, +\infty)$, we have proved (b).

8.3 Prove that Theorem 8.3 and 8.4.

(**Theorem 8.3**) Let $\{a_n\}$ be a sequence of real numbers. Then we have:

(a) $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.

Proof: If $\limsup_{n\to\infty} a_n = +\infty$, then it is clear. We may assume that $\limsup_{n\to\infty} a_n < +\infty$. Hence, $\{a_n\}$ is bounded above. We consider two cases: (i) $\limsup_{n\to\infty} a_n = a$, where a is finite and (ii) $\limsup_{n\to\infty} a_n = -\infty$.

For case (i), if $\liminf_{n\to\infty} a_n = -\infty$, then there is nothing to prove it. We may assume that $\liminf_{n\to\infty} a_n = a'$, where a' is finite. By definition of limit superior and limit inferior, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$a' - \varepsilon/2 < a_n < a + \varepsilon/2$$

which implies that $a' \leq a$ since ε is arbitrary.

For case (ii), since $\limsup_{n\to\infty} a_n = -\infty$, we have $\{a_n\}$ is not bounded below. If $\liminf_{n\to\infty} a_n = -\infty$, then there is nothing to prove it. We may assume that $\liminf_{n\to\infty} a_n = a'$, where a' is finite. By definition of limit inferior, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$a' - \varepsilon/2 < a_n$$

which contradicts that $\{a_n\}$ is not bounded below.

So, from above results, we have proved it.

(b) The sequence converges if and only if, $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both finite and equal, in which case $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Proof: ($\Rightarrow$)Given $\{a_n\}$ a convergent sequence with limit a. So, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$a - \varepsilon < a_n < a + \varepsilon.$$

By definition of limit superior and limit inferior, $a = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

($\Leftarrow$)By definition of limit superior, given $\varepsilon > 0$, there exists a positive integer N_1 such that as $n \geq N_1$, we have

$$a_n < a + \varepsilon$$

and by definition of limit superior, given $\varepsilon > 0$, there exists a positive integer N_2 such that as $n \geq N_2$, we have

$$a-\varepsilon < a_n.$$

So, as $n \geq \max(N_1, N_2)$, we have

$$a-\varepsilon < a_n < a+\varepsilon.$$

That is, $\lim_{n\to\infty} a_n = a$.

(c) The sequence diverges to $+\infty$ if and only if, $\lim\inf_{n\to\infty} a_n = \lim\sup_{n\to\infty} a_n = +\infty$.

Proof: ($\Rightarrow$)Given a sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = +\infty$. So, given $M > 0$, there is a positive integer N such that as $n \geq N$, we have

$$M \leq a_n. \qquad *$$

It implies that $\{a_n\}$ is not bounded above. So, $\lim\sup_{n\to\infty} a_n = +\infty$. In order to show that $\lim\inf_{n\to\infty} a_n = +\infty$. We first note that $\{a_n\}$ is bounded below. Hence, $\lim\inf_{n\to\infty} a_n \neq -\infty$. So, it suffices to consider that $\lim\inf_{n\to\infty} a_n$ is not finite. (So, we have $\lim\inf_{n\to\infty} a_n = +\infty$.). Assume that $\lim\inf_{n\to\infty} a_n = a$, where a is finite. Then given $\varepsilon = 1$, and an integer m, there exists a positive $K(m) > m$ such that

$$a_{K(m)} < a+1$$

which contradicts to (*) if we choose $M = a+1$. So, $\lim\inf_{n\to\infty} a_n$ is not finite.

(d) The sequence diverges to $-\infty$ if and only if, $\lim\inf_{n\to\infty} a_n = \lim\sup_{n\to\infty} a_n = -\infty$.

Proof: Note that, $\lim\sup_{n\to\infty}(-a_n) = -\lim\inf_{n\to\infty} a_n$. So, by (c), we have proved it.

(**Theorem 8.4**)Assume that $a_n \leq b_n$ for each $n = 1, 2, \ldots.$ Then we have:

$$\lim_{n\to\infty}\inf a_n \leq \lim_{n\to\infty}\inf b_n \text{ and } \lim_{n\to\infty}\sup a_n \leq \lim_{n\to\infty}\sup b_n.$$

Proof: If $\lim\inf_{n\to\infty} b_n = +\infty$, there is nothing to prove it. So, we may assume that $\lim\inf_{n\to\infty} b_n < +\infty$. That is, $\lim\inf_{n\to\infty} b_n = -\infty$ or b, where b is finite.

For the case, $\lim\inf_{n\to\infty} b_n = -\infty$, it means that the sequence $\{a_n\}$ is not bounded below. So, $\{b_n\}$ is also not bounded below. Hence, we also have $\lim\inf_{n\to\infty} a_n = -\infty$.

For the case, $\lim\inf_{n\to\infty} b_n = b$, where b is finite. We consider three cases as follows.

(i) if $\lim\inf_{n\to\infty} a_n = -\infty$, then there is nothing to prove it.

(ii) if $\lim\inf_{n\to\infty} a_n = a$, where a is finite. Given $\varepsilon > 0$, then there exists a positive integer N such that as $n \geq N$

$$a-\varepsilon/2 < a_n \leq b_n < b+\varepsilon/2$$

which implies that $a \leq b$ since ε is arbitrary.

(iii) if $\lim\inf_{n\to\infty} a_n = +\infty$, then by **Theorem 8.3 (a) and (c)**, we know that $\lim_{n\to\infty} a_n = +\infty$ which implies that $\lim_{n\to\infty} b_n = +\infty$. Also, by **Theorem 8.3 (c)**, we have $\lim\inf_{n\to\infty} b_n = +\infty$ which is absurb.

So, by above results, we have proved that $\lim\inf_{n\to\infty} a_n \leq \lim\inf_{n\to\infty} b_n$.

Similarly, we have $\lim\sup_{n\to\infty} a_n \leq \lim\sup_{n\to\infty} b_n$.

8.4 If each $a_n > 0$, prove that

$$\lim_{n\to\infty}\inf \frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty}\inf(a_n)^{1/n} \leq \lim_{n\to\infty}\sup(a_n)^{1/n} \leq \lim_{n\to\infty}\sup \frac{a_{n+1}}{a_n}.$$

Proof: By **Theorem 8.3 (a)**, it suffices to show that

$$\lim_{n\to\infty}\inf \frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty}\inf(a_n)^{1/n} \text{ and } \lim_{n\to\infty}\sup(a_n)^{1/n} \leq \lim_{n\to\infty}\sup \frac{a_{n+1}}{a_n}.$$

We first prove

$$\lim_{n\to\infty}\sup(a_n)^{1/n} \leq \lim_{n\to\infty}\sup \frac{a_{n+1}}{a_n}.$$

If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty$, then it is clear. In addition, since $\frac{a_{n+1}}{a_n}$ is positive, $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \neq -\infty$. So, we may assume that $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = a$, where a is finite.

Given $\varepsilon > 0$, then there exists a positive integer N such that as $n \geq N$, we have

$$\frac{a_{n+1}}{a_n} < a + \varepsilon$$

which implies that

$$a_{N+k} < a_N(a+\varepsilon)^k, \text{ where } k = 1,2,\ldots.$$

So,

$$(a_{N+k})^{\frac{1}{N+k}} < (a_N)^{\frac{1}{N+k}}(a+\varepsilon)^{\frac{k}{N+k}}$$

which implies that

$$\begin{aligned}\lim_{k\to\infty}\sup(a_{N+k})^{\frac{1}{N+k}} &\leq \lim_{k\to\infty}\sup(a_N)^{\frac{1}{N+k}}(a+\varepsilon)^{\frac{k}{N+k}} \\ &= a+\varepsilon.\end{aligned}$$

So,

$$\lim_{k\to\infty}\sup(a_{N+k})^{\frac{1}{N+k}} \leq a$$

since ε is arbitrary. Note that the finitely many terms do **NOT** change the value of limit superiror of a given sequence. So, we finally have

$$\lim_{n\to\infty}\sup(a_n)^{1/n} \leq a = \lim_{n\to\infty}\sup\frac{a_{n+1}}{a_n}.$$

Similarly for

$$\lim_{n\to\infty}\inf\frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty}\inf(a_n)^{1/n}.$$

Remark: These ineqaulities is much important; we suggest that the reader keep it mind. At the same time, these inequalities tells us that **the root test is more powerful than the ratio test**. We give an example to say this point. Given a series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{2^n}+\frac{1}{3^n}+\ldots$$

where

$$a_{2n-1} = \left(\frac{1}{2}\right)^n, \text{ and } a_{2n} = \left(\frac{1}{3}\right)^n, \ n = 1,2,\ldots$$

with

$$\lim_{n\to\infty}\sup(a_n)^{1/n} = \sqrt{\frac{1}{2}} < 1$$

and

$$\lim_{n\to\infty}\inf\frac{a_{n+1}}{a_n} = 0, \ \lim_{n\to\infty}\sup\frac{a_{n+1}}{a_n} = +\infty.$$

8.5 Let $a_n = n^n/n!$. Show that $\lim_{n\to\infty} a_{n+1}/a_n = e$ and use Exercise 8.4 to deduce that

$$\lim_{n\to\infty}\frac{n}{(n!)^{1/n}} = e.$$

Proof: Since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}n!}{(n+1)!n^n} = \left(1+\frac{1}{n}\right)^n \to e,$$

by **Exercise 8.4**, we have

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\frac{n}{(n!)^{1/n}} = e.$$

Remark: There are many methods to show this. We do **NOT** give the detailed proof. But there are hints.

(1) Taking log on $\left(\frac{n!}{n^n}\right)^{1/n}$, and thus consider

$$\frac{1}{n}\left(\log\frac{1}{n}+\ldots+\log\frac{n}{n}\right)\to\int_0^1 \log x dx=-1.$$

(2) **Stirling's Formula**:

$$n!=n^n e^{-n}\sqrt{2\pi n}\, e^{\frac{\theta}{12n}},\text{ where } \theta\in(0,1).$$

Note: In general, we have

$$\lim_{x\to+\infty}\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}=1,$$

where $\Gamma(x)$ is the **Gamma Function**. The reader can see the book, **Principles of Mathematical Analysis by Walter Rudin, pp 192-195.**

(3) Note that $(1+\frac{1}{x})^x \nearrow e$ and $(1+\frac{1}{x})^{x+1}\searrow e$ on $(0,\infty)$. So,

$$\left(1+\frac{1}{n}\right)^n<e<\left(1+\frac{1}{n}\right)^{n+1}$$

which implies that

$$e(n^n e^{-n})<n!<e(n^{n+1}e^{-n}).$$

(4) Using **O-Stolz's Theorem:** Let $\lim_{n\to\infty}y_n=+\infty$ and $y_n\nearrow$. If

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=a,\text{ where } a \text{ is finite or } +\infty,$$

then

$$\lim_{n\to\infty}\frac{x_n}{y_n}=a.$$

Let $x_n=\log\frac{1}{n}+\ldots+\log\frac{n}{n}$ and $y_n=n$.

Note: For the proof of **O-Stolz's Theorem,** the reader can see the book, **An Introduction to Mathematical Analysis by Loo-Keng Hua, pp 195. (Chinese Version)**

(5) Note that, if $\{a_n\}$ is a positive sequence with $\lim_{n\to\infty}a_n=a$, then

$$(a_1\cdots a_n)^{1/n}\to a \text{ as } n\to\infty.$$

Taking $a_n=(1+\frac{1}{n})^n$, then

$$(a_1\cdots a_n)^{1/n}=\left(\frac{n^n}{n!}\right)^{1/n}\left(1+\frac{1}{n}\right)\to e.$$

Note: For the proof, it is easy from the **Exercise 8.6**. We give it a proof as follows. Say $\lim_{n\to\infty}a_n=a$. If $a=0$, then by $A.P.\geq G.P.$, we have

$$(a_1\cdots a_n)^{1/n}\leq\frac{a_1+\ldots+a_n}{n}\to 0 \text{ by } \textbf{Exercise 8.6}.$$

So, we consider $a\neq 0$ as follows. Note that $\log a_n\to\log a$. So, by **Exercise 8.6,**

$$\frac{\log a_1+\ldots+\log a_n}{n}\to\log a$$

which implies that $(a_1\cdots a_n)^{1/n}\to a$.

8.6 Let $\{a_n\}$ be real-valued sequence and let $\sigma_n=(a_1+\ldots+a_n)/n$. Show that

$$\underline{\lim}_{n\to\infty}\inf a_n\leq\underline{\lim}_{n\to\infty}\inf\sigma_n\leq\underline{\lim}_{n\to\infty}\sup\sigma_n\leq\underline{\lim}_{n\to\infty}\sup a_n.$$

Proof: By **Theorem 8.3 (a)**, it suffices to show that

$$\underline{\lim}_{n\to\infty}\inf a_n\leq\underline{\lim}_{n\to\infty}\inf\sigma_n \text{ and } \underline{\lim}_{n\to\infty}\sup\sigma_n\leq\underline{\lim}_{n\to\infty}\sup a_n.$$

We first prove

$$\lim_{n\to\infty}\sup \sigma_n \leq \lim_{n\to\infty}\sup a_n.$$

If $\limsup_{n\to\infty} a_n = +\infty,$ there is nothing to prove it. We may assume that $\limsup_{n\to\infty} a_n = -\infty$ or $a,$ where a is finite.

For the case, $\limsup_{n\to\infty} a_n = -\infty,$ then by **Theorem 8.3 (d)**, we have

$$\lim_{n\to\infty} a_n = -\infty.$$

So, given $M > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$a_n \leq -M. \quad *$$

Let $n > N,$ we have

$$\begin{aligned}\sigma_n &= \frac{(a_1+\ldots+a_N)+..+a_n}{n}\\ &= \frac{a_1+\ldots+a_N}{n} + \frac{a_{N+1}+\ldots+a_n}{n}\\ &\leq \frac{a_1+\ldots+a_N}{n} + \left(\frac{n-N}{n}\right)(-M)\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\sup \sigma_n \leq -M.$$

Since M is arbitrary, we finally have

$$\lim_{n\to\infty}\sup \sigma_n = -\infty.$$

For the case, $\limsup_{n\to\infty} a_n = a,$ where a is finite. Given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$a_n < a + \varepsilon.$$

Let $n > N,$ we have

$$\begin{aligned}\sigma_n &= \frac{(a_1+\ldots+a_N)+..+a_n}{n}\\ &= \frac{a_1+\ldots+a_N}{n} + \frac{a_{N+1}+\ldots+a_n}{n}\\ &\leq \frac{a_1+\ldots+a_N}{n} + \left(\frac{n-N}{n}\right)(a+\varepsilon)\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\sup \sigma_n \leq a+\varepsilon$$

which implies that

$$\lim_{n\to\infty}\sup \sigma_n \leq a$$

since ε is arbitrary.

Hence, from above results, we have proved that $\limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} a_n.$

Similarly for $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} \sigma_n.$

Remark: We suggest that the reader keep it in mind since it is the fundamental and useful in the **theory of Fourier Series.**

8.7 Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ if a_n is given by

(a) $\cos n$

Proof: Note that, $\{a + b\pi : a, b \in Z\}$ is dense in $R.$ By $\cos n = \cos(n + 2k\pi),$ we know that

$$\lim_{n\to\infty}\sup \cos n = 1 \text{ and } \lim_{n\to\infty}\inf \cos n = -1.$$

Remark: The reader may give it a try to show that

$$\lim_{n\to\infty}\sup \sin n = 1 \text{ and } \lim_{n\to\infty}\inf \sin n = -1.$$

(b) $(1+\frac{1}{n})\cos n\pi$

Proof: Note that

$$\left(1+\frac{1}{n}\right)\cos n\pi = \begin{cases} 1 \text{ if } n = 2k \\ -1 \text{ if } n = 2k-1 \end{cases}.$$

So, it is clear that

$$\lim_{n\to\infty}\sup\left(1+\frac{1}{n}\right)\cos n\pi = 1 \text{ and } \lim_{n\to\infty}\inf\left(1+\frac{1}{n}\right)\cos n\pi = -1.$$

(c) $n\sin\frac{n\pi}{3}$

Proof: Note that as $n = 1+6k$, $n\sin\frac{n\pi}{3} = (1+6k)\sin\frac{\pi}{3}$, and as $n = 4+6k$, $n = -(4+6k)\sin\frac{\pi}{3}$. So, it is clear that

$$\lim_{n\to\infty}\sup n\sin\frac{n\pi}{3} = +\infty \text{ and } \lim_{n\to\infty}\inf n\sin\frac{n\pi}{3} = -\infty.$$

(d) $\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}$

Proof: Note that $\sin\frac{n\pi}{2}\cos\frac{n\pi}{2} = \frac{1}{2}\sin n\pi = 0$, we have

$$\lim_{n\to\infty}\sup \sin\frac{n\pi}{2}\cos\frac{n\pi}{2} = \lim_{n\to\infty}\inf \sin\frac{n\pi}{2}\cos\frac{n\pi}{2} = 0.$$

(e) $(-1)^n n/(1+n)^n$

Proof: Note that

$$\lim_{n\to\infty}(-1)^n n/(1+n)^n = 0,$$

we know that

$$\lim_{n\to\infty}\sup(-1)^n n/(1+n)^n = \lim_{n\to\infty}\inf(-1)^n n/(1+n)^n = 0.$$

(f) $\frac{n}{3} - [\frac{n}{3}]$

Proof: Note that

$$\frac{n}{3} - \left[\frac{n}{3}\right] = \begin{cases} \frac{1}{3} \text{ if } n = 3k+1 \\ \frac{2}{3} \text{ if } n = 3k+2 \\ 0 \text{ if } n = 3k \end{cases}, \text{ where } k = 0,1,2,\ldots.$$

So, it is clear that

$$\lim_{n\to\infty}\sup \frac{n}{3} - \left[\frac{n}{3}\right] = \frac{2}{3} \text{ and } \lim_{n\to\infty}\inf \frac{n}{3} - \left[\frac{n}{3}\right] = 0.$$

Note. In (f), $[x]$ denoted the largest integer $\le x$.

