CHAPTER 3

5 LET $\{w_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Then $\limsup w_n \equiv l$ can be characterized in the following ways:

a) $l = \sup E$ where E is the set of limits of subsequences of $\{w_n\}$

b) $l = \lim_{k\to\infty}\left[\sup_{n\geq k} w_n\right]$

c) $l = \sup\{\alpha : w_n > \alpha \text{ for infinitely many } n \in J\}$

d) $l = \inf\{\beta : w_n > \beta \text{ for at most finitely many } n \in J\}$

Prop. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be bounded sequences in $\mathbb{R}$ with $c_n = a_n + b_n$ for all $n \in J$. Then $\limsup c_n \leq \limsup a_n + \limsup b_n$.

proof #1 (USING a) Write $\hat{a} = \limsup a_n$, $\hat{b} = \limsup b_n$ and $\hat{c} = \limsup c_n$ and use "honest" function notation for sequences. There is a subsequence of $\{c_n\}$ converging to $\hat{c}$ i.e. there is a strictly increasing function $f : J \to J$ satisfying $\lim_{n\to\infty} c(f(n)) = \hat{c}$ i.e. the subsequence $c \circ f$ converges to $\hat{c}$. Since $a \circ f$ is a bounded sequence it has a subsequence converging to some number $\tilde{a} \leq \hat{a}$, that is, there is a strictly increasing function $g : J \to J$ such that $a \circ f \circ g$ converges to $\tilde{a}$. Similarly, there is an increasing function $h : J \to J$ such that

$b \circ f \circ g \circ h$ converges to some $\tilde{b} \le \hat{b}$.
Then $c \circ f \circ g \circ h$ is a subsequence of $c \circ f$ and hence must converge to $\hat{c}$.

But $\lim_{n\to\infty} c(f(g(h(n)))) = \lim_{n\to\infty} a(f(g(h(n)))) + \lim_{n\to\infty} b(f(g(h(n))))$

or $\hat{c} = \tilde{b} + \tilde{a} \le \hat{a} + \hat{b}$ since $\tilde{b}, \tilde{a}$ are subsequential limits of b, a respectively.

Proof #2 (using b) Fix k and note that

$$c_m = a_m + b_m \le \sup_{n\ge k} a_n + \sup_{n\ge k} b_n \quad \text{for each } m \ge k$$

Thus $\sup_{n\ge k} a_n + \sup_{n\ge k} b_n$ is an upper bound for $\{c_m\}_{m=k}^{\infty}$

By definition of <u>least</u> upper bound, we have

$$\sup_{n\ge k} c_n \le \sup_{n\ge k} a_n + \sup_{n\ge k} b_n \quad \text{for each } k.$$

Write $x_k = \sup_{n\ge k} a_n$, $y_k = \sup_{n\ge k} b_n$, $z_k = \sup_{n\ge k} c_n$

Since $\{x_k\}$, $\{y_k\}$, and $\{z_k\}$ are monotone and bdd, they converge whence

$$\hat{c} = \lim_{k\to\infty} z_k \le \lim_{k\to\infty} (x_k + y_k) = \lim_{k\to\infty} x_k + \lim_{k\to\infty} y_k = \hat{a} + \hat{b}.$$

Proof #3 (Using d) Let $\varepsilon > 0$. Then

$a_n < \hat{a} + \varepsilon/4$ for all but finitely many $n \in J$

and $b_n < \hat{b} + \varepsilon/4$ for all but finitely many $n \in J$.

$\therefore a_n + b_n < \hat{a} + \hat{b} + \varepsilon/2$ for all but finitely many $n \in J$.

$\therefore c_n > \hat{a} + \hat{b} + \varepsilon$ for at most finitely many $n \in J$.

In other words $\hat{a} + \hat{b} + \varepsilon \in \{\gamma : c_n > \gamma \text{ for at most finitely many } n\} \equiv E$.

By defn of lower bound, we have $\hat{c} = \text{glb}\, E < \hat{a} + \hat{b} + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, hence $\hat{c} \le \hat{a} + \hat{b}$.

6a) $\sum_{n=1}^{\infty} \sqrt{n+1}-\sqrt{n}$ has $s_n = \sqrt{n+1}-1$ and hence diverges.

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$ converges by comparison with $\sum \frac{1}{n^{3/2}}$.

c) $\sum_{n=1}^{\infty} (\sqrt[n]{n}-1)^n$ converges by the root test.

d) Consider the series $\sum_{n=1}^{\infty} \frac{1}{1+z^n}$

If $|z| \le 1$, then $|1+z^n| \le 2$ so $\left|\frac{1}{1+z^n}\right| \ge \frac{1}{2}$ for all n and the series diverges by the n'th term test. (If $|z|=1$, e.g. $z=i$, some of the terms of the series may not even be defined.)

If $|z|>1$, the series converges by the ratio test since $\lim_{n\to\infty}\left|\frac{1+z^n}{1+z^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{z^{n+1}}+\frac{1}{z}}{\frac{1}{z^{n+1}}+1}\right| = \frac{0}{1} < 1.$

Alternatively, you can use the ratio-comparison test: $\left[\lim_{n\to\infty} \frac{\frac{1}{z^n}}{\frac{1}{1+z^n}} = \lim_{n\to\infty} \frac{1}{z^n}+1 = 1\right]$ since the geometric series $\sum_{n=1}^{\infty} \frac{1}{z^n}$ converges.

