NOTES TO PROBLEMS FROM RUDIN

CHAPTER 1

#4 Begin the proof by $\boxed{\text{choosing}}$ an element in the given non-empty set.

#5 Let A be a non-empty subset of $\mathbb{R}$ having a lower bound.

a) $\alpha \in \mathbb{R}$ is a lower bound of A iff $-\alpha$ is an upper bound of $-A$.

proof ($\Rightarrow$) Suppose α is a lower bound of A and $\boxed{\text{let } y \in -A}$. Then $-y \in A$ by definition of $-A$. Thus $\alpha \leq -y$ by definition of lower bound. Then $y \leq -\alpha$. Since this holds for each $y \in -A$ we have shown that $-\alpha$ is an upper bound of $-A$ as desired. The proof of the converse ($\Leftarrow$) is similar.

b) Let $\beta = \text{glb.}(-A)$. Then $-\beta = \text{lub.}(A)$.

proof That $-\beta$ is an upper bound of $+A$ follows from the preceding part. $\boxed{\text{Let } x \text{ be any upper bound of } A}$. Again by part a), we conclude $-x$ is a lower bound of $-A$. Then $-x \leq \beta$ by definition of greatest lower bound. But this implies $-\beta \leq x$ for every upper bound of A i.e. that $-\beta$ is the least upper bound of A.

#6 Properties of natural exponents.

a) Properties of integral exponents (valid in any field $\mathbb{F}$). The definition is recursive. Given $a \in \mathbb{F} \setminus \{0\}$, $a^0 := 1$ and $a^{n+1} := a^n$ for all $n \in \mathbb{N}$; $a^{-n} := \frac{1}{a^n}$ for $n \in \mathbb{N}$. Inductive arguments establish the usual laws: $a^{m+n} = a^m \cdot a^n$, $(a^m)^n = a^{mn}$ and $(ab)^n = a^n b^n$ for all $m, n \in \mathbb{Z}$, $b \neq 0$.
[You may take this all for granted].

b) Integral exponents and inequalities (valid in any ordered field). Proposition Suppose $a, b > 0$

(1) If $a^{n_0} < b^{n_0}$ for one positive integer n_0, then $a^n < b^n$ for all positive integers n.

(2) If $a^{n_0} = b^{n_0}$ for one positive integer n_0, then $a^n = b^n$ for all positive integers n.

proof First assume $a < b$ and prove that $a^n < b^n$ for all positive integers n by induction. Similarly, $a = b$ implies $a^n = b^n$ for all positive integers n. All the rest follows by trichotomy.

c) Rational exponents are well-defined in $\mathbb{R}$. Suppose $b > 0$. By definition, for $n \in \mathbb{N}$, $b^{1/n}$ is the unique positive number whose n'th power is b: existence is established in the text and uniqueness follows from b)(2) above. Thus, we have $(b^{1/n})^n = (b^n)^{1/n} = b$ for all positive integers. It follows from a), that for positive integers m, n, p, q with $\frac{m}{n} = \frac{p}{q}$, we have $\left[(b^m)^{1/n}\right]^{nq} = b^{mq} = b^{np} = \left[(b^p)^{1/q}\right]^{nq}$ whence $(b^m)^{1/n} = (b^p)^{1/q}$ by part b2)

d) Laws of rational exponents in $\mathbb{R}$. If $b > 0$, m, n, p, q are positive integers, then

$$\left[b^{\frac{m}{n}} \cdot b^{p/q}\right]^{nq} = b^{mq} \cdot b^{np} = b^{mq+np} = \left[b^{\frac{m}{n}+\frac{p}{q}}\right]^{nq}$$

by a) and c) whence $b^{\frac{m}{n}} \cdot b^{p/q} = b^{\frac{m}{n}+\frac{p}{q}}$ by b2).

e) Order and rational exponents If p, q are positive integers, and $b > 1$ then $b^p > 1$ and $b^{p/q} > 1$ by Part b1). From d), it now follows that $b^r > b^s$ whenever $r > s \in \mathbb{Q}$.

f) <u>Real exponents in $\mathbb{R}$</u> For $b>1$ and $r \in \mathbb{R}$, set $B(r) = \{b^s : s \in \mathbb{Q},\ s \le r\}$. In view of e) above, when $r \in \mathbb{Q}$, b^r is the largest elt of $B(r)$; a fortiori, it is the l.u.b. of $B(r)$. We define $b^x = \sup B(x)$ for every $x \in \mathbb{R}$. To see that $b^x \cdot b^y = b^{x+y}$ for $x, y \in \mathbb{R}$ amounts to proving the following lemma.

<u>Lemma</u> Let A, B be non-empty subsets of $\mathbb{R}^+$ which are bounded. Then $\sup(A \cdot B) = (\sup A)(\sup B)$.

<u>proof</u> Here $AB := \{ab : a \in A, b \in B\}$. Write $\alpha = \sup A$, $\beta = \sup B$. It is clear that $\alpha\beta$ is an upper bound of AB. To see that it is the <u>least</u> upper bound of AB, ~~Suppose~~ $\gamma < \alpha\beta$. By continuity of multiplication, there is a $\delta > 0$ such that $xy > \gamma$ whenever $x > \alpha - \delta$ and $y > \beta - \delta$. By definition of least upper bound, such numbers x, y belong to A, B respectively. But then $xy \in AB$ and γ cannot be an upper bound of AB. This shows that $\alpha\beta$ is indeed the least upper bound of AB.

(Rudin, Chapter 1, pages 22-23)

#8 It is not legitimate to restrict attention to a specific order on $\mathbb{C}$ (e.g. that of the next problem).

#9 (Dictionary order on $\mathbb{C}$). In addressing trichotomy, you must show one <u>and</u> only one of $z<w$, $z=w$, $w<z$ holds for each pair $z, w \in \mathbb{C}$; your proof should be based on trichotomy of the usual order on $\mathbb{R}$.
To see that the least upper bound property fails, you can use $E_1 \equiv \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. Then $a \in \mathbb{C}$ is an upper bound of E_1 iff $\operatorname{Re} a \geq 0$. In particular, if a is an upper bound of E_1, then so is $a - i$ which is strictly smaller than a, whence E_1 has no lub.
Another interesting counterexample is $E_2 \equiv \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$.

#11 (Polar decomposition) If $z = rw$, then $|z| = |rw| = r$ which shows there is only one possible candidate for r. If $z \neq 0$, this forces $w = \frac{z}{|z|}$ which shows there is <u>at most</u> one such expression for z. To complete the discussion, note that $|z| \geq 0$, $\left|\frac{z}{|z|}\right| = 1$ and $|z|\left(\frac{z}{|z|}\right) = z$ and $0 = 0 \cdot 1 = 0 \cdot i$.

#12 For the inductive step, assume $|z_1 + \cdots + z_k| \leq |z_1| + \cdots + |z_k|$ for <u>all</u> choices of $z_1, \ldots, z_k \in \mathbb{C}$ and <u>let</u> $w_1, \ldots, w_{k+1} \in \mathbb{C}$. Complete the proof by showing $|w_1 + \cdots + w_{k+1}| \leq |w_1| + \cdots + |w_{k+1}|$

#13 The triangle inequality yields $|x| = |(x-y)+y| \leq |x-y| + |y|$. Thus $|x| - |y| \leq |x-y|$. Similarly $|y| - |x| \leq |x-y|$, whence $\big||x| - |y|\big| \leq |x-y|$ as desired.