Math 131C: Homework 3 Solutions

(From Rudin, Chapter 9)

Problem 3:

Suppose $A\mathbf{x} = A\mathbf{y}$. Then $A(\mathbf{x} - \mathbf{y}) = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{y}$. $\square$

Problem 4:

These results follow immediately from the definition of a vector space and linearity. You don't have to verify all the vector space axioms; since the range and kernel are subsets of vector spaces, it suffices to show they are subspaces (i.e. that they are closed under addition and scalar multiplication).

Problem 5:

Given $A \in L(\mathbb{R}^n, \mathbb{R})$, let $\mathbf{y}_A = (A\mathbf{e}_1, A\mathbf{e}_2, \ldots, A\mathbf{e}_n)$ (where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$). By linearity, $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}_A$ for all $\mathbf{x} \in \mathbb{R}^n$. The uniqueness of $\mathbf{y}_A$ is immediate.

Now by Cauchy-Schwarz, we have

$$\|A\| = \sup_{|\mathbf{x}| \leq 1} \mathbf{x} \cdot \mathbf{y}_A \leq \sup_{|\mathbf{x}| \leq 1} |\mathbf{x}||\mathbf{y}_A| = |\mathbf{y}_A|.$$

But if $\mathbf{x} = \frac{\mathbf{y}_A}{|\mathbf{y}_A|}$, we have $|\mathbf{x}| = 1$ and $A\mathbf{x} = |\mathbf{y}_A|$, so $\|A\| \geq |\mathbf{y}_A|$. So $\|A\| = |\mathbf{y}_A|$. $\square$

Problem 6:

For $(x, y) \neq (0, 0)$, we can compute $(D_j f)(x, y)$ via the usual differentiation formulas just as in Math 32A or the like. At $(0, 0)$, we consider the definition of partial derivatives:

$$(D_1 f)(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Similarly, $(D_2 f)(0, 0) = 0$. So the partial derivatives of f exist everywhere in $\mathbb{R}^2$.

However, f is not continuous at $(0, 0)$, since $f(a, a) = \frac{1}{2}$ for all $a \in \mathbb{R}$. (So there is no neighborhood of the origin on which $|f(x, y) - f(0, 0)| = |f(x, y)| < \frac{1}{2}$). $\square$

Problem 7:

Fix $\mathbf{x} \in E$ and $\epsilon > 0$. Since the partial derivatives of f are bounded in E (say by M), the Mean Value Theorem implies that $|f(\mathbf{a} + h\mathbf{e}_j) - f(\mathbf{a})| < hM$ for all $\mathbf{a} \in E$ and all $h \in \mathbb{R}$ such that $\mathbf{a} + h\mathbf{e}_j \in E$ $(1 \leq j \leq n)$. Now choose $\delta < \frac{\epsilon}{nM}$ s.t. $B_\delta(\mathbf{x}) \subset E$ (we can choose such a δ since E is open).

Now pick an arbitrary $\mathbf{y}$ in $B_\delta(\mathbf{x})$. Write $\mathbf{y} - \mathbf{x} = \sum_1^n h_j \mathbf{e}_j$, $\mathbf{v}_0 = \mathbf{0}$, and $\mathbf{v}_k = \sum_1^k h_j \mathbf{e}_j$. (Note $\mathbf{v}_n = \mathbf{y} - \mathbf{x}$). It's pretty easy to see that $h_j < \delta$ for all j. Now we have

$$
\begin{aligned}
|f(\mathbf{y}) - f(\mathbf{x})| &= |f(\mathbf{x} + (\mathbf{y} - \mathbf{x})) - f(\mathbf{x})| \\
&= |\sum_{j=1}^{n} [f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})]| \\
&\leq \sum_{j=1}^{n} |f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})| \\
&\leq \sum_{j=1}^{n} h_j M \\
&< \sum_{j=1}^{n} \delta M \quad = \quad \epsilon,
\end{aligned}
$$

where we've used the Mean Value Theorem result from the first paragraph. So $|\mathbf{y} - \mathbf{x}| < \delta$ $\Rightarrow$ $|f(\mathbf{y}) - f(\mathbf{x})| < \epsilon$, and f is continuous at $\mathbf{x}$. $\square$