8.8 Let $a_n = 2\sqrt{n} - \sum_{k=1}^{n} 1/\sqrt{k}$. Prove that the sequence $\{a_n\}$ converges to a limit p in the interval $1 < p < 2$.

Proof: Consider $\sum_{k=1}^{n} 1/\sqrt{k} := S_n$ and $\int_1^n x^{-1/2}dx := T_n$, then

$$\lim_{n\to\infty} d_n \text{ exists, where } d_n = S_n - T_n$$

by **Integral Test.** We denote the limit by d, then

$$0 \leq d < 1 \qquad *$$

by **Theorem 8.23 (i)**. Note that $\{d_n - f(n)\}$ is a positive increasing sequence, so we have

$$d > 0. \qquad **$$

Since

$$T_n = 2\sqrt{n} - 2$$

which implies that

$$\lim_{n\to\infty}\left(2\sqrt{n} - \sum_{k=1}^{n} 1/\sqrt{k}\right) = \lim_{n\to\infty} a_n = 2 - d = p.$$

By (*) and (**), we have proved that $1 < p < 1$.

Remark: (1) The use of **Integral Test** is very useful since we can know the behavior of a given series by integral. However, in many cases, the integrand may be so complicated that it is not easy to calculate. For example: Prove that the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \text{ where } p > 1.$$

Of course, it can be checked by **Integral Test**. But there is the Theorem called **Cauchy Condensation Theorem** much powerful than **Integral Test** in this sense. In addition, the reader can think it twice that in fact, **Cauchy condensation Theorem is equivalent to Integral Test.**

(**Cauchy Condensation Theorem**)Let $\{a_n\}$ be a positive decreasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges if, and only if, } \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges.}$$

Note: (1) The proof is not hard; the reader can see the book, **Principles of Mathematical Analysis by Walter Rudin, pp 61-63.**

(2) There is an extension of **Cauchy Condensation Theorem (Oskar Schlomilch)**: Suppose that $\{a_k\}$ be a positive and decreasing sequence and $\{m_k\}(\subseteq N)$ is a sequence. If there exists a $c > 0$ such that

$$0 < m_{k+2} - m_{k+1} \leq c(m_{k+1} - m_k) \text{ for all } k,$$

then

$$\sum_{k=1}^{\infty} a_k \text{ converges if, and only if, } \sum_{k=0}^{\infty} (m_{k+1} - m_k) a_{m_k}.$$

Note: The proof is similar with **Cauchy Condensation Theorem**, so we omit it.

(2) There is a similar Theorem, we write it as a reference. If $t \geq a$, $f(t)$ is a non-negative increasing function, then as $x \geq a$, we have

$$\left| \sum_{a \leq n \leq x} f(n) - \int_a^x f(t)dt \right| \leq f(x).$$

Proof: The proof is easy by drawing a graph. So, we omit it.

P.S.: The theorem is useful when we deal with some sums. For example,

$$f(t) = \log t.$$

Then

$$\left|\sum_{1\le n\le x}\log n - x\log x + x - 1\right| \le \log x.$$

In particular, as $x \in N,$ we thus have

$$n\log n - n + 1 - \log n \le \log n! \le n\log n - n + 1 + \log n$$

which implies that

$$n^{n-1}e^{-n+1} \le n! \le n^{n+1}e^{-n+1}.$$

In each of Exercise 8.9. through 8.14, show that the real-valed sequence $\{a_n\}$ is convergent. The given conditions are assumed to hold for all $n \ge 1.$ In Exercise 8.10 through 8.14, show that $\{a_n\}$ has the limit L indicated.

8.9 $|a_n| < 2,\ |a_{n+2} - a_{n+1}| \le \frac{1}{8}|a_{n+1}^2 - a_n^2|.$

Proof: Since

$$\begin{aligned}|a_{n+2} - a_{n+1}| &\le \frac{1}{8}|a_{n+1}^2 - a_n^2| \\ &= \frac{1}{8}|a_{n+1} - a_n||a_{n+1} + a_n| \\ &\le \frac{1}{2}|a_{n+1} - a_n| \text{ since } |a_n| < 2\end{aligned}$$

we know that

$$|a_{n+1} - a_n| \le \left(\frac{1}{2}\right)^{n-1}|a_2 - a_1| \le \left(\frac{1}{2}\right)^{n-3}.$$

So,

$$\begin{aligned}|a_{n+k} - a_n| &\le \sum_{j=1}^{k}|a_{n+j} - a_{n+j-1}| \\ &\le \sum_{j=1}^{k}\left(\frac{1}{2}\right)^{n+j-4} \\ &\le \left(\frac{1}{2}\right)^{n-2} \to \infty \text{ as } n \to \infty.\end{aligned}$$

Hence, $\{a_n\}$ is a Cauchy sequence. So, $\{a_n\}$ is a convergent sequence.

Remark: (1) If $|a_{n+1} - a_n| \le b_n$ for all $n \in N,$ and $\sum b_n$ converges, then $\sum a_n$ converges.

Proof: Since the proof is similar with the Exercise, we omit it.

(2) In (1), the condition $\sum b_n$ converges **CANNOT** omit. For example,
(i) Let $a_n = \sin\left(\sum_{k=1}^{n}\frac{1}{k}\right)$ Or
(ii) a_n is defined as follows:

$$a_1 = 1,\ a_2 = 1/2,\ a_3 = 0,\ a_4 = 1/4,\ a_5 = 1/2,\ a_6 = 3/4,\ a_7 = 1, \text{ and so on.}$$

8.10 $a_1 \ge 0,\ a_2 \ge 0,\ a_{n+2} = (a_n a_{n+1})^{1/2},\ L = (a_1 a_2^2)^{1/3}.$

Proof: If one of a_1 or a_2 is 0, then $a_n = 0$ for all $n \ge 2.$ So, we may assume that $a_1 \ne 0$ and $a_2 \ne 0.$ So, we have $a_n \ne 0$ for all $n.$ Let $b_n = \frac{a_{n+1}}{a_n},$ then

$$b_{n+1} = 1/\sqrt{b_n} \text{ for all } n$$

which implies that

$$b_{n+1} = (b_1)^{\left(\frac{-1}{2}\right)^n} \to 1 \text{ as } n \to \infty.$$

Consider

$$\Pi_{j=2}^{n+1} b_j = \Pi_{j=1}^{n} (b_j)^{-1/2}$$

which implies that

$$\left(a_1^{1/2} a_2\right)^{-2/3} a_{n+1} = \left(\frac{1}{b_{n+1}}\right)^{2/3}$$

which implies that

$$\lim_{n\to\infty} a_{n+1} = (a_1 a_2^2)^{1/3}.$$

Remark: There is another proof. We write it as a reference.

Proof: If one of a_1 or a_2 is 0, then $a_n = 0$ for all $n \geq 2.$ So, we may assume that $a_1 \neq 0$ and $a_2 \neq 0.$ So, we have $a_n \neq 0$ for all $n.$ Let $a_2 \geq a_1.$ Since $a_{n+2} = (a_n a_{n+1})^{1/2},$ then inductively, we have

$$a_1 \leq a_3 \leq \ldots \leq a_{2n-1} \leq \ldots \leq a_{2n} \leq \ldots \leq a_4 \leq a_2.$$

So, both of $\{a_{2n}\}$ and $\{a_{2n-1}\}$ converge. Say

$$\lim_{n\to\infty} a_{2n} = x \text{ and } \lim_{n\to\infty} a_{2n-1} = y.$$

Note that $a_1 \neq 0$ and $a_2 \neq 0,$ so $x \neq 0,$ and $y \neq 0.$ In addition, $x = y$ by $a_{n+2} = (a_n a_{n+1})^{1/2}.$ Hence, $\{a_n\}$ converges to $x.$

By $a_{n+2} = (a_n a_{n+1})^{1/2},$ and thus

$$\Pi_{j=1}^{n} a_{j+2}^2 = \Pi_{j=1}^{n} a_j a_{j+1} = (a_1 a_2^2 a_{n+1}) \Pi_{j=1}^{n-2} a_{j+2}^2$$

which implies that

$$a_{n+1} a_{n+2}^2 = a_1 a_2^2$$

which implies that

$$\lim_{n\to\infty} a_n = x = (a_1 a_2^2)^{1/3}.$$

8.11 $a_1 = 2,\ a_2 = 8,\ a_{2n+1} = \frac{1}{2}(a_{2n} + a_{2n-1}),\ a_{2n+2} = \frac{a_{2n} a_{2n-1}}{a_{2n+1}},\ L = 4.$

Proof: First, we note that

$$a_{2n+1} = \frac{a_{2n} + a_{2n-1}}{2} \geq \sqrt{a_{2n} a_{2n-1}} \text{ by } A.P. \geq G.P. \qquad *$$

for $n \in N.$ So, by $a_{2n+2} = \frac{a_{2n} a_{2n-1}}{a_{2n+1}}$ and (*),

$$a_{2n+2} = \frac{a_{2n} a_{2n-1}}{a_{2n+1}} \leq \sqrt{a_{2n} a_{2n-1}} \leq a_{2n+1} \text{ for all } n \in N.$$

Hence, by **Mathematical Induction**, it is easy to show that

$$a_4 \leq a_6 \leq \ldots \leq a_{2n+2} \leq \ldots \leq a_{2n+1} \leq \ldots \leq a_5 \leq a_3$$

for all $n \in N.$ It implies that both of $\{a_{2n}\}$ and $\{a_{2n-1}\}$ converge, say

$$\lim_{n\to\infty} a_{2n} = x \text{ and } \lim_{n\to\infty} a_{2n-1} = y.$$

With help of $a_{2n+1} = \frac{1}{2}(a_{2n} + a_{2n-1}),$ we know that $x = y.$ In addition, by $a_{2n+2} = \frac{a_{2n} a_{2n-1}}{a_{2n+1}},$ $a_1 = 2,$ and $a_2 = 8,$ we know that $x = 4.$

8.12 $a_1 = \frac{-3}{2},\ 3a_{n+1} = 2 + a_n^3,\ L = 1.$ Modify a_1 to make $L = -2.$

Proof: By **Mathematical Induction**, it is easy to show that

$$-2 \leq a_n \leq 1 \text{ for all } n. \qquad *$$

So,

$$3(a_{n+1} - a_n) = a_n^3 - 3a_n + 2 \geq 0$$

by (*) and $f(x) = x^3 - 3x + 2 = (x-1)^2(x+2) \geq 0$ on $[-2,1]$. Hence, $\{a_n\}$ is an increasing sequence with a upper bound 1. So, $\{a_n\}$ is a convergent sequence with limit L. So, by $3a_{n+1} = 2 + a_n^3$,

$$L^3 - 3L + 2 = 0$$

which implies that

$$L = 1 \text{ or } -2.$$

So, $L = 1$ sinc $a_n \nearrow$ and $a_1 = -3/2$.

In order to make $L = -2$, it suffices to let $a_1 = -2$, then $a_n = -2$ for all n.

8.13 $a_1 = 3$, $a_{n+1} = \frac{3(1+a_n)}{3+a_n}$, $L = \sqrt{3}$.

Proof: By **Mathematical Induction**, it is easy to show that

$$a_n \geq \sqrt{3} \text{ for all } n. \qquad *$$

So,

$$a_{n+1} - a_n = \frac{3 - a_n^2}{3 + a_n} \leq 0$$

which implies that $\{a_n\}$ is a decreasing sequence. So, $\{a_n\}$ is a convergent sequence with limit L by (*). Hence,

$$L = \frac{3(1+L)}{3+L}$$

which implies that

$$L = \pm\sqrt{3}.$$

So, $L = \sqrt{3}$ since $a_n \geq \sqrt{3}$ for all n.

8.14 $a_n = \frac{b_{n+1}}{b_n}$, where $b_1 = b_2 = 1$, $b_{n+2} = b_n + b_{n+1}$, $L = \frac{1+\sqrt{5}}{2}$.

Hint. Show that $b_{n+2}b_n - b_{n+1}^2 = (-1)^{n+1}$ and deduce that $|a_n - a_{n+1}| < n^{-2}$, if $n > 4$.

Proof: By **Mathematical Induction**, it is easy to show that

$$b_{n+2}b_n - b_{n+1}^2 = (-1)^{n+1} \text{ for all } n$$

and

$$b_n \geq n \text{ if } n > 4$$

Thus, (Note that $b_n \neq 0$ for all n)

$$|a_{n+1} - a_n| = \left|\frac{b_{n+2}}{b_{n+1}} - \frac{b_{n+1}}{b_n}\right| = \left|\frac{(-1)^{n+1}}{b_n b_{n+1}}\right| \leq \frac{1}{n(n+1)} < \frac{1}{n^2} \text{ if } n > 4.$$

So, $\{a_n\}$ is a Cauchy sequence. In other words, $\{a_n\}$ is a convergent sequence, say $\lim_{n\to\infty} b_n = L$. Then by $b_{n+2} = b_n + b_{n+1}$, we have

$$\frac{b_{n+2}}{b_{n+1}} = \frac{b_n}{b_{n+1}} + 1$$

which implies that (Note that $(0 \neq)L \geq 1$ since $a_n \geq 1$ for all n)

$$L = \frac{1}{L} + 1$$

which implies that

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

So, $L = \frac{1+\sqrt{5}}{2}$ since $L \geq 1$.

Remark: (1) The sequence $\{b_n\}$ is the famous sequence named **Fabonacci sequence**. There are many researches around it. Also, it is related with so called **Golden Section**, $\frac{\sqrt{5}-1}{2} = 0.618\ldots$

(2) The reader can see the book, **An Introduction To The Theory Of Numbers by G. H. Hardy and E. M. Wright, Chapter X.** Then it is clear by **continued fractions.**

(3) There is another proof. We write it as a reference.

Proof: (**STUDY**) Since $b_{n+2} = b_n + b_{n+1}$, we may think

$$x^{n+2} = x^n + x^{n+1},$$

and thus consider $x^2 = x + 1$. Say α and β are roots of $x^2 = x + 1$, with $\alpha < \beta$. Then let

$$F_n = \frac{\beta^n - \alpha^n}{\beta - \alpha},$$

we have

$$F_n = b_n.$$

So, it is easy to show that $L = \frac{1+\sqrt{5}}{2}$. We omit the details.

Note: The reader should be noted that there are many methods to find the formula of **Fabonacci sequence** like F_n. For example, using the concept of **Eigenvalues** if we can find a suitable matrix.

Series

8.15 Test for convergence (p and q denote fixed rela numbers).

(a) $\sum_{n=1}^{\infty} n^3 e^{-n}$

Proof: By **Root Test**, we have

$$\lim_{n\to\infty} \sup \left(\frac{n^3}{e^n} \right)^{1/n} = 1/e < 1.$$

So, the series converges.

(b) $\sum_{n=2}^{\infty} (\log n)^p$

Proof: We consider 2 cases: (i) $p \geq 0$, and (ii) $p < 0$.
For case (i), the series diverges since $(\log n)^p$ does not converge to zero.
For case (ii), the series diverges by **Cauchy Condensation Theorem** (or **Integral Test**.)