7) Write $a = \sum_{n=1}^{\infty} a_n$ and $b = \sum_{n=1}^{\infty} \frac{1}{n}$.

By the Cauchy Schwartz inequality we have

$$s_n = \sum_{j=1}^{n} \frac{a_j}{\sqrt{j}} = (a_1, \dots, a_n)\cdot\left(\frac{1}{\sqrt{1}}, \dots, \frac{1}{\sqrt{n}}\right)$$
$$\le \sqrt{a}\sqrt{b} \text{ for each } n \in \mathbb{J}.$$

Thus $\sum_{j=1}^{\infty} \frac{a_j}{\sqrt{j}}$ has bdd partial sums and non-negative terms & must converge.

9a) $\sum_{n=1}^{\infty} n^3 z^n$ has radius of convergence 1

b) $\sum_{n=1}^{\infty} \frac{2^n}{n!} z^n$ has radius of convergence ∞

c) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n$ has radius of convergence $\frac{1}{2}$

d) $\sum_{n=1}^{\infty} \frac{n^3}{3^n} z^n$ has radius of convergence 3.

These can all be done by the root test: for b) note that $n! \geq \left(\frac{n}{2}\right)^{n/2}$ so $0 \leq \sqrt[n]{\frac{2^n}{n!}} \leq 2\sqrt{\frac{2}{n}}$ whence $\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n!}} = 0$ by the squeeze principle. You can also use the ratio test: the radius of convergence of $\sum_{n=1}^{\infty} c_n z^n$ is $\lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$ <u>if this limit exists</u> or is infinite. However the radius of convergence <u>may exceed</u> $\frac{1}{\limsup \left|\frac{c_{n+1}}{c_n}\right|}$ (which equals $\liminf \left(\left|\frac{c_n}{c_{n+1}}\right|\right)$) if $\lim_{n\to\infty} \frac{c_n}{c_{n+1}}$ does not exist.

10) If $\{c_n\} \subseteq \mathbb{Z}$ and $c_n \neq 0$ for infinitely many n, then $\sqrt[n]{|c_n|} \geq \sqrt[n]{1} = 1$ for infinitely many n, whence $\limsup \sqrt[n]{|c_n|} \geq 1$ and $R \leq \frac{1}{1} = 1$.

14a SUPPOSE FIRST $\lim_{n\to\infty} s_n = 0$ and define $\sigma_n = \frac{\sum_{j=0}^{n} s_j}{n+1}$ as in the text. Let $\varepsilon > 0$ be given. Choose M to be an upper bound for $\{|s_n| : n \in \mathbb{J} \cup \{0\}\}$ and N such that $|s_n| < \frac{\varepsilon}{2}$ for all $n \geq N$. Take $K \geq \frac{2NM}{\varepsilon}$. Then if $n \geq K$, we have $|\sigma_n| \leq \frac{\sum_{j=0}^{N} |s_j|}{n+1} + \frac{\sum_{j=N+1}^{n} |s_j|}{n+1}$

$$\leq \frac{NM}{K} + \frac{\varepsilon}{2} < \varepsilon$$

b) If $s_n = (-1)^n$, then the sequence $\{s_n\}$ diverges but $|\sigma_n| \leq \frac{1}{n+1}$ for all n.

[20] Suppose $\{p_n\}_{n=1}^{\infty}$ ~~.....~~ is Cauchy and its subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ converges to L.

Given $\varepsilon > 0$, find $N \in J$ such that $m, n \geq N$ implies $d(p_n, p_m) < \varepsilon/2$. Suppose $n \geq N$. Choose $k_0 > N$ such that $d(p_{n_{k_0}}, L) < \varepsilon/2$.

Then $d(p_n, L) \leq d(p_n, p_{n_{k_0}}) + d(p_{n_{k_0}}, L) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

[This write-up shows that the choice of N does not need to depend on the rate at which the subsequence approaches L].

[21] Suppose $\{E_n\}_{n=1}^{\infty}$ are closed non-empty subsets of a complete metric space (X, d) with $\lim_{n\to\infty} \operatorname{diam} E_n = 0$ and $E_{n+1} \subseteq E_n$ for all n. For each n, choose $x_n \in E_n$. Given $\varepsilon > 0$, choose N such that $\operatorname{diam} E_n < \varepsilon$ for all $n \geq N$. Then if ~~.....~~ $n, m \geq N$, we would have $x_n, x_m \in E_N$ whence $d(x_n, x_m) \leq \operatorname{diam} E_N < \varepsilon$.

This shows $\{x_n\}$ is Cauchy. By completeness of X, $\lim_{n\to\infty} x_n \equiv p$ exists in X. Fix an integer K. Then given $\varepsilon > 0$, there is an integer $n > K$ such that $d(x_n, p) < \varepsilon$. But then $x_n \in E_K$ so $N_\varepsilon(p)$ contains a point in E_K. The arbitrariness of ε shows $p \in \overline{E_K} = E_K$. The arbitrariness of K shows $p \in \bigcap_{n=1}^{\infty} E_n$ so this set is not empty.

Assuming q also belonged to $\bigcap_{n=1}^{\infty} E_n$, we would have $d(p, q) \leq \operatorname{diam} E_n$ for all n. From $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, we get $d(p, q) = 0$ whence $p = q$.