Problem 8:

Let $\mathbf{x} = (x_1, \ldots, x_n)$, and let E_j be the j-cross-section of E through $\mathbf{x}$; i.e.,

$$E_j = \{x \in \mathbb{R} \mid (x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_n) \in E\}.$$

Now define $f_j : E_j \to \mathbb{R}$ by $f_j(x) = f(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_n)$. Since f has a local maximum at $\mathbf{x}$, f_j has a local maximum at $x_j \in E_j$ for all j. By single-variable calculus, this implies $f_j'(x_j) = 0$ for all j. But since $f_j'(x_j) = D_j f(\mathbf{x})$ (VERIFY!), this implies $f'(\mathbf{x}) = \mathbf{0}$ by Theorem 9.17 in Rudin. $\square$

Problem 9:

To show f is constant, fix some $\mathbf{x}_0 \in E$ and consider the set $A = \{\mathbf{x} \in E \mid f(\mathbf{x}) = f(\mathbf{x}_0)\}$. We want to show that $A = E$, but since E is connected and A is nonempty, it suffices to show that A is both open and closed in the relative topology on E. Since E is open, the relative topology coincides with the usual topology on $\mathbb{R}^n$.

First we show A is open. Take any $\mathbf{x} \in A$. Since E is open, there is an open ball $B_r(\mathbf{x}) \subset E$. Now $B_r(\mathbf{x})$ is convex, so Theorem 9.19 (or its corollary) in Rudin implies that f is constant on $B_r(\mathbf{x})$. So, since $\mathbf{x} \in A$, this implies $B_r(\mathbf{x}) \subset A$. So A is open.

To show A is closed, use an exactly analogous argument to show that A^c is open. So A is both closed and open, which implies that $A = E$. $\square$

Problem 10:

(This whole problem is easier to think about if you draw a picture in two dimensions.) Fix any $\mathbf{x}_0 = (x_1^{(0)}, \ldots, x_n^{(0)}) \in E$. We want to show that for all $\mathbf{x} = (x_1, x_2^{(0)}, \ldots, x_n^{(0)}) \in E$, $f(\mathbf{x}) = f(\mathbf{x}_0)$. As in Problem 8, define $f_1(x) = f(x, x_2^{(0)}, \ldots, x_n^{(0)})$. Note that $f_1'(x) = (D_1 f)(x, x_2^{(0)}, \ldots, x_n^{(0)})$, so $f_1'(x) = 0$ for all x at which f_1 is defined. Now take any x_1 s.t. $(x_1, x_2^{(0)}, \ldots, x_n^{(0)}) \in E$. By convexity, we have $(x, x_2^{(0)}, \ldots, x_n^{(0)}) \in E$ for all $x_1^{(0)} \leq x \leq x_1$, so f_1 is defined for all such x. Since $f_1'(x) = 0$ for all these

x, the Mean Value Theorem shows that $f_1(x_1^{(0)}) = f_1(x_1)$. So $f(\mathbf{x}_0) = f(x_1, x_2^{(0)}, \ldots, x_n^{(0)})$, as desired. $\square$

Note that the proof just given only required that E be "convex in the first variable"; i.e., we can relax the convexity condition to

$$(a, x_2, \ldots, x_n) \in E \text{ and } (b, x_2, \ldots, x_n) \in E \implies (x, x_2, \ldots, x_n) \in E \ \forall \ a \leq x \leq b.$$

But, as noted, the proof doesn't work for arbitrary connected regions. Draw the union of the following three regions in $\mathbb{R}^2$:

$$A = \{(x, y) \mid -2 < x < 2,\ 0 < y < 1\}$$
$$B = \{(x, y) \mid -2 < x < -1,\ -2 < y \leq 0\}$$
$$C = \{(x, y) \mid 1 < x < 2,\ -2 < y \leq 0\}.$$

Take

$$f(x, y) = \begin{cases} 0, & (x, y) \in A \\ -y, & (x, y) \in B \\ y, & (x, y) \in C \end{cases}$$

Then clearly $\frac{\partial f}{\partial x} \equiv 0$ on the union of these regions, but f does not depend only on y. The problem is that we can't apply the Mean Value Theorem because of the "gap" between B and C.