(c) $\sum_{n=1}^{\infty} p^n n^p$ $(p > 0)$

Proof: By **Root Test**, we have

$$\lim_{n\to\infty} \sup \left(\frac{p^n}{n^p} \right)^{1/n} = p.$$

So, as $p > 1$, the series diverges, and as $p < 1$, the series converges. For $p = 1$, it is clear that the series $\sum n$ diverges. Hence,

$$\sum_{n=1}^{\infty} p^n n^p \text{ converges if } p \in (0,1)$$

and

$$\sum_{n=1}^{\infty} p^n n^p \text{ diverges if } p \in [1, \infty).$$

(d) $\sum_{n=2}^{\infty} \frac{1}{n^p - n^q}$ $(0 < q < p)$

Proof: Note that $\frac{1}{n^p-n^q} = \frac{1}{n^p}\frac{1}{1-n^{q-p}}$. We consider 2 cases: (i) $p > 1$ and (ii) $p \leq 1$.
For case (i), by **Limit Comparison Test** with $\frac{1}{n^p}$,

$$\lim_{n\to\infty} \frac{\frac{1}{n^p-n^q}}{\frac{1}{n^p}} = 1,$$

the series converges.
For case (ii), by **Limit Comparison Test** with $\frac{1}{n^p}$,

$$\lim_{n\to\infty} \frac{\frac{1}{n^p-n^q}}{\frac{1}{n^p}} = 1,$$

the series diverges.

(e) $\sum_{n=1}^{\infty} n^{-1-1/n}$

Proof: Since $n^{-1-1/n} \geq n^{-1}$ for all n, the series diverges.

(f) $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n}$ $(0 < q < p)$

Proof: Note that $\frac{1}{p^n-q^n} = \frac{1}{p^n}\frac{1}{1-\left(\frac{q}{p}\right)^n}$. We consider 2 cases: (i) $p > 1$ and (ii) $p \leq 1$.
For case (i), by **Limit Comparison Test** with $\frac{1}{p^n}$,

$$\lim_{n\to\infty} \frac{\frac{1}{p^n-q^n}}{\frac{1}{p^n}} = 1,$$

the series converges.
For case (ii), by **Limit Comparison Test** with $\frac{1}{p^n}$,

$$\lim_{n\to\infty} \frac{\frac{1}{p^n-q^n}}{\frac{1}{p^n}} = 1,$$

the series diverges.

(g) $\sum_{n=1}^{\infty} \frac{1}{n\log(1+1/n)}$

Proof: Since

$$\lim_{n\to\infty} \frac{1}{n\log(1+1/n)} = 1,$$

we know that the series diverges.

(h) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$

Proof: Since the identity $a^{\log b} = b^{\log a}$, we have

$$\begin{aligned}(\log n)^{\log n} &= n^{\log\log n}\\ &\geq n^2 \text{ as } n \geq n_0.\end{aligned}$$

So, the series converges.

(i) $\sum_{n=3}^{\infty} \frac{1}{n\log n(\log\log n)^p}$

Proof: We consider 3 cases: (i) $p \leq 0$, (ii) $0 < p < 1$ and (iii) $p > 1$.

For case (i), since

$$\frac{1}{n\log n(\log\log n)^p} \geq \frac{1}{n\log n} \text{ for } n \geq 3,$$

we know that the series diverges by the divergence of $\sum_{n=3}^{\infty}\frac{1}{n\log n}$.

For case (ii), we consider (choose n_0 large enough)

$$\begin{aligned}\sum_{j=n_0}^{\infty}\frac{2^j}{2^j\log 2^j(\log\log 2^j)^p} &= \frac{1}{\log 2}\sum_{j=n_0}^{\infty}\frac{1}{j(\log j\log 2)^p} \\ &\geq \sum_{j=n_0}^{\infty}\frac{1}{j(\log j)^p},\end{aligned}$$

then, by **Cauchy Condensation Theorem,** the series diverges since $\sum_{j=n_0}^{\infty}\frac{1}{j(\log j)^p}$ diverges by using **Cauchy Condensation Theorem** again.

For case (iii), we consider (choose n_0 large enough)

$$\begin{aligned}\sum_{j=n_0}^{\infty}\frac{2^j}{2^j\log 2^j(\log\log 2^j)^p} &= \frac{1}{\log 2}\sum_{j=n_0}^{\infty}\frac{1}{j(\log j\log 2)^p} \\ &\leq 2\sum_{j=n_0}^{\infty}\frac{1}{j(\log j\log 2)^p} \\ &\leq 4\sum_{j=n_0}^{\infty}\frac{1}{j(\log j)^p},\end{aligned}$$

then, by **Cauchy Condensation Theorem,** the series converges since $\sum_{j=n_0}^{\infty}\frac{1}{j(\log j)^p}$ converges by using **Cauchy Condensation Theorem** again.

Remark: There is another proof by **Integral Test**. We write it as a reference.

Proof: It is easy to check that $f(x) = \frac{1}{x\log x(\log\log x)^p}$ is continous, positive, and decreasing to zero on $[a,\infty)$ where $a > 0$ for each fixed p. Consider

$$\int_a^{\infty}\frac{dx}{x\log x(\log\log x)^p} = \int_{\log\log a}^{\infty}\frac{dy}{y^p}$$

which implies that the series converges if $p > 1$ and diverges if $p \leq 1$ by **Integral Test**.

(j) $\sum_{n=3}^{\infty}\left(\frac{1}{\log\log n}\right)^{\log\log n}$

Proof: Let $a_n = \left(\frac{1}{\log\log n}\right)^{\log\log n}$ for $n \geq 3$ and $b_n = 1/n$, then

$$\begin{aligned}\frac{a_n}{b_n} &= n\left(\frac{1}{\log\log n}\right)^{\log\log n} \\ &= e^{-(y\log y - e^y)} \to +\infty.\end{aligned}$$

So, by **Limit Comparison Test**, the series diverges.

(k) $\sum_{n=1}^{\infty}\left(\sqrt{1+n^2}-n\right)$

Proof: Note that

$$\sqrt{1+n^2}-n = \frac{1}{\sqrt{1+n^2}+n} \geq \frac{1}{\left(1+\sqrt{2}\right)n} \text{ for all } n.$$

So, the series diverges.

(l) $\sum_{n=2}^{\infty} n^p\left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right)$

Proof: Note that

$$n^p\left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) = \frac{1}{n^{\frac{3}{2}-p}}\left(\sqrt{\frac{n}{n-1}}\,\frac{1}{1+\sqrt{\frac{n-1}{n}}}\right).$$

So, as $p < 1/2,$ the series converges and as $p \geq 1/2,$ the series diverges by **Limit Comparison Test.**

(m) $\sum_{n=1}^{\infty}\left((n)^{1/n} - 1\right)^n$

Proof: With help of **Root Test**,

$$\lim_{n\to\infty}\sup\left[\left((n)^{1/n} - 1\right)^n\right]^{1/n} = 0(< 1),$$

the series converges.

(n) $\sum_{n=1}^{\infty} n^p\left(\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1}\right)$

Proof: Note that

$$n^p\left(\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1}\right)$$

$$= \frac{1}{n^{\frac{3}{2}-p}}\left[\frac{n^{\frac{3}{2}}}{\left(\sqrt{n} + \sqrt{n+1}\right)\left(\sqrt{n} + \sqrt{n-1}\right)\left(\sqrt{n-1} + \sqrt{n+1}\right)}\right].$$

So, as $p < 1/2,$ the series converges and as $p \geq 1/2,$ the series diverges by **Limit Comparison Test.**

8.16 Let $S = \{n_1, n_2, \dots\}$ denote the collection of those positive integers that do not involve the digit 0 is their decimal representation. (For example, $7 \in S$ but $101 \notin S.$) Show that $\sum_{k=1}^{\infty} 1/n_k$ converges and has a sum less than 90.

Proof: Define $S_j = \left\{\text{the } j - \text{digit number}\right\}(\subseteq S).$ Then $\#S_j = 9^j$ and $S = \cup_{j=1}^{\infty} S_j.$ Note that

$$\sum_{k\in S_j} 1/n_k < \frac{9^j}{10^{j-1}}.$$

So,

$$\sum_{k=1}^{\infty} 1/n_k \leq \sum_{j=1}^{\infty}\frac{9^j}{10^{j-1}} = 90.$$

In addition, it is easy to know that $\sum_{k=1}^{\infty} 1/n_k \neq 90.$ Hence, we have proved that $\sum_{k=1}^{\infty} 1/n_k$ converges and has a sum less than 90.

8.17 Given integers $a_1, a_2, \dots$ such that $1 \leq a_n \leq n-1,\ n = 2, 3, \dots$ Show that the sum of the series $\sum_{n=1}^{\infty} a_n/n!$ is rational if and only if there exists an integer N such that $a_n = n-1$ for all $n \geq N.$ Hint: For sufficency, show that $\sum_{n=2}^{\infty}(n-1)/n!$ is a telescoping series with sum 1.

Proof: $(\Leftarrow)$Assume that there exists an integer N such that $a_n = n-1$ for all $n \geq N.$ Then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_n}{n!} &= \sum_{n=1}^{N-1}\frac{a_n}{n!}+\sum_{n=N}^{\infty}\frac{a_n}{n!}\\ &= \sum_{n=1}^{N-1}\frac{a_n}{n!}+\sum_{n=N}^{\infty}\frac{n-1}{n!}\\ &= \sum_{n=1}^{N-1}\frac{a_n}{n!}+\sum_{n=N}^{\infty}\frac{1}{(n-1)!}-\frac{1}{n!}\\ &= \sum_{n=1}^{N-1}\frac{a_n}{n!}+\frac{1}{(N-1)!}\in Q.\end{aligned}$$

($\Rightarrow$)Assume that $\sum_{n=1}^{\infty} a_n/n!$ is rational, say $\frac{q}{p}$, where $g.c.d.(p,q) = 1.$ Then

$$p!\sum_{n=1}^{\infty}\frac{a_n}{n!}\in Z.$$

That is, $p!\sum_{n=p+1}^{\infty}\frac{a_n}{n!}\in Z.$ Note that

$$p!\sum_{n=p+1}^{\infty}\frac{a_n}{n!}\le p!\sum_{n=p+1}^{\infty}\frac{n-1}{n!} = \frac{p!}{p!} = 1 \text{ since } 1\le a_n\le n-1.$$

So, $a_n = n-1$ for all $n\ge p+1.$ That is, there exists an integer N such that $a_n = n-1$ for all $n\ge N.$

Remark: From this, we have proved that e is irrational. The reader should be noted that we can use **Theorem 8.16** to show that e is irrational by considering $e^{-1}.$ Since it is easy, we omit the proof.

8.18 Let p and q be fixed integers, $p\ge q\ge 1,$ and let

$$x_n = \sum_{k=qn+1}^{pn}\frac{1}{k},\ s_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}.$$

(a) Use formula (8) to prove that $\lim_{n\to\infty}x_n = \log(p/q).$

Proof: Since

$$\sum_{k=1}^{n}\frac{1}{k} = \log n + r + O\left(\frac{1}{n}\right),$$

we know that

$$\begin{aligned}x_n &= \sum_{k=1}^{pn}\frac{1}{k}-\sum_{k=1}^{qn}\frac{1}{k}\\ &= \log(p/q)+O\left(\frac{1}{n}\right)\end{aligned}$$

which implies that $\lim_{n\to\infty}x_n = \log(p/q).$

(b) When $q = 1, p = 2,$ show that $s_{2n} = x_n$ and deduce that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \log 2.$$

Proof: We prove it by **Mathematical Induction** as follows. As $n = 1,$ it holds trivially. Assume that $n = m$ holds, i.e.,

$$s_{2m} = \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k} = \sum_{k=m+1}^{2m} \frac{1}{k} = x_m$$

consider $n = m + 1$ as follows.

$$\begin{aligned} x_{m+1} &= \sum_{k=(m+1)+1}^{2(m+1)} \frac{1}{k} \\ &= x_m - \frac{1}{m+1} + \frac{1}{2m+1} + \frac{1}{2m+2} \\ &= s_{2m} + \frac{1}{2m+1} - \frac{1}{2m+2} \\ &= s_{2(m+1)}. \end{aligned}$$

So, by **Mathematical Induction**, we have proved that $s_{2n} = x_n$ for all n.

By $s_{2n} = x_n$ for all n, we have

$$\lim_{n\to\infty} s_{2n} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2 = \lim_{n\to\infty} x_n.$$

(c) rearrange the series in (b), writing alternately p positive terms followed by q negative terms and use (a) to show that this rearrangement has sum

$$\log 2 + \frac{1}{2}\log(p/q).$$

Proof: We prove it by using **Theorem 8.13.** So, we can consider the new series $\sum_{k=1}^{\infty} a_k$ as follows:

$$a_k = \left[\left(\frac{1}{2(k-1)p+1} + \ldots + \frac{1}{2kp-1} \right) - \left(\frac{1}{2(k-1)q} + \ldots + \frac{1}{2kq} \right) \right]$$

Then

$$\begin{aligned} S_n &= \sum_{k=1}^{n} a_k \\ &= \sum_{k=1}^{2np} \frac{1}{k} - \sum_{k=1}^{np} \frac{1}{2k} - \sum_{k=1}^{nq} \frac{1}{2k} \\ &= \log 2np + \gamma + O\left(\frac{1}{n}\right) - \frac{1}{2}\log np - \frac{\gamma}{2} + O\left(\frac{1}{n}\right) - \frac{1}{2}\log nq - \frac{\gamma}{2} + O\left(\frac{1}{n}\right) \\ &= \log 2np - \log n\sqrt{pq} + O\left(\frac{1}{n}\right) \\ &= \log 2\sqrt{\frac{p}{q}} + O\left(\frac{1}{n}\right). \end{aligned}$$

So,

$$\lim_{n\to\infty} S_n = \log 2 + \frac{1}{2}\log(p/q)$$

by **Theorem 8.13**.

Remark: There is a reference around rearrangement of series. The reader can see the book, **Infinite Series by Chao Wen-Min, pp 216-220. (Chinese Version)**

(d) Find the sum of $\sum_{n=1}^{\infty}(-1)^{n+1}(1/(3n-2) - 1/(3n-1))$.

Proof: Write

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} (-1)^{k+1} \left(\frac{1}{3k-2} - \frac{1}{3k-1} \right) \\
&= \sum_{k=1}^{n} (-1)^{k} \frac{1}{3k-1} + \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{3k-2} \\
&= -\sum_{k=1}^{n} (-1)^{3k-1} \frac{1}{3k-1} - \sum_{k=1}^{n} (-1)^{3k-2} \frac{1}{3k-2} \\
&= -\left[\sum_{k=1}^{n} (-1)^{3k-1} \frac{1}{3k-1} + \sum_{k=1}^{n} (-1)^{3k-2} \frac{1}{3k-2} \right] \\
&= -\left[\sum_{k=1}^{3n} \frac{(-1)^{k}}{k} - \sum_{k=1}^{n} \frac{(-1)^{3k}}{3k} \right] \\
&= -\left[\sum_{k=1}^{3n} \frac{(-1)^{k}}{k} - \frac{1}{3} \sum_{k=1}^{n} \frac{(-1)^{k}}{k} \right] \\
&= \sum_{k=1}^{3n} \frac{(-1)^{k+1}}{k} - \frac{1}{3} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \\
&\to \frac{2}{3} \log 2.
\end{aligned}
$$

So, the series has the sum $\frac{2}{3} \log 2$.

Remark: There is a refernece around rearrangement of series. The reader can see the book, **An Introduction to Mathematical Analysis by Loo-Keng Hua, pp 323-325. (Chinese Version)**

8.19 Let $c_n = a_n + ib_n$, where $a_n = (-1)^n/\sqrt{n}$, $b_n = 1/n^2$. Show that $\sum c_n$ is conditioinally convergent.

Proof: It is clear that $\sum c_n$ converges. Consider

$$\sum |c_n| = \sum \sqrt{\frac{1}{n^2} + \frac{1}{n^4}} = \sum \frac{1}{n} \sqrt{1 + \frac{1}{n^2}} \geq \sum \frac{1}{n}$$

Hence, $\sum |c_n|$ diverges. That is, $\sum c_n$ is conditioinally convergent.

Remark: We say $\sum c_n$ converges if, and only if, the real part $\sum a_n$ converges and the imaginary part $\sum b_n$ converges, where $c_n = a_n + ib_n$.

8.20 Use Theorem 8.23 to derive the following formulas:

(a) $\sum_{k=1}^{n} \frac{\log k}{k} = \frac{1}{2} \log^2 n + A + O\left(\frac{\log n}{n} \right)$ (A is constant)

Proof: Let $f(x) = \frac{\log x}{x}$ define on $[3, \infty)$, then $f'(x) = \frac{1 - \log x}{x^2} < 0$ on $[3, \infty)$. So, it is clear that $f(x)$ is a positive and continuous function on $[3, \infty)$, with

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0 \text{ by } \textbf{L-Hospital Rule}.$$

So, by **Theorem 8.23**, we have

$$\sum_{k=3}^{n}\frac{\log k}{k} = \int_3^n \frac{\log x}{x}dx + C + O\left(\frac{\log n}{n}\right), \text{ where } C \text{ is a constant}$$
$$= \frac{1}{2}\log^2 n - \frac{1}{2}\log^2 3 + C + O\left(\frac{\log n}{n}\right), \text{ where } C \text{ is a constant}$$

which implies that

$$\sum_{k=1}^{n}\frac{\log k}{k} = \frac{1}{2}\log^2 n + A + O\left(\frac{\log n}{n}\right),$$

where $A = C + \frac{\log 2}{2} - \frac{1}{2}\log^2 3$ is a constant.

