MATH 131C: HOMEWORK 5 SOLUTIONS

(From Rudin, Chapter 9)

Problem 18:

(a) If we define $f(x,y) = (u(x,y), v(x,y))$, then the range of f is $\mathbb{R}^2$. The slickest way to see this is to note that if $z = x + iy$, then $u = \Re(z^2)$ and $v = \Im(z^2)$ (where $\Re$ and $\Im$ denote real and imaginary parts, respectively). Then since the map $z \mapsto z^2$ maps $\mathbb{C}$ onto $\mathbb{C}$, it follows that the range of f is all of $\mathbb{R}^2$. If you don't want to appeal to complex variables, you can also show this by brute-force algebra.

(b) The Jacobian of f *is* in fact 0 at the origin. However, it is nonzero at all other points of $\mathbb{R}^2$, so f is one-to-one on a neighborhood of every point of $\mathbb{R}^2 \setminus \{(0,0)\}$. But it's easily seen that $f(x,y) = f(-x,-y)$ for all (x,y), so f is not injective on $\mathbb{R}^2 \setminus \{(0,0)\}$. (This is the same as the fact that $z \mapsto z^2$ is not injective on $\mathbb{C} \setminus \{0\}$.)

(c) Once again, this is much easier if you appeal to the complex-variable interpretation of f, so identify $\mathbb{R}^2$ with $\mathbb{C}$. Since $u + iv = (x+iy)^2$, we can write

$$x + iy = \sqrt{u + iv},$$

where $\sqrt{}$ denotes a branch of the square root function defined in a neighborhood of $u + iv$ (note that we can't find such a branch if $u = v = 0$). In a neighborhood of $(0, \pi/3)$, we can take the standard branch of the square root, which is represented most easily in polar form (recall $e^{i\theta} = \cos\theta + i\sin\theta$):

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\frac{\theta}{2}}.$$

Using some trig identities (viz., the half-angle formulas), we can rewrite this in the upper half-plane as:

$$\sqrt{u + iv} = \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} + i\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}},$$

where the square roots on the right hand side are just the usual square roots of positive real numbers. Rewriting this in $\mathbb{R}^2$-notation:

$$f^{-1}(u,v) = \left(\sqrt{\frac{\sqrt{u^2+v^2}+u}{2}}, \sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}\right).$$

This is a good way to brush up on playing with complex numbers. I'll let you compute the derivatives in question; it's just calculus.

(d) The images under f of lines parallel to the coordinate axes are parabolas, unless of course we're considering the coordinate axes themselves, in which case we get we get the nonnegative u-axis as the image of the x-axis and the nonpositive u-axis as the image of the y-axis.

Problem 19:

First subtract the second equation from the first and compare with the third to see that u must be

0 or 3. To show that we can solve for x, y, u, in terms of z, apply the Implicit Function Theorem:

Let $f(x, y, z, u) = (3x + y - z + u^2, x - y + 2z + u, 2x + 2y - 3z + 2u) = (f_1, f_2, f_3)$. Then define

$$A_{x,y,u} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial u} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Then $\det A_{x,y,u} = 8u - 12 \neq 0$ since $u = 0$ or $u = 3$, so the Implicit Function Theorem gives the desired result. The next two claims follow by analogous reasoning.

Finally, to show that we cannot solve for x, y, z in terms of u, note that the system can be expressed as

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -u^2 \\ -u \\ -2u \end{bmatrix},$$

where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

But $\det A = 0$, so the system cannot be solved.

Problem 21:

(a) Compute $\nabla f = 6(x^2 - x, y^2 + y)$. Then clearly the gradient is zero at precisely the points $(0, 0)$, $(1, 0)$, $(0, -1)$, $(1, -1)$. $(1, 0)$ is a local minimum and $(0, -1)$ is a local maximum, while the other two are saddle points. This can easily be seen either by examining the Hessian of f or by examining f directly near the points in question.

(b) We'll first try to describe the set S. With some cleverness, you can notice that $f(x, -x) = 0$ for all $x \in \mathbb{R}$. This should suggest that we can factor f as

$$f(x, y) = (x + y)P(x, y),$$

where P is a polynomial. P turns out to be

$$P(x, y) = 2x^2 - 3x - 2xy + 3y + 2y^2.$$

The set of zeroes of P is an ellipse; so that S is the union of this ellipse and the line $y = -x$. (Try plotting this in Mathematica or some other such program.)

