

Rudin:

(1) Chapter 2, Problem 6

Done in class on Thursday September 26. Here $E \subset X$ is a subset of a metric space and E' is the set of limit points, in X , of E .

(a) Prove that E' is closed.

If $p \in X$ is a limit point of E' then for each $r > 0$, $B(p, r) \cap E' \ni q$ is not empty. Since q is a limit point of E and $r - d(p, q) > 0$, $B(q, r - d(p, q)) \cap E$ is an infinite set. By the triangle inequality, $B(q, r - d(p, q)) \subset B(p, r)$ so $B(p, r) \cap E$ is also infinite and p is therefore a limit point of E , i.e. $p \in E'$. Thus E' contains each of its limit points and it is therefore closed.

(b) Prove that E and $\overline{E}$ have the same limit points.

If p is a limit point of E then it is a limit point of $\overline{E}$ since $E \subset \overline{E}$. If p is a limit point of $\overline{E}$ then $B(p, \frac{1}{n}) \cap (E \setminus \{0\})$ decreases with n ; either it is infinite for all n or it is empty for large n . We show that the second case cannot occur. Indeed this would imply that $B(p, \frac{1}{n}) \cap (E' \setminus \{p\})$ is infinite for all n and hence that p is a limit point of E' ; by the preceding result it is then a limit point of E contradicting the assumption that it is not. Thus a limit point of $\overline{E}$ is a limit point of E .

(c) Do E and E' have the same limit points?

No, not in general. A limit point of E' must be a limit point of E but the converse need not be true. For example consider $E = \{1/n \in \mathbb{R}; n \in \mathbb{N}\}$. This has a single limit point, 0 so $E' = \{0\}$ has no limit points at all.

(2) Chapter 2, Problem 8

(a) Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E ?

Yes. If $E \subset \mathbb{R}^2$ is open then $B(p, r) \subset E$ for some $r' > 0$ and all $0 < r < r'$. Since $B(p, r) \subset \mathbb{R}^2$ is infinite if $r > 0$ it follows that p is a limit point of E .

(b) Same question for E closed?

No, not in general. For instance the set containing a single point $\{0\}$ is closed but has no limit points.

(3) Chapter 2, Problem 9.

Let E° denote all the interior points of $E \subset X$, meaning that $p \in E^\circ$ if $B(p, r) \subset E$ for some $r > 0$.

(a) Prove that E° is always open

If $p \in E^\circ$ then $B(p, r) \subset E$ for some $r > 0$ and if $q \in B(p, r)$ then, by the triangle inequality, $B(q, r - d(p, q)) \subset E$ so $B(p, r) \subset E^\circ$ and hence E° is open.

(b) Prove that E is open if and only if $E = E^\circ$.

Certainly if E is open then $E = E^\circ$ since for each $p \in E$ there exists $r > 0$ such that $B(p, r) \subset E$. Conversely if $E^\circ = E$ then this holds for each $p \in E$ so E is open.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

If $G \subset E$ is open then for each $p \in G$ there exists $r > 0$ such that $B(p, r) \subset G$, hence $B(p, r) \subset E$ so $p \in E^\circ$ and it follows that $G \subset E^\circ$.

- (d) Prove that the complement of E° is the closure of the complement of E .

The complement $(E^\circ)^c$ consists of the points $p \in E$ such that $B(p, r) \cap E^c \neq \emptyset$ for all $r > 0$. Since $p \notin E^c$ this implies that p is a limit point of E^c so

$$(E^\circ)^c \subset \overline{E^c}.$$

Conversely if $p \in \overline{E^c}$ then, by Problem 6 above, either $p \in E^c$ or $p \in (E^c)'$ (or both). In the first case certainly $p \in (E^\circ)^c$, since $E^\circ \subset E$. So we may assume $p \in E$, i.e. $p \notin E^c$, and $p \in (E^c)'$. Then for each $r > 0$ $B(p, r) \cap (E^c) \neq \emptyset$ (since p is a limit point not in the set) and this means $B(p, r)$ is NOT a subset of E for any $r > 0$, hence $p \notin E^\circ$. Thus $\overline{E^c} \subset (E^\circ)^c$ and these two sets are therefore equal.

- (e) Do E and $\overline{E}$ have the same interiors?

Not necessarily. For instance $(0, 1) \cup (1, 2) \subset \mathbb{R}$ is open, so equal to its interior, but its closure is $[0, 2]$ with interior $(0, 2)$ which contains the extra point 1. It is always the case that $E^\circ \subset (\overline{E})^\circ$.

- (f) Do E and E° have the same closures?

Again in general no. For example if $E = \{0\} \subset \mathbb{R}$ its interior is empty but it is closed and non-empty. Clearly the closure of E contains the closure of E° .

- (4) Chapter 2, Problem 11.

- (a) d_1 is not a metric since for the three points 0, $\frac{1}{2}$ and $\frac{1}{4}$

$$\frac{1}{4} = (0 - \frac{1}{2})^2 > (0 - \frac{1}{4})^2 + (\frac{1}{4} - \frac{1}{2})^2 = \frac{1}{8}.$$

- (b) d_2 is a metric. It satisfies the first two axioms trivially. To see the triangle inequality first note that

$$|x - y| \leq |x - z| + |z - y|$$

for any three real numbers. Taking square-roots of both sides (using the monotonicity of $\sqrt{\cdot}$) we find

$$\begin{aligned} d_2(x, y) &= \sqrt{|x - y|} \leq \sqrt{|x - z| + |z - y|} \\ &= \sqrt{(d_2(x, z))^2 + (d_2(z, y))^2} \leq d_2(x, z) + d_2(z, y) \end{aligned}$$

by the usual triangle inequality.

- (c) d_3 is not a metric since $d_3(x, -x) = 0$ for all x .

- (d) d_4 is not a metric since $d_4(1, 2) \neq d_4(2, 1)$.

- (e) d_5 is a metric. Certainly it is symmetric and $d_5(x, y) = 0$ implies $|x - y| = 0$ and hence $x = y$. To get the triangle inequality we need to find the sign of

$$\frac{|x - z|}{1 + |x - z|} + \frac{|y - z|}{1 + |y - z|} - \frac{|x - y|}{1 + |x - y|}.$$

Multiplying by the product of the demoninators (which are all strictly positive) this is the same as the sign of

$$\begin{aligned}
 & (1 + |x - y|)(1 + |y - z|)|x - z| + (1 + |x - y|)(1 + |x - z|)|y - z| \\
 & \quad - (1 + |x - z|)(1 + |y - z|)|x - y| \\
 & = |x - y||y - z||x - z| + 2|y - z||x - z| + (|x - z| + |y - z| - |x - y|)
 \end{aligned}$$

All three terms here are non-negative, the last being the triangle inequality. Thus d_5 does also satisfy the triangle inequality.

Remark: If $d(x, y)$ is a metric then so is

$$\frac{d(x, y)}{1 + d(x, y)}.$$

The proof in general essentially the same.