(b) $\sum_{k=2}^{n}\frac{1}{k\log k} = \log(\log n) + B + O\left(\frac{1}{n\log n}\right)$ (B is constant)

Proof: Let $f(x) = \frac{1}{x\log x}$ defined on $[2,\infty)$, then $f'(x) = -\left(\frac{1}{x\log x}\right)^2(1+\log x) < 0$ on $[2,\infty)$. So, it is clear that $f(x)$ is a positive and continuous function on $[3,\infty)$, with

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{1}{x\log x} = 0.$$

So, by **Theorem 8.23**, we have

$$\sum_{k=2}^{n}\frac{1}{k\log k} = \int_2^n \frac{dx}{x\log x} + C + O\left(\frac{1}{n\log n}\right), \text{ where } C \text{ is a constant}$$
$$= \log\log n + B + O\left(\frac{1}{n\log n}\right), \text{ where } C \text{ is a constant}$$

where $B = C - \log\log 2$ is a constant.

8.21 If $0 < a \le 1$, $s > 1$, define $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$.

(a) Show that this series converges absolutely for $s > 1$ and prove that

$$\sum_{h=1}^{k}\zeta\left(s,\frac{h}{k}\right) = k^s\zeta(s) \text{ if } k = 1,2,\ldots$$

where $\zeta(s) = \zeta(s,1)$ is the Riemann zeta function.

Proof: First, it is clear that $\zeta(s,a)$ converges absolutely for $s > 1$. Consider

$$\begin{aligned}
\sum_{h=1}^{k}\zeta\left(s,\frac{h}{k}\right) &= \sum_{h=1}^{k}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{h}{k})^s} \\
&= \sum_{h=1}^{k}\sum_{n=0}^{\infty}\frac{k^s}{(kn+h)^s} \\
&= \sum_{n=0}^{\infty}\sum_{h=1}^{k}\frac{k^s}{(kn+h)^s} \\
&= k^s\sum_{n=0}^{\infty}\sum_{h=1}^{k}\frac{1}{(kn+h)^s} \\
&= k^s\sum_{n=0}^{\infty}\frac{1}{(n+1)^s} \\
&= k^s\zeta(s).
\end{aligned}$$

(b) Prove that $\sum_{n=1}^{\infty}(-1)^{n-1}/n^s = (1-2^{1-s})\zeta(s)$ if $s > 1$.

Proof: Let $\left\{S_n = \sum_{j=1}^{n}\frac{(-1)^{j-1}}{j^s}\right\}$, and thus consider its subsequence $\{S_{2n}\}$ as follows:

$$S_{2n} = \sum_{j=1}^{2n}\frac{1}{j^s} - 2\sum_{j=1}^{n}\frac{1}{(2j)^s}$$
$$= \sum_{j=1}^{2n}\frac{1}{j^s} - 2^{1-s}\sum_{j=1}^{n}\frac{1}{j^s}$$

which implies that

$$\lim_{n\to\infty} S_{2n} = (1-2^{1-s})\zeta(s).$$

Since $\{S_n\}$ converges, we know that $\{S_{2n}\}$ also converges and has the same value. Hence,

$$\sum_{n=1}^{\infty}(-1)^{n-1}/n^s = (1-2^{1-s})\zeta(s).$$

8.22 Given a convergent series $\sum a_n$, where each $a_n \geq 0$. Prove that $\sum \sqrt{a_n}\, n^{-p}$ converges if $p > 1/2$. Give a counterexample for $p = 1/2$.

Proof: Since

$$\frac{a_n + n^{-2p}}{2} \geq \sqrt{a_n n^{-2p}} = \sqrt{a_n}\, n^{-p},$$

we have $\sum \sqrt{a_n}\, n^{-p}$ converges if $p > 1/2$ since

$$\sum a_n \text{ converges and } \sum n^{-2p} \text{ converges if } p > 1/2.$$

For $p = 1/2$, we consider $a_n = \frac{1}{n(\log n)^2}$, then

$$\sum a_n \text{ converges by } \textbf{Cauchy Condensation Theorem}$$

and

$$\sum \sqrt{a_n}\, n^{-1/2} = \sum \frac{1}{n\log n} \text{ diverges by } \textbf{Cauchy Condensation Theorem}.$$

8.23 Given that $\sum a_n$ diverges. Prove that $\sum na_n$ also diverges.

Proof: Assume $\sum na_n$ converges, then its partial sum $\sum_{k=1}^{n} ka_k$ is bounded. Then by **Dirichlet Test**, we would obtain

$$\sum (ka_k)\left(\frac{1}{k}\right) = \sum a_k \text{ converges}$$

which contradicts to $\sum a_n$ diverges. Hence, $\sum na_n$ diverges.

8.24 Given that $\sum a_n$ converges, where each $a_n > 0$. Prove that

$$\sum (a_n a_{n+1})^{1/2}$$

also converges. Show that the converse is also true if $\{a_n\}$ is monotonic.

Proof: Since

$$\frac{a_n + a_{n+1}}{2} \geq (a_n a_{n+1})^{1/2},$$

we know that

$$\sum (a_n a_{n+1})^{1/2}$$

converges by $\sum a_n$ converges.

Conversly, since $\{a_n\}$ is monotonic, it must be decreasing since $\sum a_n$ converges. So, $a_n \geq a_{n+1}$ for all $n.$ Hence,

$$(a_n a_{n+1})^{1/2} \geq a_{n+1} \text{ for all } n.$$

So, $\sum a_n$ converges since $\sum (a_n a_{n+1})^{1/2}$ converges.

8.25 Given that $\sum a_n$ converges absolutely. Show that each of the following series also converges absolutely:

(a) $\sum a_n^2$

Proof: Since $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty.$ So, given $\varepsilon = 1,$ there exists a positive integer N such that as $n \geq N,$ we have

$$|a_n| < 1$$

which implies that

$$a_n^2 < |a_n| \text{ for } n \geq N.$$

So, $\sum a_n^2$ converges if $\sum |a_n|$ converges. Of course, $\sum a_n^2$ converges absolutely.

(b) $\sum \frac{a_n}{1+a_n}$ (if no $a_n = -1$)

Proof: Since $\sum |a_n|$ converges, we have $\lim_{n\to\infty} a_n = 0.$ So, there exists a positive integer N such that as $n \geq N,$ we have

$$1/2 < |1 + a_n|.$$

Hence, as $n \geq N,$

$$\left| \frac{a_n}{1 + a_n} \right| < 2|a_n|$$

which implies that $\sum \left| \frac{a_n}{1+a_n} \right|$ converges. So, $\sum \frac{a_n}{1+a_n}$ converges absolutely.

(c) $\sum \frac{a_n^2}{1+a_n^2}$

Proof: It is clear that

$$\frac{a_n^2}{1 + a_n^2} \leq a_n^2.$$

By (a), we have proved that $\sum \frac{a_n^2}{1+a_n^2}$ converges absolutely.

8.26 Determine all real values of x for which the following series converges.

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) \frac{\sin nx}{n}.$$

Proof: Consider its partial sum

$$\sum_{k=1}^{n} \frac{\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right)}{k} \sin kx$$

as follows.

As $x = 2m\pi,$ the series converges to zero. So it remains to consider $x \neq 2m\pi$ as follows. Define

$$a_k = \frac{1 + \frac{1}{2} + \ldots + \frac{1}{k}}{k}$$

and

$$b_k = \sin kx,$$

then

$$\begin{aligned} a_{k+1} - a_k &= \frac{1 + \frac{1}{2} + \ldots + \frac{1}{k} + \frac{1}{k+1}}{k+1} - \frac{1 + \frac{1}{2} + \ldots + \frac{1}{k}}{k} \\ &= \frac{k(1 + \frac{1}{2} + \ldots + \frac{1}{k} + \frac{1}{k+1}) - (k+1)(1 + \frac{1}{2} + \ldots + \frac{1}{k})}{k(k+1)} \\ &= \frac{\frac{k}{k+1} - (1 + \frac{1}{2} + \ldots + \frac{1}{k})}{k(k+1)} < 0 \end{aligned}$$

and

$$\left|\sum_{k=1}^{n} b_k\right| \leq \left|\frac{1}{\sin(\frac{x}{2})}\right|.$$

So, by **Dirichlet Test**, we know that

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \frac{(1 + \frac{1}{2} + \ldots + \frac{1}{k})}{k} \sin kx$$

converges.

From above results, we have shown that the series converges for all $x \in R$.

8.27. Prove that following statements:

(a) $\sum a_n b_n$ converges if $\sum a_n$ converges and if $\sum (b_n - b_{n+1})$ converges absolutely.

Proof: Consider s**ummation by parts**, i.e., **Theorem 8.27**, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k).$$

Since $\sum a_n$ converges, then $|A_n| \leq M$ for all $n.$ In addition, by **Theorem 8.10,** $\lim_{n\to\infty} b_n$ exists. So, we obtain that

$$(1).\ \lim_{n\to\infty} A_n b_{n+1} \text{ exists}$$

and

$$(2).\ \sum_{k=1}^{n} |A_k (b_{k+1} - b_k)| \leq M \sum_{k=1}^{n} |b_{k+1} - b_k| \leq M \sum_{k=1}^{\infty} |b_{k+1} - b_k|.$$

(2) implies that

$$(3).\ \sum_{k=1}^{n} A_k (b_{k+1} - b_k) \text{ converges.}$$

By (1) and (3), we have shown that $\sum_{k=1}^{n} a_k b_k$ converges.

Remark: In 1871, **Paul du Bois Reymond (1831-1889)** gave the result.

(b) $\sum a_n b_n$ converges if $\sum a_n$ has bounded partial sums and if $\sum (b_n - b_{n+1})$ converges absolutely, provided that $b_n \to 0$ as $n \to \infty$.

Proof: By **summation by parts**, we have

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k).$$

Since $b_n \to 0$ as $n \to \infty$ and $\sum a_n$ has bounded partial sums, say $|A_n| \leq M$ for all $n.$ Then

$$(1).\ \lim_{n\to\infty} A_n b_{n+1} \text{ exists.}$$

In addition,

$$(2).\ \sum_{k=1}^{n} |A_k(b_{k+1}-b_k)| \le M\sum_{k=1}^{n}|b_{k+1}-b_k| \le M\sum_{k=1}^{\infty}|b_{k+1}-b_k|.$$

(2) implies that

$$(3).\ \sum_{k=1}^{n} A_k(b_{k+1}-b_k) \text{ converges.}$$

By (1) and (3), we have shown that $\sum_{k=1}^{n} a_k b_k$ converges.

Remark: (1) The result is first discovered by **Richard Dedekind (1831-1916).**

(2) There is an exercise by (b), we write it as a reference. Show the convergence of the series $\sum_{k=1}^{\infty} \frac{(-1)^{[\sqrt{k}]}}{k}$.

Proof: Let $a_k = \frac{(-1)^{[\sqrt{k}]}}{k^{2/3}}$ and $b_k = \frac{1}{k^{1/3}}$, then in order to show the convergence of $\sum_{k=1}^{\infty} \frac{(-1)^{[\sqrt{k}]}}{k}$, it suffices to show that $\left\{\sum_{k=1}^{n} a_k := S_n\right\}$ is bounded sequence. Given $n \in N$, there exists $j \in N$ such that $j^2 \le N < (j+1)^2$. Consider

$$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_8 + a_9 + \ldots. + a_{15} + \ldots + a_{j^2} + \ldots + a_n$$

$$\le \begin{cases} 3a_3 + 5a_4 + 7a_{15} + 9a_{16} + .. + (4k-1)a_{(2k)^2-1} + (4k+1)a_{(2k)^2} \text{ if } j = 2k,\ k \ge 2 \\ 3a_3 + 5a_4 + 7a_{15} + 9a_{16} + .. + (4k-3)a_{(2k-2)^2} \text{ if } j = 2k-1,\ k \ge 3 \end{cases}$$

then as n large enough,

$$S_n \le \begin{cases} (-3a_4 + 5a_4) + (-7a_{16} + 9a_{16}) + \ldots + \left(-(4k-1)a_{(2k)^2} + (4k+1)a_{(2k)^2}\right) \\ (-3a_4 + 5a_4) + (-7a_{16} + 9a_{16}) + \ldots + \left(-(4k-5)a_{(2k-2)^2} + (4k-3)a_{(2k-2)^2}\right) \end{cases}$$

which implies that as n large enough,

$$S_n \le 2\sum_{j=2}^{\infty} a_{(2j)^2} = 2\sum_{j=2}^{\infty} \frac{1}{(2j)^{4/3}} := M_1 \qquad *$$

Similarly, we have

$$M_2 \le S_n \text{ for all } n \qquad **$$

By (*) and (**), we have shown that $\left\{\sum_{k=1}^{n} a_k := S_n\right\}$ is bounded sequence.

Note: (1) By above method, it is easy to show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{[\sqrt{k}]}}{k^p}$$

converges for $p > 1/2$. For $0 < p \le 1/2$, the series diverges by

$$\frac{1}{(n^2)^p} + \ldots + \frac{1}{(n^2+2n)^p} \ge \frac{2n+1}{(n^2+n)^p} \ge \frac{2n+1}{(n^2+n)^p} \ge \frac{2n+1}{(n+1)^{2p}} \ge \frac{2n+1}{n+1} > 1.$$

(2) There is a similar question, show the divergence of the series $\sum_{k=1}^{\infty} \frac{(-1)^{[\log k]}}{k}$.

Proof: We use **Theorem 8.13** to show it by inserting parentheses as follows. We insert parentheses such that the series $\sum \frac{(-1)^{[\log k]}}{k}$ forms $\sum(-1)^k b_k$. If we can show $\sum(-1)^k b_k$

diverges, then $\sum \frac{(-1)^{[\log k]}}{k}$ also diverges. Consider

$$b_k = \frac{1}{m} + \ldots + \frac{1}{m+r}, \qquad *$$

where

$$\begin{aligned}
&(1).\ [\log m] = N \\
&(2).\ [\log(m-1)] = N-1 \Rightarrow [\log e(m-1)] = N \\
&(3).\ [\log(m+r)] = N \\
&(4).\ [\log(m+r+1)] = N+1 \Rightarrow [\log \tfrac{m+r+1}{e}] = N
\end{aligned}.$$

By (2) and (4),

$$\frac{m+r+1}{e} > m-1 \Rightarrow r+1 \geq m \text{ if } m \text{ is large enough.}$$

By (1) and (3),

$$2m \geq r.$$

So, as k large enough ($\Leftrightarrow$ m is large enough),

$$b_k \geq \frac{r+1}{m+r} \geq \frac{m}{3m} = \frac{1}{3} \text{ by (*).}$$

It implies that $\sum(-1)^k b_k$ diverges since b_k does **NOT** tends to zero as k goes infinity.So, we have proved that the series $\sum \frac{(-1)^{[\log k]}}{k}$ diverges.

(3) There is a good exercise by **summation by parts**, we write it as a reference. Assume that $\sum_{k=1}^{\infty} a_k b_k$ converges and $b_n \nearrow$ with $\lim_{n\to\infty} b_n = \infty$. Show that $b_n \sum_{k=n}^{\infty} a_k$ converges.

Proof: First, we show that the convergence of $\sum_{k=1}^{\infty} a_k$ by **Dirichlet Test** as follows. Since $b_n \nearrow \infty$, there exists a positive integer n_0 such that as $n > n_0$, we have $b_n > 0$. So, we have $\left\{\frac{1}{b_{n+n_0}}\right\}_{n=1}^{\infty}$ is decreasing to zero. So

$$\sum_{k=1}^{\infty} a_{k+n_0} = \sum_{k=1}^{\infty} (a_{k+n_0} b_{k+n_0}) \left(\frac{1}{b_{k+n_0}}\right)$$

converges by **Dirichlet Test**.