Now by the Implicit Function Theorem, the only candidates for points that have no neighborhoods in which $f(x, y) = 0$ can be solved for y in terms of x (or vice versa) will be points at which $D_1 f = 0$ *and* $D_2 f = 0$, i.e. $\nabla f = 0$. (This is a NOT(p OR q) $\iff$ (NOT p) AND (NOT q) thing). Using part (a), it's easy to see that the only points of S at which $\nabla f = 0$ are $(0, 0)$ and $(1, -1)$. These both lie on the line $y = -x$ and the ellipse $P(x, y) = 0$, so the curves "cross" at these points. If you examine a plot of the points in S, it's pretty clear that it is neither a graph of y as a function of x nor a graph of x as a function of y. So these are the points we're looking for. As an example of how to be more rigorous about this, we'll consider the point $(0, 0)$. Pick a positive $\varepsilon \ll 1$. Then for any $x_0 \in (-\varepsilon, \varepsilon)$, the point $(x_0, -x_0)$ satisfies $f(x, y) = 0$. But there is also a y_0 s.t. $P(x_0, y_0) = 0 \Rightarrow f(x_0, y_0) = 0$. It's pretty easy to see that this y_0 must have the same sign as x_0, so $y_0 \neq -x_0$. So, in short, we can't solve for y in terms of x in some small neighborhood of the $(0,0)$. An analogous argument shows we can't solve for x in terms of y, and the point $(1,-1)$ is treated similarly. (I realize this is probably confusing; it really helps to draw a picture).

Problem 24:

Compute the Jacobian matrix of $\mathbf{f}$:

$$[\mathbf{f}'(x,y)] = \frac{1}{(x^2+y^2)^2}\begin{bmatrix} 4xy^2 & -4x^2y \\ y^3-x^2y & x^3-y^2x \end{bmatrix}.$$

Noticing that the first column is $\frac{-x}{y}$ times the second, we see that $\mathbf{f}'(x,y)$ has rank 1. The range of $\mathbf{f}$ is an ellipse, as you can see from the relation

$$f_1^2 + 4f_2^2 = 1.$$

Problem 25:

(a) By the hint, SA is a projection in $\mathbb{R}^n$. The $\mathbf{z}_j$ used to define S are linearly independent (VERIFY!), so S is injective. Then it's immediate that $\mathscr{N}(SA) = \mathscr{N}(A)$. Finally, $SA\mathbf{z}_j = \mathbf{z}_j$ so that $\mathscr{R}(SA) = \mathscr{R}(S)$, since clearly $\mathscr{R}(SA) \subseteq \mathscr{R}(S)$.

(b) This is immediate from the Rank-Nullity Theorem, but we can do the problem as Rudin suggests without using this result. We'll use the following result from the proof of 9.31 (b) in Rudin:

$$P \text{ a projection in } \mathbb{R}^n \implies \dim \mathscr{N}(P) + \dim \mathscr{R}(P) = n.$$

Since we noted in part (a) that the $\mathbf{z}_j$ are linearly independent, we know that $\operatorname{rank}(S) = r = \operatorname{rank}(A)$. The desired result now follows immediately from part (a).

Problem 28:

To show φ is continuous, it suffices to consider points (x,t) with $t \geq 0$ since φ is odd in t. The following is just a sketch of the continuity argument; I expect a bit more detail on the actual homeworks. First consider a point (x_0, t_0) with $0 < x_0 < \sqrt{t_0}$. Argue that you can find an $\varepsilon > 0$ such that $0 < x < \sqrt{t}$ for all $(x,y) \in B_\varepsilon((x_0,y_0))$. Then $\varphi(x,t) = x$ on this ball, and the continuity follows immediately. Treat the cases $\sqrt{t_0} < x < 2\sqrt{t_0}$ and $2\sqrt{t_0} < x$ similarly. Now consider a point of the form $(\sqrt{t_0}, t_0)$ (i.e. $x = \sqrt{t_0}$). Since the functions $\varphi_1(x,t) = x$ and $\varphi_2(x,t) = -x + 2\sqrt{t}$ are both continuous, choose $\delta > 0$ so that both $|\varphi_1(x,t) - \varphi_1(\sqrt{t_0}, t_0)| < \varepsilon$ and $|\varphi_2(x,t) - \varphi_2(\sqrt{t_0}, t_0)| < \varepsilon$ for a given $\varepsilon > 0$. This δ will "work" for φ in the usual sense. Treat the other "boundary" cases similarly.

Showing that $(D_2\varphi)(x,0) = 0$ for all x is routine; just use the definition of partial derivatives.

Now $|t| < \frac{1}{4} \Rightarrow 2\sqrt{|t|} < 1$, so we have

$$f(t) = \int_{-1}^{1} \varphi(x,t)\,dx = \int_0^1 \varphi(x,t)\,dx = \operatorname{sgn}(t)\left[\int_0^{\sqrt{|t|}} x\,dx \; + \; \int_{\sqrt{|t|}}^{2\sqrt{|t|}} (-x + 2\sqrt{|t|})\,dx\right] = t,$$

where

$$\operatorname{sgn}(t) \; = \; \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0. \end{cases}$$

So this implies $f'(t) = 1$ for $|t| < \frac{1}{4}$, and hence

$$1 \;=\; f'(0) \;\neq\; \int_{-1}^{1} (D_2\varphi)(x,0)\,dx \;= 0.$$

This suggests you should take care when differentiating under the integral sign.