For the convergence of $b_n \sum_{k=n}^{\infty} a_k$, let $n > n_0$, then

$$b_n \sum_{k=n}^{\infty} a_k = \sum_{k=n}^{\infty} a_k b_k \frac{b_n}{b_k}$$

and define $c_k = a_k b_k$ and $d_k = \frac{b_n}{b_k}$. Note that $\{d_k\}$ is decreasing to zero. Define $C_k = \sum_{j=1}^{k} c_j$ and thus we have

$$\begin{aligned}
b_n \sum_{k=n}^{m} a_k &= \sum_{k=n}^{m} a_k b_k \frac{b_n}{b_k} \\
&= \sum_{k=n}^{m} (C_k - C_{k-1}) d_k \\
&= \sum_{k=n}^{m-1} C_k (d_k - d_{k+1}) + C_m d_m - C_{n-1} d_n.
\end{aligned}$$

So,

$$\begin{aligned} b_n \sum_{k=n}^{\infty} a_k &= \sum_{k=n}^{\infty} a_k b_k \frac{b_n}{b_k} \\ &= \sum_{k=n}^{\infty} C_k (d_k - d_{k+1}) + C_\infty d_\infty - C_{n-1} d_n \\ &= \sum_{k=n}^{\infty} C_k (d_k - d_{k+1}) - C_{n-1} d_n \end{aligned}$$

by $C_\infty = \lim_{k\to\infty} C_k$ and $\lim_{k\to\infty} d_k = 0.$ In order to show the existence of $\lim_{n\to\infty} b_n \sum_{k=n}^{\infty} a_k$, it suffices to show the existence of $\lim_{n\to\infty} \sum_{k=n}^{\infty} C_k (d_k - d_{k+1}).$ Since the series $\sum_{k=n}^{\infty} C_k (d_k - d_{k+1})$ exists, $\lim_{n\to\infty} \sum_{k=n}^{\infty} C_k (d_k - d_{k+1}) = 0.$ From above results, we have proved the convergence of $\lim_{n\to\infty} b_n \sum_{k=n}^{\infty} a_k$.

Note: We also show that $\lim_{n\to\infty} b_n \sum_{k=n}^{\infty} a_k = 0$ by preceding sayings.

Supplement on the convergence of series.

(A) Show the divergence of $\sum 1/k.$ We will give some methods listed below. If the proof is easy, we will omit the details.

(1) Use **Cauchy Criterion** for series. Since it is easy, we omit the proof.

(2) Just consider

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n} &\geq 1 + \frac{1}{2} + 2\frac{1}{4} + \ldots + 2^{n-1}\frac{1}{2^n} \\ &= 1 + \frac{n}{2} \to \infty. \end{aligned}$$

Remark: We can consider

$$1 + \left(\frac{1}{2} + \ldots + \frac{1}{10}\right) + \left(\frac{1}{11} + \ldots + \frac{1}{100}\right) + \ldots \geq 1 + \frac{9}{10} + \frac{90}{100} + \ldots$$

Note: The proof comes from **Jing Yu**.

(3) Use **Mathematical Induction** to show that

$$\frac{1}{k-1} + \frac{1}{k} + \frac{1}{k+1} \geq \frac{3}{k} \text{ if } k \geq 3.$$

Then

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots . \geq 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \ldots$$

Remark: The proof comes from **Bernoulli.**

(4) Use **Integral Test**. Since the proof is easy, we omit it.

(5) Use **Cauchy condensation Theorem**. Since the proof is easy, we omit it.

(6) **Euler Summation Formula**, the reader can give it a try. We omit the proof.

(7) The reader can see the book, **Princilpes of Mathematical Analysis by Walter Rudin, Exercise 11-(b) pp 79.**

Suppose $a_n > 0,$ $S_n = a_1 + \ldots + a_n,$ and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

Proof: If $a_n \to 0$ as $n \to \infty,$ then by **Limit Comparison Theorem**, we know that $\sum \frac{a_n}{1+a_n}$ diverges. If $\{a_n\}$ does not tend to zero. Claim that $\frac{a_n}{1+a_n}$ does not tend to zero.

Suppose **NOT**, it means that $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0.$ That is,

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{a_n}} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$

which contradicts our assumption. So, $\sum \frac{a_n}{1+a_n}$ diverges by claim.

(b) Prove that

$$\frac{a_{N+1}}{S_{N+1}} + \ldots + \frac{a_{N+k}}{S_{N+k}} \geq 1 - \frac{S_N}{S_{N+k}}$$

and deduce that $\sum \frac{a_n}{S_n}$ diverges.

Proof: Consider

$$\frac{a_{N+1}}{S_{N+1}} + \ldots + \frac{a_{N+k}}{S_{N+k}} \geq \frac{a_{N+1} + \ldots + a_{N+k}}{S_{N+k}} = 1 - \frac{S_N}{S_{N+k}}, \qquad *$$

then $\sum \frac{a_n}{S_n}$ diverges by Cauchy Criterion with (*).

Remark: Let $a_n = 1,$ then $\sum \frac{a_n}{S_n} = \sum 1/n$ diverges.

(c) Prove that

$$\frac{a_n}{S_n^2} \leq \frac{1}{S_{n-1}} - \frac{1}{S_n}$$

and deduce that $\sum \frac{a_n}{S_n^2}$ converges.

Proof: Consider

$$\frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{a_n}{S_{n-1}S_n} \geq \frac{a_n}{S_n^2},$$

and

$$\sum \frac{1}{S_{n-1}} - \frac{1}{S_n} \text{ converges by telescoping series with } \frac{1}{S_n} \to 0.$$

So, $\sum \frac{a_n}{S_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \text{ and } \sum \frac{a_n}{1+n^2a_n}?$$

Proof: For $\sum \frac{a_n}{1+na_n}$: as $a_n = 1$ for all $n,$ the series $\sum \frac{a_n}{1+na_n} = \sum \frac{1}{1+n}$ diverges. As

$$a_n = \begin{matrix} 0 \text{ if } n \neq k^2 \\ 1 \text{ if } n = k^2 \end{matrix} \quad ,$$

the series $\sum \frac{a_n}{1+na_n} = \sum \frac{1}{1+k^2}$ converges.

For $\sum \frac{a_n}{1+n^2a_n}$: Consider

$$\frac{a_n}{1+n^2a_n} = \frac{1}{\frac{1}{a_n}+n^2} \leq \frac{1}{n^2},$$

so $\sum \frac{a_n}{1+n^2a_n}$ converges.

(8) Consider $\sum \sin\frac{1}{n}$ diverges.

Proof: Since

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1,$$

the series $\sum \frac{1}{n}$ diverges by **Limit Comparison Theorem.**

Remark: In order to show the series $\sum \sin \frac{1}{n}$ diverges, we consider Cauchy Criterion as follows.

$$n \sin\left(\frac{1}{2n}\right) \leq \sin\left(\frac{1}{n+1}\right) + \ldots + \sin\left(\frac{1}{n+n}\right)$$

and given $x \in R,$ for $n = 0, 1, 2, \ldots,$ we have

$$|\sin nx| \leq n|\sin x|.$$

So,

$$\sin \frac{1}{2} \leq \sin\left(\frac{1}{n+1}\right) + \ldots + \sin\left(\frac{1}{n+n}\right)$$

for all $n.$ Hence, $\sum \sin \frac{1}{n}$ diverges.

Note: There are many methods to show the divergence of the series $\sum \sin \frac{1}{n}.$ We can use **Cauchy Condensation Theorem** to prove it. Besides, by (11), it also owrks.

(9) **O-Stolz's Theorem.**

Proof: Let $S_n = \sum_{j=1}^{n} \frac{1}{j}$ and $X_n = \log n.$ Then by **O-Stolz's Theorem,** it is easy to see

$$\lim_{n \to \infty} S_n = \infty.$$

(10) Since $\Pi_{k=1}^{n} 1 + \frac{1}{k}$ diverges, the series $\sum 1/k$ diverges by **Theorem 8.52**.

(11) **Lemma:** If $\{a_n\}$ is a decreasing sequence and $\sum a_n$ converges. Then $\lim_{n \to \infty} na_n = 0.$

Proof: Since $a_n \to 0$ and $\{a_n\}$ is a decreasing sequence, we conclude that $a_n \geq 0.$ Since $\sum a_n$ converges, given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$a_n + .. + a_{n+k} < \varepsilon/2 \text{ for all } k \in N$$

which implies that

$$(k+1)a_{n+k} < \varepsilon/2 \text{ since } a_n \searrow.$$

Let $k = n,$ then as $n \geq N,$ we have

$$(n+1)a_{2n} < \varepsilon/2$$

which implies that as $n \geq N$

$$2(n+1)a_{2n} < \varepsilon$$

which implies that

$$\lim_{n \to \infty} 2na_{2n} = 0 \text{ since } \lim_{n \to \infty} a_n = 0. \qquad *$$

Similarly, we can show that

$$\lim_{n \to \infty} (2n+1)a_{2n+1} = 0. \qquad **$$

So, by (*) adn (**), we have proved that $\lim_{n \to \infty} na_n = 0.$

Remark: From this, it is clear that $\sum \frac{1}{n}$ diverges. In addition, we have the convergence of $\sum n(a_n - a_{n+1}).$ We give it a proof as follows.

Proof: Write

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} k(a_k - a_{k+1}) \\
&= \sum_{k=1}^{n} a_k - na_{n+1},
\end{aligned}
$$

then

$$\lim_{n\to\infty} S_n \text{ exists}$$

since

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k \text{ exists and } \lim_{n\to\infty} na_n = 0.$$

(B) Prove that $\sum \frac{1}{p}$ diverges, where p is a prime.

Proof: Given $N,$ let $p_1, \ldots, p_k$ be the primes that divide at least one integer$\leq N.$ Then

$$
\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n} &\leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \ldots\right) \\
&= \prod_{j=1}^{k} \frac{1}{1 - \frac{1}{p_j}} \\
&\leq \exp\left(\sum_{j=1}^{k} \frac{2}{p_j}\right)
\end{aligned}
$$

by $(1-x)^{-1} \leq e^{2x}$ if $0 \leq x \leq 1/2.$ Hence, $\sum \frac{1}{p}$ diverges since $\sum \frac{1}{n}$ diverges.

Remark: There are many proofs about it. The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 91-93. (Chinese Version)**

(C) Discuss some series related with $\sum \frac{\sin k}{k}.$

STUDY: (1) We have shown that the series $\sum \sin \frac{1}{k}$ diverges.

(2) The series $\sum \sin(na + b)$ diverges where $a \neq n\pi$ for all $n \in Z$ and $b \in R.$

Proof: Suppose that $\sum \sin(na + b)$ converges, then $\lim_{n\to\infty} \sin(na + b) = 0.$ Hence, $\lim_{n\to\infty} |\sin[(n+1)a + b] - \sin(na + b)| = 0.$ Consider

$$
\begin{aligned}
&|\sin[(n+1)a + b] - \sin(na + b)| \\
&= \left|2\cos\left(na + b + \frac{a}{2}\right)\sin\left(\frac{a}{2}\right)\right| \\
&= \left|2\left[\cos(na + b)\cos\left(\frac{a}{2}\right) - \sin(na + b)\sin\left(\frac{a}{2}\right)\right]\sin\left(\frac{a}{2}\right)\right|
\end{aligned}
$$

which implies that

$$\begin{aligned}
&\lim_{n\to\infty}|\sin[(n+1)a+b]-\sin(na+b)| \\
&= \left|\lim_{n\to\infty}\sin[(n+1)a+b]-\sin(na+b)\right| \\
&= \left|\lim_{n\to\infty}\sup 2\left[\cos(na+b)\cos\left(\frac{a}{2}\right)-\sin(na+b)\sin\left(\frac{a}{2}\right)\right]\right|\left|\sin\left(\frac{a}{2}\right)\right| \\
&= \left|\lim_{n\to\infty}\sup 2\left[\cos(na+b)\cos\left(\frac{a}{2}\right)\right]\right|\left|\sin\left(\frac{a}{2}\right)\right| \\
&= |\sin a| \neq 0
\end{aligned}$$

which is impossible. So, $\sum\sin(na+b)$ diverges.

Remark: (1) By the same method, we can show the divergence of $\sum\cos(na+b)$ if $a\neq n\pi$ for all $n\in Z$ and $b\in R$.

(2) The reader may give it a try to show that,

$$\sum_{n=0}^{p}\cos(na+b) = \frac{\sin\frac{p+1}{2}b}{\sin\frac{b}{2}}\sin\left(a+\frac{p}{2}b\right) \qquad *$$

and

$$\sum_{n=0}^{p}\sin(na+b) = \frac{\sin\frac{p+1}{2}b}{\sin\frac{b}{2}}\cos\left(a+\frac{p}{2}b\right) \qquad **$$

by considering $\sum_{n=0}^{p}e^{i(na+b)}$. However, it is not easy to show the divergence by (*) and (**).

(3) The series $\sum\frac{\sin k}{k}$ converges conditionally.

Proof: First, it is clear that $\sum\frac{\sin k}{k}$ converges by **Dirichlet's Test** since $|\sum\sin k|\leq\left|\frac{1}{\sin\frac{1}{2}}\right|$. In order to show that the divergence of $\sum\left|\frac{\sin k}{k}\right|$, we consider its partial sums as follows: Since

$$\sum_{k=1}^{3n+3}\left|\frac{\sin k}{k}\right| = \sum_{k=0}^{n}\left|\frac{\sin 3k+1}{3k+1}\right|+\left|\frac{\sin 3k+2}{3k+2}\right|+\left|\frac{\sin 3k+3}{3k+3}\right|$$

and note that there is one value is bigger than 1/2 among three values $|\sin 3k+1|$, $|\sin 3k+2|$, and $|\sin 3k+3|$. So,

$$\sum_{k=1}^{3n+3}\left|\frac{\sin k}{k}\right| \geq \sum_{k=0}^{n}\frac{\frac{1}{2}}{3k+3}$$

which implies the divergence of $\sum\left|\frac{\sin k}{k}\right|$.

Remark: The series is like **Dirichlet Integral** $\int_0^\infty\frac{\sin x}{x}dx$. Also, we know that **Dirichlet Integral** converges conditionally.

(4) The series $\sum\frac{|\sin k|^r}{k}$ diverges for any $r\in R$.

Proof: We prove it by three cases as follows.
(a) As $r\leq 0$, we have

$$\sum\frac{|\sin k|^r}{k}\geq\sum\frac{1}{k}.$$

So, $\sum\frac{|\sin k|^r}{k}$ diverges in this case.
(b) As $0<r\leq 1$, we have

$$\sum \frac{|\sin k|^r}{k} \geq \sum \frac{|\sin k|}{k}.$$

So, $\sum \frac{|\sin k|^r}{k}$ diverges in this case by (3).

(c) As $r > 1,$ we have

$$\sum_{k=1}^{3n+3} \frac{|\sin k|^r}{k} = \sum_{k=0}^{n} \frac{|\sin 3k+1|^r}{3k+1} + \frac{|\sin 3k+2|^r}{3k+2} + \frac{|\sin 3k+3|^r}{3k+3}$$

$$\geq \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)^r}{3k+3}.$$

So, $\sum \frac{|\sin k|^r}{k}$ diverges in this case.

(5) The series $\sum \frac{\sin^{2p-1} k}{k},$ where $p \in N,$ converges.

Proof: We will prove that there is a positive integer $M(p)$ such that

$$\left| \sum_{k=1}^{n} \sin^{2p-1} k \right| \leq M(p) \text{ for all } n. \qquad *$$

So, if we can show (*), then by **Dirichlet's Test**, we have proved it. In order to show (*), we claim that $\sin^{2p-1} k$ can be written as a linear combination of $\sin k,$ $\sin 3k, \ldots,$ $\sin(2p-1)k.$ So,

$$\left| \sum_{k=1}^{n} \sin^{2p-1} k \right| = \left| \sum_{k=1}^{n} a_1 \sin k + a_2 \sin 3k + \ldots + a_p \sin(2p-1)k \right|$$

$$\leq |a_1| \left| \sum_{k=1}^{n} \sin k \right| + \ldots + |a_p| \left| \sum_{k=1}^{n} \sin(2p-1)k \right|$$

$$\leq \frac{|a_1|}{\left|\sin \frac{1}{2}\right|} + \ldots + \frac{|a_p|}{\left|\sin \frac{2p-1}{2}\right|} := M(p) \text{ by } \textbf{Theorem 8.30.}$$

We show the claim by **Mathematical Induction** as follows. As $p = 1,$ it trivially holds. Assume that as $p = s$ holds, i.e.,

$$\sin^{2s-1} k = \sum_{j=1}^{s} a_j \sin(2j-1)k$$

then as $p = s+1,$ we have

$$\begin{aligned}
\sin^{2s+1}k &= \sin^2 k(\sin k)^{2s-1} \\
&= \sin^2 k\left(\sum_{j=1}^{s} a_j \sin(2j-1)k\right) \text{ by induction hypothesis} \\
&= \sum_{j=1}^{s} a_j[\sin^2 k \sin(2j-1)k] \\
&= \sum_{j=1}^{s} a_j\left[\frac{1-\cos 2k}{2}\sin(2j-1)k\right] \\
&= \frac{1}{2}\left[\sum_{j=1}^{s} a_j \sin(2j-1)k - \sum_{j=1}^{s} a_j \cos 2k \sin(2j-1)k\right] \\
&= \frac{1}{2}\left\{\sum_{j=1}^{s} a_j \sin(2j-1)k - \frac{1}{2}\sum_{j=1}^{s} a_j[\sin(2j+1)k + \sin(2j-3)k]\right\}
\end{aligned}$$

which is a linear combination of $\sin k, \ldots, \sin(2s+1)k$. Hence, we have proved the claim by **Mathematical Induction.**

Remark: By the same argument, the series

$$\sum_{k=1}^{n} \cos^{2p-1} k$$

is also bounded, i.e., there exists a positive number $M(p)$ such that

$$\sum_{k=1}^{n} |\cos^{2p-1} k| \leq M(p).$$

(6) Define $\sum_{k=1}^{n} \frac{\sin kx}{k} := F_n(x)$, then $\{F_n(x)\}$ is boundedly convergent on R.

Proof: Since $F_n(x)$ is a periodic function with period 2π, and $F_n(x)$ is an odd function. So, it suffices to consider $F_n(x)$ is defined on $[0,\pi]$. In addition, $F_n(0) = 0$ for all n. Hence, the domain I that we consider is $(0,\pi]$. Note that $\frac{\sin kx}{k} = \int_0^x \cos kt dt$. So,

$$\begin{aligned}
F_n(x) &= \sum_{k=1}^{n} \frac{\sin kx}{k} \\
&= \int_0^x \sum_{k=1}^{n} \cos kt dt \\
&= \int_0^x \frac{\sin(n+\frac{1}{2})t - \sin(\frac{1}{2})t}{2\sin(\frac{1}{2})t} dt \\
&= \int_0^x \frac{\sin(n+\frac{1}{2})t}{t} dt + \int_0^x \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right) dt - \frac{x}{2} \\
&= \int_0^{(n+\frac{1}{2})x} \frac{\sin t}{t} dt + \int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right) dt - \frac{x}{2}
\end{aligned}$$

which implies that

$$|F_n(x)| \leq \left|\int_0^{(n+\frac{1}{2})x} \frac{\sin t}{t} dt\right| + \left|\int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right) dt\right| + \frac{\pi}{2}.$$

For the part $\left|\int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin t}{t} dt\right|$: Since $\int_0^\infty \frac{\sin t}{t} dt$ converges, there exists a positive M_1 such that

$$\left|\int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin t}{t} dt\right| \leq M_1 \text{ for all } x \in I \text{ and for all } n.$$

For the part $\left|\int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)(\sin(n+\frac{1}{2})t)dt\right|$: Consider

$$\begin{aligned}
&\left|\int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right)dt\right| \\
&\leq \int_0^x \frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}} dt \text{ since } t-2\sin\frac{t}{2} > 0 \text{ on } I \\
&\leq \int_0^\pi \frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}} dt := M_2 \text{ since } \lim_{t\to 0^+} \frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}} = 0.
\end{aligned}$$

Hence,

$$|F_n(x)| \leq M_1 + M_2 + \frac{\pi}{2} \text{ for all } x \in I \text{ and for all } n.$$

So, $\{F_n(x)\}$ is uniformly bounded on I. It means that $\{F_n(x)\}$ is uniformly bounded on R.

In addition, since

$$F_n(x) = \int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin t}{t} dt + \int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right)dt - \frac{x}{2},$$

fixed $x \in I$, we have

$$\int_0^\infty \frac{\sin t}{t} dt \text{ exists.}$$

and by **Riemann-Lebesgue Lemma**, in the text book, **pp 313**,

$$\lim_{n\to\infty} \int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right)dt = 0.$$

So, we have proved that

$$\lim_{n\to\infty} F_n(x) = \int_0^\infty \frac{\sin t}{t} dt - \frac{x}{2} \text{ where } x \in (0,\pi].$$

Hence, $\{F_n(x)\}$ is pointwise convergent on I. It means that $\{F_n(x)\}$ is pointwise convergent on R.

Remark: (1) For definition of being boundedly convergent on a set S, the reader can see the text book, pp **227**.

(2) In the proof, we also shown the value of **Dirichlet Integral**

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

by letting $x = \pi$.

(3) There is another proof on uniform bound. We write it as a reference.

Proof: The domain that we consider is still $(0,\pi]$. Let $\delta > 0$, and consider two cases as follows.

(a) $x \geq \delta > 0$: Using summation by parts,

$$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| \leq \left|\frac{1}{n+1}\sum_{k=1}^{n}\frac{\sin kx}{k}\right| + \left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\sin jx\right)\left(\frac{1}{k+1}-\frac{1}{k}\right)\right|$$
$$\leq \frac{1}{n+1}\frac{1}{\sin(\frac{\delta}{2})} + \frac{1}{\sin(\frac{\delta}{2})}\left(1-\frac{1}{n+1}\right)$$
$$= \frac{1}{\sin(\frac{\delta}{2})}.$$

(b) $0 < x \leq \delta$: Let $N = [\frac{1}{x}],$ consider two cases as follows.
As $n < N,$ then

$$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| \leq n|x| < N|x| \leq 1 \qquad *$$

and as $n \geq N,$ then

$$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right|$$
$$\leq \left|\sum_{k=1}^{N-1}\frac{\sin kx}{k}\right| + \left|\sum_{k=N}^{n}\frac{\sin kx}{k}\right|$$
$$\leq 1 + \left|\sum_{k=N}^{n}\frac{\sin kx}{k}\right| \text{ by (*)}$$
$$\leq 1 + \left|\frac{1}{n+1}\sum_{k=1}^{n}\frac{\sin kx}{k}\right| + \left|\frac{1}{N}\sum_{k=1}^{N-1}\frac{\sin kx}{k}\right| + \left|\sum_{k=N}^{n}\left(\sum_{j=1}^{k}\sin jx\right)\left(\frac{1}{k+1}-\frac{1}{k}\right)\right|$$

by summation by parts

$$\leq 1 + \frac{1}{(n+1)\sin\frac{x}{2}} + \frac{1}{N\sin\frac{x}{2}} + \left(\frac{1}{N}-\frac{1}{n+1}\right)\frac{1}{\sin\frac{x}{2}}$$
$$= 1 + \frac{2}{[\frac{1}{x}]\sin\frac{x}{2}}.$$

Note that $\lim_{x\to 0^+}\frac{2}{[\frac{1}{x}]\sin\frac{x}{2}} = 4.$ So, we may choose a $\delta' = \delta$ such that

$$\frac{2}{[\frac{1}{x}]\sin\frac{x}{2}} \leq 5 \text{ for all } x \in (0,\delta').$$

By preceding sayings, we have proved that $\{F_n(x)\}$ is uniformly bounded on $I.$ It means that $\{F_n(x)\}$ is uniformly bounded on $R.$

(D) In 1911, **Otto Toeplitz** proves the following. Let $\{a_n\}$ and $\{x_n\}$ be two sequences such that $a_n > 0$ for all n with $\lim_{n\to\infty}\frac{1}{a_1+\ldots+a_n} = 0$ and $\lim_{n\to\infty}x_n = x.$ Then

$$\lim_{n\to\infty}\frac{a_1x_1+\ldots+a_nx_n}{a_1+\ldots+a_n} = x.$$

Proof: Let $S_n = \sum_{k=1}^{n}a_k$ and $T_n = \sum_{k=1}^{n}a_kx_k,$ then

$$\lim_{n\to\infty}\frac{T_{n+1}-T_n}{S_{n+1}-S_n} = \lim_{n\to\infty}\frac{a_{n+1}x_{n+1}}{a_{n+1}} = \lim_{n\to\infty}x_{n+1} = x.$$

So, by **O-Stolz's Theorem**, we have prove it.

Remark: (1) Let $a_n = 1,$ then it is an extension of **Theorem 8.48.**

(2) Show that

$$\lim_{n\to\infty}\frac{\sin\theta+\ldots+\sin\frac{\theta}{n}}{1+\ldots+\frac{1}{n}}=\theta.$$

Proof: Write

$$\frac{\sin\theta+\ldots+\sin\frac{\theta}{n}}{1+\ldots+\frac{1}{n}}=\frac{(\frac{1}{1})1\sin\theta+\ldots+(\frac{1}{n})n\sin\frac{\theta}{n}}{1+\ldots+\frac{1}{n}},$$

the by **Toeplitz's Theorem,** we have proved it.

(E) **Theorem 8.16** emphasizes the decrease of the sequence $\{a_n\}$, we may ask if we remove the condition of decrease, is it true? The answer is **NOT** necessary. For example, let

$$a_n=\frac{1}{n}+\frac{(-1)^{n+1}}{2n}.(>0)$$

(F) Some questions on series.

(1) Show the convergence of the series $\sum_{n=1}^{\infty}\log n\sin\frac{1}{n}$.

Proof: Since $n\sin\frac{1}{n}<1$ for all n, $\log n\sin\frac{1}{n}<0$ for all n. Hence, we consider the new series

$$\sum_{n=1}^{\infty}-\log n\sin\frac{1}{n}=\sum_{n=1}^{\infty}\log\frac{\sin 1/n}{1/n}$$

as follows. Let $a_n=\log\frac{\sin 1/n}{1/n}$ and $b_n=\log\left(1+\frac{1}{n^2}\right)$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{1}{6}.$$

In addition,

$$\sum b_n\leq\sum\frac{1}{n^2}$$

by $e^x\geq 1+x$ for all $x\in R$. From the convergence of $\sum b_n$, we have proved that the convergence of $\sum a_n$ by **Limit Comparison Test.**

(2) Suppose that $a_n\in R$, and the series $\sum_{n=1}^{\infty}a_n^2$ converges. Prove that the series $\sum_{n=1}^{\infty}\frac{a_n}{n}$ converges absolutely.

Proof: By $A.P.\geq G.P.$, we have

$$\frac{a_n^2+\frac{1}{n^2}}{2}\geq\left|\frac{a_n}{n}\right|$$

which implies that $\sum_{n=1}^{\infty}\frac{a_n}{n}$ converges absolutely.

Remark: We metion that there is another proof by using **Cauchy-Schwarz inequality**. the difference of two proofs is that one considers a_n, and another considers the partial sums S_n.

Proof: By **Cauchy-Schwarz inequality,**

$$\left(\sum_{k=1}^{n}\frac{|a_n|}{k}\right)^2\leq\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)$$

which implies that $\sum_{n=1}^{\infty}\frac{a_n}{n}$ converges absolutely.

Double sequences and series

8.28 Investigate the existence of the two iterated limits and the double limit of the double sequence f defined by the followings. **Answer**. Double limit exists in (a), (d), (e), (g). Both iterated limits exists in (a), (b), (h). Only one iterated limit exists in (c), (e). Neither iterated limit exists in (d), (f).

(a) $f(p,q) = \frac{1}{p+q}$

Proof: It is easy to know that the double limit exists with $\lim_{p,q\to\infty} f(p,q) = 0$ by definition. We omit it. In addition, $\lim_{p\to\infty} f(p,q) = 0.$ So, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q)) = 0.$ Similarly, $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$ Hence, we also have the existence of two iterated limits.

(b) $f(p,q) = \frac{p}{p+q}$

Proof: Let $q = np,$ then $f(p,q) = \frac{1}{n+1}.$ It implies that the double limit does not exist. However, $\lim_{p\to\infty} f(p,q) = 1,$ and $\lim_{q\to\infty} f(p,q) = 0.$ So, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q)) = 1,$ and $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$

(c) $f(p,q) = \frac{(-1)^p p}{p+q}$

Proof: Let $q = np,$ then $f(p,q) = \frac{(-1)^p}{n+1}.$ It implies that the double limit does not exist. In addition, $\lim_{q\to\infty} f(p,q) = 0.$ So, $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$ However, since $\lim_{p\to\infty} f(p,q)$ does not exist, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q))$ does not exist.

(d) $f(p,q) = (-1)^{p+q}(\frac{1}{p} + \frac{1}{q})$

Proof: It is easy to know $\lim_{p,q\to\infty} f(p,q) = 0.$ However, $\lim_{q\to\infty} f(p,q)$ and $\lim_{p\to\infty} f(p,q)$ do not exist. So, neither iterated limit exists.

(e) $f(p,q) = \frac{(-1)^p}{q}$

Proof: It is easy to know $\lim_{p,q\to\infty} f(p,q) = 0.$ In addition, $\lim_{q\to\infty} f(p,q) = 0.$ So, $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$ However, since $\lim_{p\to\infty} f(p,q)$ does not exist, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q))$ does not exist.

(f) $f(p,q) = (-1)^{p+q}$

Proof: Let $p = nq,$ then $f(p,q) = (-1)^{(n+1)q}.$ It means that the double limit does not exist. Also, since $\lim_{p\to\infty} f(p,q)$ and $\lim_{q\to\infty} f(p,q)$ do not exist, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q))$ and $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q))$ do not exist.

(g) $f(p,q) = \frac{\cos p}{q}$

Proof: Since $|f(p,q)| \le \frac{1}{q},$ then $\lim_{p,q\to\infty} f(p,q) = 0,$ and $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$ However, since $\{\cos p : p \in N\}$ dense in $[-1,1],$ we know that $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q))$ does not exist.

(h) $f(p,q) = \frac{p}{q^2}\sum_{n=1}^{q} \sin\frac{n}{p}$

Proof: Rewrite

$$f(p,q) = \frac{p\sin\frac{q}{2p}\sin\frac{q+1}{2p}}{q^2\sin\frac{1}{2p}}$$

and thus let $p = nq,$ $f(p,q) = \frac{\sin\frac{1}{2n}\sin\left(\frac{q+1}{2nq}\right)}{nq\sin\frac{1}{2nq}}.$ It means that the double limit does not exist. However, $\lim_{p\to\infty} f(p,q) = \frac{q+1}{2q}$ since $\sin x \sim x$ as $x \to 0.$ So, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q)) = \frac{1}{2}.$

Also, $\lim_{q\to\infty} f(p,q) = \lim_{q\to\infty}\left(p\sin\frac{1}{2p}\right)\left(\frac{\sin\frac{q}{2p}\sin\frac{q+1}{2p}}{q^2}\right) = 0$ since $|\sin x| \leq 1$. So, $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0$.

8.29 Prove the following statements:

(a) A double series of positive terms converges if, and only if, the set of partial sums is bounded.

Proof: $(\Rightarrow)$Suppose that $\sum_{m,n} f(m,n)$ converges, say $\sum_{m,n} f(m,n) = A_1$, then it means that $\lim_{p,q\to\infty} s(p,q) = A_1$. Hence, given $\varepsilon = 1$, there exists a positive integer N such that as $p,q \geq N$, we have

$$|s(p,q)| \leq |A_1| + 1.$$

So, let $A_2 = \max\{s(p,q) : 1 \leq p,q < N\}$, we have $|s(p,q)| \leq \max(A_1,A_2)$ for all p,q. Hence, we have proved the set of partial sums is bounded.

$(\Leftarrow)$Suppose that the set of partial sums is bounded by M, i.e., if $S = \{s(p,q) : p,q \in N\}$, then $\sup S := A \leq M$. Hence, given $\varepsilon > 0$, then there exists a $s(p_1,q_1) \in S$ such that

$$A - \varepsilon < s(p_1,q_1) \leq A.$$

Choose $N = \max(p_1,q_1)$, then

$$A - \varepsilon < s(p,q) \leq A \text{ for all } p,q \geq N$$

since every term is positive. Hence, we have proved $\lim_{p,q\to\infty} s(p,q) = A$. That is, $\sum_{m,n} f(m,n)$ converges.

(b) A double series converges if it converges absolutely.

Proof: Let $s_1(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q}|f(m,n)|$ and $s_2(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} f(m,n)$, we want to show that the existence of $\lim_{p,q\to\infty} s_2(p,q)$ by the existence of $\lim_{p,q\to\infty} s_1(p,q)$ as follows.

Since $\lim_{p,q\to\infty} s_1(p,q)$ exists, say its limit a. Then $\lim_{p\to\infty} s_1(p,p) = a$. It implies that $\lim_{p\to\infty} s_2(p,p)$ converges, say its limit b. So, given $\varepsilon > 0$, there exists a positive integer N such that as $p,q \geq N$

$$|s_1(p,p) - s_1(q,q)| < \varepsilon/2$$

and

$$|s_2(N,N) - b| < \varepsilon/2.$$

So, as $p \geq q \geq N$,

$$\begin{aligned}
|s_2(p,q) - b| &= |[s_2(N,N) - b] + [s_2(p,q) - s_2(N,N)]| \\
&< \varepsilon/2 + |s_2(p,q) - s_2(N,N)| \\
&< \varepsilon/2 + s_1(p,p) - s_1(N,N) \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Similarly for $q \geq p \geq N$. Hence, we have shown that

$$\lim_{p,q\to\infty} s_2(p,q) = b.$$

That is, we have prove that a double series converges if it converges absolutely.

(c) $\sum_{m,n} e^{-(m^2+n^2)}$ converges.

Proof: Let $f(m,n) = e^{-(m^2+n^2)}$, then by **Theorem 8.44**, we have proved that

$\sum_{m,n} e^{-(m^2+n^2)}$ converges since $\sum_{m,n} e^{-(m^2+n^2)} = \sum_m e^{-m^2} \sum_n e^{-n^2}$.

Remark: $\sum_{m,n=1}^{\infty} e^{-(m^2+n^2)} = \sum_{m=1}^{\infty} e^{-m^2} \sum_{n=1}^{\infty} e^{-n^2} = \left(\frac{e}{e^2-1}\right)^2$.

8.30 Asume that the double series $\sum_{m,n} a(n)x^{mn}$ converges absolutely for $|x| < 1$. Call its sum $S(x)$. Show that each of the following series also converges absolutely for $|x| < 1$ and has sum $S(x)$:

$$\sum_{n=1}^{\infty} a(n)\frac{x^n}{1-x^n},\ \sum_{n=1}^{\infty} A(n)x^n,\ \text{where } A(n) = \sum_{d|n} a(d).$$

Proof: By **Theorem 8.42**,

$$\sum_{m,n} a(n)x^{mn} = \sum_{n=1}^{\infty} a(n)\sum_{m=1}^{\infty} x^{mn} = \sum_{n=1}^{\infty} a(n)\frac{x^n}{1-x^n} \text{ if } |x| < 1.$$

So, $\sum_{n=1}^{\infty} a(n)\frac{x^n}{1-x^n}$ converges absolutely for $|x| < 1$ and has sum $S(x)$.

Since every term in $\sum_{m,n} a(n)x^{mn}$, the term appears once and only once in $\sum_{n=1}^{\infty} A(n)x^n$. The converse also true. So, by **Theorem 8.42** and **Theorem 8.13**, we know that

$$\sum_{n=1}^{\infty} A(n)x^n = \sum_{m,n} a(n)x^{mn} = S(x).$$

8.31 If α is real, show that the double series $\sum_{m,n}(m+in)^{-\alpha}$ converges absolutely if, and only if, $\alpha > 2$. Hint. Let $s(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q}|m+in|^{-\alpha}$. The set

$$\{m+in : m = 1,2,\dots p, n = 1,2,\dots,p\}$$

consists of p^2 complex numbers of which one has absolute value $\sqrt{2}$, three satisfy $|1+2i| \le |m+in| \le 2\sqrt{2}$, five satisfy $|1+3i| \le |m+in| \le 3\sqrt{2}$, etc. Verify this geometricall and deduce the inequlity

$$2^{-\alpha/2}\sum_{n=1}^{p}\frac{2n-1}{n^\alpha} \le s(p,p) \le \sum_{n=1}^{p}\frac{2n-1}{(n^2+1)^{\alpha/2}}.$$

Proof: Since the hint is trivial, we omit the proof of hint. From the hint, we have

$$\sum_{n=1}^{p}\frac{2n-1}{\left(n\sqrt{2}\right)^\alpha} \le s(p,p) = \sum_{m=1}^{p}\sum_{n=1}^{p}|m+in|^{-\alpha} \le \sum_{n=1}^{p}\frac{2n-1}{(1+n^2)^{\alpha/2}}.$$

Thus, it is clear that the double series $\sum_{m,n}(m+in)^{-\alpha}$ converges absolutely if, and only if, $\alpha > 2$.

8.32 (a) Show that the Cauchy product of $\sum_{n=0}^{\infty}(-1)^{n+1}/\sqrt{n+1}$ with itself is a divergent series.

Proof: Since

$$
\begin{aligned}
c_n &= \sum_{k=0}^{n} a_k b_{n-k} \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} \\
&= (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}
\end{aligned}
$$

and let $f(k) = \sqrt{(n-k+1)(k+1)} = \sqrt{-(k-\frac{n}{2})^2 + (\frac{n+2}{2})^2} \leq \frac{n+2}{2}$ for $k = 0, 1, \ldots, n.$ Hence,

$$
\begin{aligned}
|c_n| &= \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}} \\
&\geq \frac{2(n+1)}{n+2} \to 2 \text{ as } n \to \infty.
\end{aligned}
$$

That is, the Cauchy product of $\sum_{n=0}^{\infty} (-1)^{n+1}/\sqrt{n+1}$ with itself is a divergent series.

(b) Show that the Cauchy product of $\sum_{n=0}^{\infty} (-1)^{n+1}/(n+1)$ with itself is the series

$$
2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).
$$

Proof: Since

$$
\begin{aligned}
c_n &= \sum_{k=0}^{n} a_k b_{n-k} \\
&= \sum_{k=0}^{n} \frac{(-1)^n}{(n-k+1)(k+1)} \\
&= (-1)^n \sum_{k=0}^{n} \frac{1}{n+2}\left(\frac{1}{k+1} + \frac{1}{n-k+1}\right) \\
&= \frac{2(-1)^n}{n+2} \sum_{k=0}^{n} \frac{1}{k+1},
\end{aligned}
$$

we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_n &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{n+2} \sum_{k=0}^{n} \frac{1}{k+1} \\
&= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n+1}\right) \\
&= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).
\end{aligned}
$$

(c) Does this converge ? Why?

Proof: Yes by the same argument in **Exercise 8.26**.

8.33 Given two absolutely convergent power series, say $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, having sums $A(x)$ and $B(x)$, respectively, show that $\sum_{n=0}^{\infty} c_n x^n = A(x)B(x)$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proof: By **Theorem 8.44** and **Theorem 8.13**, it is clear.

Remark: We can use **Mertens' Theorem**, then it is clear.

8.34 A series of the form $\sum_{n=1}^{\infty} a_n/n^s$ is called a Dirichlet series. Given two absolutely convergent Dirichlet series, say $\sum_{n=1}^{\infty} a_n/n^s$ and $\sum_{n=1}^{\infty} b_n/n^s$, having sums $A(s)$ and $B(s)$, respectively, show that $\sum_{n=1}^{\infty} c_n/n^s = A(s)B(s)$, where $c_n = \sum_{d|n} a_d b_{n/d}$.

Proof: By **Theorem 8.44** and **Theorem 8.13**, we have

$$\left(\sum_{n=1}^{\infty} a_n/n^s\right)\left(\sum_{n=1}^{\infty} b_n/n^s\right) = \left(\sum_{n=1}^{\infty} C_n\right)$$

where

$$\begin{aligned} C_n &= \sum_{d|n} a_d d^{-s} b_{n/d} (n/d)^{-s} \\ &= n^{-s} \sum_{d|n} a_d b_{n/d} \\ &= c_n/n^s. \end{aligned}$$

So, we have proved it.

8.35 $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, $s > 1$, show that $\zeta^2(s) = \sum_{n=1}^{\infty} d(n)/n^s$, where $d(n)$ is the number of positive divisors of n (including 1 and n).

Proof: It is clear by **Exercise 8.34**. So, we omit the proof.

Ces'aro summability

8.36 Show that each of the following series has $(C, 1)$ sum 0 :

(a) $1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + 1 - - + + \cdots$.

Proof: It is clear that $|s_1 + \ldots + s_n| \leq 1$ for all n, where s_n means that the nth partial sum of given series. So,

$$\left|\frac{s_1 + \ldots + s_n}{n}\right| \leq \frac{1}{n}$$

which implies that the given series has $(C, 1)$ sum 0.

(b) $\frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2} - 1 + + - \cdots$.

Proof: It is clear that $|s_1 + \ldots + s_n| \leq \frac{1}{2}$ for all n, where s_n means that the nth partial sum of given series. So,

$$\left|\frac{s_1 + \ldots + s_n}{n}\right| \leq \frac{1}{2n}$$

which implies that the given series has $(C, 1)$ sum 0.

(c) $\cos x + \cos 3x + \cos 5x + \cdots$ (x real, $x \neq m\pi$).

Proof: Let $s_n = \cos x + \ldots + \cos(2n-1)x$, then

$$\begin{aligned} s_n &= \sum_{j=1}^{n} \cos(2k-1)x \\ &= \frac{\sin 2nx}{2\sin x}. \end{aligned}$$

So,

$$\begin{aligned} \left| \frac{\sum_{j=1}^{n} s_j}{n} \right| &= \left| \frac{\sum_{j=1}^{n} \sin 2jx}{2n\sin x} \right| \\ &= \left| \frac{\sin nx \sin(n+1)x}{2n\sin x \sin x} \right| \\ &\leq \frac{1}{2n(\sin x)^2} \to 0 \end{aligned}$$

which implies that the given series has $(C,1)$ sum 0.

8.37 Given a series $\sum a_n,$ let

$$s_n = \sum_{k=1}^{n} a_k,\ t_n = \sum_{k=1}^{n} ka_k,\ \sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k.$$

Prove that:

(a) $t_n = (n+1)s_n - n\sigma_n$

Proof: Define $S_0 = 0,$ and thus

$$\begin{aligned} t_n &= \sum_{k=1}^{n} ka_k \\ &= \sum_{k=1}^{n} k(s_k - s_{k-1}) \\ &= \sum_{k=1}^{n} ks_k - \sum_{k=1}^{n} ks_{k-1} \\ &= \sum_{k=1}^{n} ks_k - \sum_{k=1}^{n-1}(k+1)s_k \\ &= \sum_{k=1}^{n} ks_k - \sum_{k=1}^{n}(k+1)s_k + (n+1)s_n \\ &= (n+1)s_n - \sum_{k=1}^{n} s_k \\ &= (n+1)s_n - n\sigma_n. \end{aligned}$$

(b) If $\sum a_n$ is $(C,1)$ summable, then $\sum a_n$ converges if, and only if, $t_n = o(n)$ as $n \to \infty$.

Proof: Assume that $\sum a_n$ converges. Then $\lim_{n\to\infty} s_n$ exists, say its limit $a.$ By (a), we have

$$\frac{t_n}{n} = \frac{n+1}{n}s_n - \sigma_n.$$

Then by **Theorem 8.48**, we also have $\lim_{n\to\infty} \sigma_n = a.$ Hence,

$$\begin{aligned}
\lim_{n\to\infty}\frac{t_n}{n} &= \lim_{n\to\infty}\frac{n+1}{n}s_n-\sigma_n \\
&= \lim_{n\to\infty}\frac{n+1}{n}\lim_{n\to\infty}s_n-\lim_{n\to\infty}\sigma_n \\
&= 1\cdot a-a \\
&= 0
\end{aligned}$$

which is $t_n = o(n)$ as $n \to \infty$.

Conversely, assume that $t_n = o(n)$ as $n \to \infty,$ then by (a), we have

$$\frac{n}{n+1}\frac{t_n}{n}+\frac{n}{n+1}\sigma_n = s_n$$

which implies that (note that $\lim_{n\to\infty}\sigma_n$ exists by hypothesis)

$$\begin{aligned}
\lim_{n\to\infty}s_n &= \lim_{n\to\infty}\frac{n}{n+1}\frac{t_n}{n}+\frac{n}{n+1}\sigma_n \\
&= \lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\frac{t_n}{n}+\lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\sigma_n \\
&= 1\cdot 0+1\cdot\lim_{n\to\infty}\sigma_n \\
&= \lim_{n\to\infty}\sigma_n
\end{aligned}$$

That is, $\sum a_n$ converges.

(c) $\sum a_n$ is $(C,1)$ summable if, and only if, $\sum t_n/n(n+1)$ converges.

Proof: Consider

$$\begin{aligned}
\frac{t_n}{n(n+1)} &= \frac{s_n}{n}-\frac{\sigma_n}{n+1} \\
&= \frac{n\sigma_n-(n-1)\sigma_{n-1}}{n}-\frac{\sigma_n}{n+1} \\
&= \frac{n}{n+1}\sigma_n-\frac{n-1}{n}\sigma_{n-1}
\end{aligned}$$

which implies that

$$\sum_{k=1}^{n}\frac{t_k}{k(k+1)} = \frac{n}{n+1}\sigma_n. \qquad *$$

($\Rightarrow$)Suppose that $\sum a_n$ is $(C,1)$ summable, i.e., $\lim_{n\to\infty}\sigma_n$ exists. Then $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{t_k}{k(k+1)}$ exists by (*).

($\Leftarrow$)Suppose that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{t_k}{k(k+1)}$ exists. Then $\lim_{n\to\infty}\sigma_n$ exists by (*). Hence, $\sum a_n$ is $(C,1)$ summable.

8.38 Given a monotonic $\{a_n\}$ of positive terms, such that $\lim_{n\to\infty}a_n = 0.$ Let

$$s_n = \sum_{k=1}^{n}a_k,\ u_n = \sum_{k=1}^{n}(-1)^k a_k,\ v_n = \sum_{k=1}^{n}(-1)^k s_k.$$

Prove that:

(a) $v_n = \frac{1}{2}u_n+(-1)^n s_n/2.$

Proof: Define $s_0 = 0,$ and thus consider

$$
\begin{aligned}
u_n &= \sum_{k=1}^{n}(-1)^k a_k \\
&= \sum_{k=1}^{n}(-1)^k (s_k - s_{k-1}) \\
&= \sum_{k=1}^{n}(-1)^k s_k + \sum_{k=1}^{n}(-1)^{k+1} s_{k-1} \\
&= \sum_{k=1}^{n}(-1)^k s_k + \sum_{k=1}^{n}(-1)^k s_k + (-1)^{n+1} s_n \\
&= 2v_n + (-1)^{n+1} s_n
\end{aligned}
$$

which implies that

$$v_n = \frac{1}{2}u_n + (-1)^n s_n/2.$$

(b) $\sum_{n=1}^{\infty}(-1)^n s_n$ is $(C,1)$ summable and has **Ces'aro sum** $\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n$.

Proof: First, $\lim_{n\to\infty} u_n$ exists since it is an alternating series. In addition, since $\lim_{n\to\infty} a_n = 0,$ we know that $\lim_{n\to\infty} s_n/n = 0$ by **Theorem 8.48**. Hence,

$$\frac{v_n}{n} = \frac{u_n}{2n} + (-1)^n \frac{s_n}{2n} \to 0 \text{ as } n \to \infty.$$

Consider by (a),

$$
\begin{aligned}
\frac{\sum_{k=1}^{n} v_k}{n} &= \frac{\frac{1}{2}\left(\sum_{k=1}^{n} u_k\right) + \frac{1}{2}\left(\sum_{k=1}^{n}(-1)^k s_k\right)}{n} \\
&= \frac{\sum_{k=1}^{n} u_k}{2n} + \frac{v_n}{2n} \\
&\to \frac{1}{2}\lim_{n\to\infty} u_k \\
&= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n
\end{aligned}
$$

by **Theorem 8.48**.

(c) $\sum_{n=1}^{\infty}(-1)^n (1 + \frac{1}{2} + \ldots + \frac{1}{n}) = -\log\sqrt{2}\ (C,1).$

Proof: By (b) and $\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\log 2$, it is clear.

Infinite products

8.39 Determine whether or not the following infinite products converges. Find the value of each convergent product.

(a) $\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right)$

Proof: Consider

$$1 - \frac{2}{n(n+1)} = \frac{(n-1)(n+2)}{n(n+1)},$$

we have

$$\prod_{n=2}^{n}\left(1-\frac{2}{k(k+1)}\right) = \prod_{n=2}^{n}\frac{(k-1)(k+2)}{k(k+1)}$$
$$= \frac{1\bullet 4}{2\bullet 3}\frac{2\bullet 5}{3\bullet 4}\frac{3\bullet 6}{4\bullet 5}\bullet\bullet\bullet\frac{(n-1)(n+2)}{n(n+1)}$$
$$= \frac{n+2}{3n}$$

which implies that

$$\prod_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right) = \frac{1}{3}.$$

(b) $\prod_{n=2}^{\infty}(1-n^{-2})$

Proof: Consider

$$1-n^{-2} = \frac{(n-1)(n+1)}{nn},$$

we have

$$\prod_{k=2}^{n}(1-k^{-2}) = \prod_{k=2}^{n}\frac{(k-1)(k+1)}{kk}$$
$$= \frac{n+1}{2n}$$

which implies that

$$\prod_{n=2}^{\infty}(1-n^{-2}) = 1/2.$$

(c) $\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$

Proof: Consider

$$\frac{n^3-1}{n^3+1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}$$
$$= \frac{(n-1)(n^2+n+1)}{(n+1)\left[(n-1)^2+(n-1)+1\right]}$$

we have (let $f(k) = (k-1)^2+(k-1)+1$),

$$\prod_{k=2}^{n}\frac{k^3-1}{k^3+1} = \prod_{k=2}^{n}\frac{(k-1)(k^2+k+1)}{(k+1)\left[(k-1)^2+(k-1)+1\right]}$$
$$= \frac{2}{3}\frac{n^2+n+1}{n(n+1)}$$

which implies that

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1} = \frac{2}{3}.$$

(d) $\prod_{n=0}^{\infty}(1+z^{(2^n)})$ if $|z| < 1$.

Proof: Consider

$$\prod_{k=0}^{n}1+z^{(2^k)} = (1+z)(1+z^2)\bullet\bullet\bullet(1+z^{(2^n)})$$

which implies that

$$(1-z)\prod_{k=0}^{n} 1+z^{(2^k)} = 1-z^{(2^{n+1})}$$

which implies that (if $|z| < 1$)

$$\prod_{k=0}^{n} 1+z^{(2^k)} = \frac{1-z^{(2^{n+1})}}{1-z} \to \frac{1}{1-z} \text{ as } n \to \infty.$$

So,

$$\prod_{n=0}^{\infty}(1+z^{(2^n)}) = \frac{1}{1-z}.$$

8.40 If each partial sum s_n of the convergent series $\sum a_n$ is not zero and if the sum itself is not zero, show that the infinite product $a_1 \prod_{n=2}^{\infty}(1-a_n/s_{n-1})$ converges and has the value $\sum_{n=1}^{\infty} a_n$.

Proof: Consider

$$\begin{aligned} a_1 \prod_{k=2}^{n}(1+a_k/s_{k-1}) &= a_1 \prod_{k=2}^{n} \frac{s_{k-1}+a_k}{s_{k-1}} \\ &= a_1 \prod_{k=2}^{n} \frac{s_k}{s_{k-1}} \\ &= s_n \to \sum a_n \neq 0. \end{aligned}$$

So, the infinite product $a_1 \prod_{n=2}^{\infty}(1-a_n/s_{n-1})$ converges and has the value $\sum_{n=1}^{\infty} a_n$.

8.41 Find the values of the following products by establishing the following identities and summing the series:

(a) $\prod_{n=2}^{\infty}(1-\frac{1}{2^n-2}) = 2\sum_{n=1}^{\infty} 2^{-n}$.

Proof: Consider

$$1-\frac{1}{2^n-2} = \frac{2^n-1}{2^n-2} = \frac{1}{2}\frac{2^n-1}{2^{n-1}-1},$$

we have

$$\begin{aligned} \prod_{k=2}^{n}\left(1-\frac{1}{2^k-2}\right) &= \prod_{k=2}^{n} \frac{1}{2}\frac{2^k-1}{2^{k-1}-1} \\ &= 2^{-(n-1)} \prod_{k=2}^{n} \frac{2^k-1}{2^{k-1}-1} \\ &= 2^{-(n-1)}(2^n-1) \\ &= 2^{-(n-1)}(2^{n-1}+\ldots+1) \\ &= 1+\ldots+\frac{1}{2^{n-1}} \\ &= \sum_{k=1}^{n} \frac{1}{2^{k-1}} \\ &= 2\sum_{k=1}^{n} \frac{1}{2^k}. \end{aligned}$$

So,

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{2^n-2}\right) = 2\sum_{n=1}^{\infty} 2^{-n}$$
$$= 2.$$

(b) $\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right) = 2\sum_{n=1}^{\infty}\frac{1}{n(n+1)}.$

Proof: Consider

$$1+\frac{1}{n^2-1} = \frac{n^2}{n^2-1} = \frac{nn}{(n-1)(n+1)},$$

we have

$$\begin{aligned}\prod_{k=2}^{n}\left(1+\frac{1}{k^2-1}\right) &= \prod_{k=2}^{n}\frac{kk}{(k-1)(k+1)}\\ &= 2\frac{n}{n+1}\\ &= 2\left(1-\frac{1}{n+1}\right)\\ &= 2\sum_{k=1}^{n}\frac{1}{k(k+1)}.\end{aligned}$$

So,

$$\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right) = 2\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$
$$= 2.$$

8.42 Determine all real x for which the product $\prod_{n=1}^{\infty}\cos(x/2^n)$ converges and find the value of the product when it does converge.

Proof: If $x \neq m\pi,$ where $m \in Z,$ then $\sin\frac{x}{2^n} \neq 0$ for all $n \in N.$ Hence,

$$\prod_{k=1}^{n}\cos(x/2^k) = \frac{2^n\sin\frac{x}{2^n}}{2^n\sin\frac{x}{2^n}}\prod_{k=1}^{n}\cos(x/2^k) = \frac{\sin x}{2^n\sin\frac{x}{2^n}} \to \frac{\sin x}{x}.$$

If $x = m\pi,$ where $m \in Z.$ Then as $m = 0,$ it is clear that the product converges to 1. So, we consider $m \neq 0$ as follows. Since $x = m\pi,$ choosing n large enough, i.e., as $n \geq N$ so that $\sin\frac{x}{2^n} \neq 0.$ Hence,

$$\begin{aligned}\prod_{k=1}^{n}\cos(x/2^k) &= \prod_{k=1}^{N-1}\cos(x/2^k)\prod_{k=N}^{n}\cos(x/2^k)\\ &= \prod_{k=1}^{N-1}\cos(x/2^k)\frac{\sin(x/2^{N-1})}{2^{n-N+1}\sin(x/2^n)}\end{aligned}$$

and note that

$$\lim_{n\to\infty}\frac{\sin(x/2^{N-1})}{2^{n-N+1}\sin(x/2^n)} = \frac{\sin(x/2^{N-1})}{x/2^{N-1}}.$$

Hence,

$$\prod_{k=1}^{\infty}\cos(x/2^k) = \frac{\sin(x/2^{N-1})}{x/2^{N-1}}\prod_{k=1}^{N-1}\cos(x/2^k).$$

So, by above sayings, we have prove that the convergence of the product for all $x \in R$.

8.43 (a) Let $a_n = (-1)^n/\sqrt{n}$ for $n = 1,2,\ldots$ Show that $\prod(1+a_n)$ diverges but that $\sum a_n$ converges.

Proof: Clearly, $\sum a_n$ converges since it is alternating series. Consider

$$\prod_{k=2}^{2n} 1 + a_k = \prod_{k=2}^{2n} 1 + \frac{(-1)^k}{\sqrt{k}}$$

$$= \left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\cdots\left(1-\frac{1}{\sqrt{2n-1}}\right)\left(1+\frac{1}{\sqrt{2n}}\right)$$

$$\leq \left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{4}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\cdots\left(1-\frac{1}{\sqrt{2n}}\right)\left(1+\frac{1}{\sqrt{2n}}\right)$$

$$= \left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2n}\right) \qquad *$$

and note that

$$\prod_{k=2}^{n} 1 - \frac{1}{2k} := p_n$$

is decreasing. From the divergence of $\sum \frac{1}{2k}$, we know that $p_n \to 0$. So,

$$\prod_{k=2}^{\infty} 1 + a_k = 0.$$

That is, $\prod_{k=2}^{\infty} 1 + a_k$ diverges to zero.

(b) Let $a_{2n-1} = -1/\sqrt{n}$, $a_{2n} = 1/\sqrt{n} + 1/n$ for $n = 1,2,\ldots$ Show that $\prod(1+a_n)$ converges but $\sum a_n$ diverges.

Proof: Clearly, $\sum a_n$ diverges. Consider

$$\prod_{k=2}^{2n} 1 + a_k = (1+a_2)(1+a_3)(1+a_4)\cdots(1+a_{2n})$$

$$= 3(1+a_3)(1+a_4)\cdots(1+a_{2n})$$

$$= 3\left(1-\frac{1}{2\sqrt{2}}\right)\cdots\left(1-\frac{1}{n\sqrt{n}}\right) \qquad *$$

and

$$\prod_{k=2}^{2n+1} 1 + a_k = (1+a_2)(1+a_3)(1+a_4)\cdots(1+a_{2n})(1+a_{2n+1})$$

$$= 3\left(1-\frac{1}{2\sqrt{2}}\right)\cdots\left(1-\frac{1}{n\sqrt{n}}\right)\left(1-\frac{1}{\sqrt{n+1}}\right) \qquad **$$

By (*) and (**), we know that

$$\prod(1+a_n) \text{ converges}$$

since $\prod_{k=2}^{n}\left(1-\frac{1}{k\sqrt{k}}\right)$ converges.

8.44 Assume that $a_n \geq 0$ for each $n = 1,2,\ldots$ Assume further that

$$\frac{a_{2n+2}}{1+a_{2n+2}} < a_{2n+1} < \frac{a_{2n}}{1+a_{2n}} \text{ for } n = 1,2,\ldots$$

Show that $\prod_{k=1}^{\infty}\left(1+(-1)^k a_k\right)$ converges if, and only if, $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

Proof: First, we note that if $\frac{a}{1+a} > b$, then $(1+a)(1-b) > 1$, and if $b > \frac{1+c}{c}$, then $1 > (1-b)(1+c)$. Hence, by hypothesis, we have

$$1 < (1+a_{2n})(1-a_{2n+1}) \qquad *$$

and

$$1 > (1+a_{2n+2})(1-a_{2n+1}). \qquad **$$

($\Leftarrow$)Suppose that $\sum_{k=1}^{\infty}(-1)^k a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Consider **Cauchy Condition for product,**

$$\left|\left(1+(-1)^{p+1}a_{p+1}\right)\left(1+(-1)^{p+2}a_{p+2}\right)\cdots(1+(-1)^{p+q}a_{p+q})-1\right| \text{ for } q = 1,2,3,\ldots.$$

If $p+1 = 2m$, and $q = 2l$, then

$$\begin{aligned}
&\left|\left(1+(-1)^{p+1}a_{p+1}\right)\left(1+(-1)^{p+2}a_{p+2}\right)\cdots(1+(-1)^{p+q}a_{p+q})-1\right| \\
&= |(1+a_{2m})(1-a_{2m+1})\cdots(1+a_{2m+2l})-1| \\
&\leq 1+a_{2m}-1 \text{ by (*) and (**)} \\
&= a_{2m} \to 0.
\end{aligned}$$

Similarly for other cases, so we have proved that $\prod_{k=1}^{\infty}\left(1+(-1)^k a_k\right)$ converges by **Cauchy Condition for product.**

($\Rightarrow$)This is a counterexample as follows. Let $a_n = (-1)^n\left[\left(\exp\frac{(-1)^n}{\sqrt{n}}\right)-1\right] \geq 0$ for all n, then it is easy to show that

$$\frac{a_{2n+2}}{1+a_{2n+2}} < a_{2n+1} < \frac{a_{2n}}{1+a_{2n}} \text{ for } n = 1,2,\ldots$$

In addition,

$$\prod_{k=1}^{n}\left(1+(-1)^k a_k\right) = \prod_{k=1}^{n}\exp\frac{(-1)^k}{\sqrt{k}} = \exp\left(\sum_{k=1}^{n}\frac{(-1)^k}{\sqrt{k}}\right) \to \exp(-\log 2) \text{ as } n\to\infty.$$

However, consider

$$\begin{aligned}
&\sum_{k=1}^{n}(a_{2k}-a_{2k-1}) \\
&= \sum_{k=1}^{n}\left[\exp\left(\frac{1}{\sqrt{2k}}\right)-\exp\left(\frac{-1}{\sqrt{2k-1}}\right)\right] \\
&= \sum_{k=1}^{n}\exp(b_k)\left(\frac{1}{\sqrt{2k}}+\frac{1}{\sqrt{2k-1}}\right), \text{ where } b_k \in \left(\frac{-1}{\sqrt{2k-1}},\frac{1}{\sqrt{2k}}\right) \\
&\geq \sum_{k=1}^{n}\exp(-1)\left(\frac{1}{\sqrt{2k}}+\frac{1}{\sqrt{2k-1}}\right) \to \infty \text{ as } n\to\infty.
\end{aligned}$$

So, by **Theorem 8.13**, we proved the divergence of $\sum_{k=1}^{\infty}(-1)^k a_k$.

8.45 A complex-valued sequence $\{f(n)\}$ is called **multiplicative** if $f(1) = 1$ and if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. (See Section 1.7) It is called **completely multiplicative** if

$$f(1) = 1 \text{ and if } f(mn) = f(m)f(n) \text{ for all } m \text{ and } n.$$

(a) If $\{f(n)\}$ is **multiplicative** and if the series $\sum f(n)$ converges absolutely, prove that

$$\sum_{n=1}^{\infty} f(n) = \prod_{k=1}^{\infty} \{1 + f(p_k) + f(p_k^2) + \ldots\},$$

where p_k denote the kth prime, the product being absolutely convergent.

Proof: We consider the partial product $P_m = \prod_{k=1}^{m} \{1 + f(p_k) + f(p_k^2) + \ldots\}$ and show that $P_m \to \sum_{n=1}^{\infty} f(n)$ as $m \to \infty$. Writing each factor as a geometric series we have

$$P_m = \prod_{k=1}^{m} \{1 + f(p_k) + f(p_k^2) + \ldots\},$$

a product of a finite number of absolutely convergent series. When we multiple these series together and rearrange the terms such that a typical term of the new absolutely convergent series is

$$f(n) = f(p_1^{a_1}) \cdots f(p_m^{a_m}), \text{ where } n = p_1^{a_1} \cdots p_m^{a_m},$$

and each $a_i \geq 0$. Therefore, we have

$$P_m = \sum_{1} f(n),$$

where $\sum_1$ is summed over those n having all their prime factors $\leq p_m$. By the **unique factorization theorem (Theorem 1.9),** each such n occors once and only once in $\sum_1$. Substracting P_m from $\sum_{n=1}^{\infty} f(n)$, we get

$$\sum_{n=1}^{\infty} f(n) - P_m = \sum_{n=1}^{\infty} f(n) - \sum_{1} f(n) = \sum_{2} f(n)$$

where $\sum_2$ is summed over those n having at least one prime factor $> p_m$. Since these n occors among the integers $> p_m$, we have

$$\left| \sum_{n=1}^{\infty} f(n) - P_m \right| \leq \sum_{n > p_m} |f(n)|.$$

As $m \to \infty$ the last sum tends to 0 because $\sum_{n=1}^{\infty} f(n)$ converges, so $P_m \to \sum_{n=1}^{\infty} f(n)$.

To prove that the product converges absolutely we use **Theorem 8.52**. The product has the form $\prod(1 + a_k)$, where

$$a_k = f(p_k) + f(p_k^2) + \ldots.$$

The series $\sum |a_k|$ converges since it is dominated by $\sum_{n=1}^{\infty} |f(n)|$. Thereofore, $\prod(1 + a_k)$ also converges absolutely.

Remark: The method comes from **Euler**. By the same method, it also shows that there are infinitely many primes. The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 91-93. (Chinese Version)**

(b) If, in addition, $\{f(n)\}$ is **completely multiplicative,** prove that the formula in (a) becomes

$$\sum_{n=1}^{\infty} f(n) = \prod_{k=1}^{\infty} \frac{1}{1 - f(p_k)}.$$

Note that Euler's product for $\zeta(s)$ (Theorem 8.56) is the special case in which $f(n) = n^{-s}$.

Proof: By (a), if $\{f(n)\}$ is **completely multiplicative,** then rewrite

$$
\begin{aligned}
1 + f(p_k) + f(p_k^2) + \ldots &= \sum_{n=0}^{\infty} [f(p_k)]^n \\
&= \frac{1}{1 - f(p_k)}
\end{aligned}
$$

since $|f(p_k)| < 1$ for all p_k. (Suppose **NOT,** then $|f(p_k)| \geq 1 \Rightarrow |f(p_k^n)| = |f(p_k)|^n \geq 1$ contradicts to $\lim_{n\to\infty} f(n) = 0.$).

Hence,

$$\sum_{n=1}^{\infty} f(n) = \prod_{k=1}^{\infty} \frac{1}{1 - f(p_k)}.$$

8.46 This exercise outlines a simple proof of the formula $\zeta(2) = \pi^2/6.$ Start with the inequality $\sin x < x < \tan x,$ valid for $0 < x < \pi/2,$ taking recipocals, and square each member to obtain

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x.$$

Now put $x = k\pi/(2m+1),$ where k and m are integers, with $1 \leq k \leq m,$ and sum on k to obtain

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1}.$$

Use the formula of **Exercise 1.49(c)** to deduce the ineqaulity

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{2m(m+1)\pi^2}{3(2m+1)^2}$$

Now let $m \to \infty$ to obtain $\zeta(2) = \pi^2/6.$

Proof: The proof is clear if we follow the hint and **Exercise 1.49 (c)**, so we omit it.

8.47 Use an argument similar to that outlined in **Exercise 8.46** to prove that $\zeta(4) = \pi^4/90.$

Proof: The proof is clear if we follow the **Exercise 8.46** and **Exercise 1.49 (c)**, so we omit it.

Remark: (1) From this, it is easy to compute the value of $\zeta(2s),$ where $s \in \{n : n \in N\}.$ In addition, we will learn some new method such as Fourier series and so on, to find the value of **Riemann zeta** function.

(2) Ther is an open problem that $\zeta(2s-1),$ where $s \in \{n \in N : n > 1\}.$